Exercise 2.16. Suppose that X is a set, $J=(a,b)\subset\mathbb{R}$ is an interval, and $f: J\times X\to\mathbb{C}$ is a function satisfying:

1. For each $x\in X$, the function $t\to f(t,x)$ is differentiable on J,
2. There is a summable function $g: X\to[0,\infty)$ such that
$$\left|\dot{f}(t,x)\right| := \left|\frac{d}{dt}f(t,x)\right| \le g(x) \text{ for all } x\in X.$$
3. There is a $t_0\in J$ such that $\sum_{x\in X}|f(t_0,x)|<\infty$.

Show:

a) for all $t\in J$ that $\sum_{x\in X}|f(t,x)|<\infty$.
b) Let $F(t):=\sum_{x\in X}f(t,x)$, show F is differentiable on J and that
$$\dot{F}(t)=\sum_{x\in X}\dot{f}(t,x).$$

(Hint: Use the mean value theorem.)

Exercise 2.17. Let $\{a_n\}_{n=-\infty}^{\infty}$ be a summable sequence of complex numbers, i.e.
$$\sum_{n=-\infty}^{\infty}|a_n|<\infty.$$
For $t\ge 0$ and $x\in\mathbb{R}$, define
$$f(t,x)=\sum_{n=-\infty}^{\infty}a_n e^{-tn^2}e^{inx},$$
where as usual $e^{ix}=\cos(x)+i\sin(x)$. Prove the following facts about $f:$

1. $f(t,x)$ is continuous for $(t,x)\in[0,\infty)\times\mathbb{R}$.
2. $\partial f(t,x)/\partial t$, $\partial f(t,x)/\partial x$ and $\partial^2 f(t,x)/\partial x^2$ exist for $t>0$ and $x\in\mathbb{R}$.
3. f satisfies the heat equation, namely
$$\partial f(t,x)/\partial t=\partial^2 f(t,x)/\partial x^2 \text{ for } t>0 \text{ and } x\in\mathbb{R}.$$

2.7.4. *Inequalities.*

Exercise 2.18. Generalize Proposition 2.25 as follows. Let $a\in[-\infty,0]$ and $f:\mathbb{R}\cap[a,\infty)\to[0,\infty)$ be a continuous strictly increasing function such that $\lim_{s\to\infty}f(s)=\infty$, $f(a)=0$ if $a>-\infty$ or $\lim_{s\to-\infty}f(s)=0$ if $a=-\infty$. Also let $g=f^{-1}$, $b=f(0)\ge 0$,
$$F(s)=\int_0^s f(s')ds' \text{ and } G(t)=\int_0^t g(t')dt'.$$
Then for all $s,t\ge 0$,
$$st\le F(s)+G(t\vee b)\le F(s)+G(t)$$
and equality holds iff $t=f(s)$. In particular, taking $f(s)=e^s$, prove Young's inequality stating
$$st\le e^s+(t\vee 1)\ln(t\vee 1)-(t\vee 1)\le e^s+t\ln t-t.$$
Hint: Refer to the following pictures.

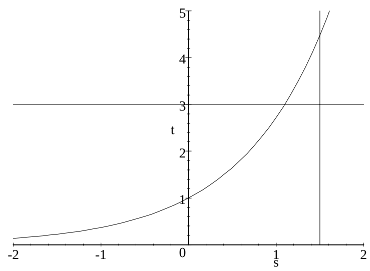

FIGURE 3. Comparing areas when $t \geq b$ goes the same way as in the text.

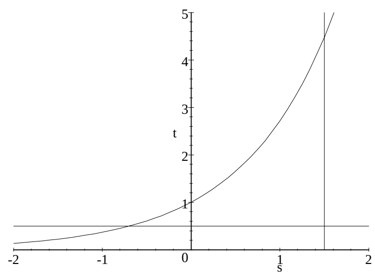

FIGURE 4. When $t \leq b$, notice that $g(t) \leq 0$ but $G(t) \geq 0$. Also notice that $G(t)$ is no longer needed to estimate st.

3. Metric and Banach Spaces I

3.1. Basic metric space notions.

Definition 3.1. A function $d : X \times X \to [0, \infty)$ is called a metric if

1. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
2. (Non-degenerate) $d(x, y) = 0$ if and only if $x = y \in X$
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

As primary examples, any normed space $(X, \|\cdot\|)$ is a metric space with $d(x, y) := \|x - y\|$. Thus the space $\ell^p(\mu)$ is a metric space for all $p \in [1, \infty]$. Also any subset of a metric space is a metric space. For example a surface Σ in $\mathbb{R}^3$ is a metric space with the distance between two points on Σ being the usual distance in $\mathbb{R}^3$.

Definition 3.2. Let (X, d) be a metric space. The **open ball** $B(x, \delta) \subset X$ centered at $x \in X$ with radius $\delta > 0$ is the set

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

We will often also write $B(x, \delta)$ as $B_x(\delta)$. We also define the **closed ball** centered at $x \in X$ with radius $\delta > 0$ as the set $C_x(\delta) := \{y \in X : d(x, y) \leq \delta\}$.

Definition 3.3. A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to be convergent if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$.

Exercise 3.1. Show that x in Definition 3.3 is necessarily unique.

Definition 3.4. A set $F \subset X$ is closed iff every convergent sequence $\{x_n\}_{n=1}^{\infty}$ which is contained in F has its limit back in F. A set $V \subset X$ is open iff V^c is closed. We will write $F \sqsubset X$ to indicate the F is a closed subset of X and $V \subset_o X$ to indicate the V is an open subset of X. We also let τ_d denote the collection of open subsets of X relative to the metric d.

Exercise 3.2. Let $\mathcal{F}$ be a collection of closed subsets of X, show $\cap\mathcal{F} := \cap_{F\in\mathcal{F}}F$ is closed. Also show that finite unions of closed sets are closed, i.e. if $\{F_k\}_{k=1}^{n}$ are closed sets then $\cup_{k=1}^{n}F_k$ is closed. (By taking compliments, this shows that the collection of open sets, τ_d, is closed under finite intersections and arbitrary unions.)

Exercise 3.3. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B_x(\delta) \subset V$. In particular show $B_x(\delta)$ is open for all $x \in X$ and $\delta > 0$.

Definition 3.5. Given a set A contained a metric space X, let

$$\bar{A} := \{x \in X : \exists\ \{x_n\} \subset A \ni x = \lim_{n\to\infty} x_n\}.$$

That is to say $\bar{A}$ contains all **limit points** of A.

Exercise 3.4. Given $A \subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A} = \cap\{F : A \subset F \subset X \text{ with } F \text{ closed}\}.$$

That is to say $\bar{A}$ is the smallest closed set containing A.

3.2. **Continuity.** Suppose that (X, d) and (Y, ρ) are two metric spaces and $f : X \to Y$ is a function.

Definition 3.6. A function $f : X \to Y$ is continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$d(f(x), f(x')) < \epsilon \text{ provided that } \rho(x, x') < \delta.$$

The function f is said to be continuous if f is continuous at all points $x \in X$.

The following lemma gives three other ways to characterize continuous functions.

Lemma 3.7. *Suppose that (X, ρ) and (Y, d) are two metric spaces and $f : X \to Y$ is a function. Then following are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(V) \in \tau_\rho$ for all $V \in \tau_d$, i.e. $f^{-1}(V)$ is open in X if V is open in Y.*
3. *$f^{-1}(C)$ is closed in X if C is closed in Y.*
4. *For all convergent sequences $\{x_n\} \subset X$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Proof. 1. $\Rightarrow$ 2. For all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ if $\rho(x, x') < \delta$. i.e.

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$$

So if $V \subset_0 Y$ and $x \in f^{-1}(V)$ we may choose $\epsilon > 0$ such that $B_{f(x)}(\epsilon) \subseteq V$ then

$$B_x(\delta) \subseteq f^{-1}(B_{f(x)}(\epsilon)) \subseteq f^{-1}(V)$$

showing that $f^{-1}(V)$ is open.

2. $\iff$ 3. If C is closed in Y, then $C^c \subset_o Y$ and hence $f^{-1}(C^c) \subset_o X$. Since $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$, this shows that $f^{-1}(C)$ is the complement of an open set and hence closed. Similarly one shows that 3. $\Rightarrow$ 2.

2. $\Rightarrow$ 1. Let $\epsilon > 0$ and $x \in X$, then, since $f^{-1}(B_{f(x)}(\epsilon)) \subset_o X$,there exists $\delta > 0$ such that $B_x(\delta) \subseteq f^{-1}(B_{f(x)}(\epsilon))$ i.e. if $\rho(x,x') < \delta$ then $d(f(x'), f(x)) < \epsilon$.

1. $\Rightarrow$ 4. If f is continuous and $x_n \to x$ in X, let $\epsilon > 0$ and choose $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ when $\rho(x,x') < \delta$. There exists an $N > 0$ such that $\rho(x, x_n) < \delta$ for all $n \geq N$ and therefore $d(f(x), f(x_n)) < \epsilon$ for all $n \geq N$. That is to say $\lim_{n\to\infty} f(x_n) = f(x)$ as $n \to \infty$.

4. $\Rightarrow$ 1.We will show that not 1. $\Rightarrow$ not 4. not 1 implies there exists $\epsilon > 0$, a point $x \in X$ and a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $d(f(x), f(x_n)) \geq \epsilon$ while $\rho(x, x_n) < \frac{1}{n}$. Clearly this sequence $\{x_n\}$ violates 4. ■

The next lemma supplies some examples of continuous functions on metric spaces.

Lemma 3.8. *For any non empty subset $A \subset X$, let $d_A(x) \equiv \inf\{d(x,a) | a \in A\}$, then*

$$|d_A(x) - d_A(y)| \leq d(x,y)\ \forall\, x,y \in X. \tag{3.1}$$

In particular, d_A is a continuous function on X. Moreover, by Lemma 3.7, for all $\epsilon > 0$ the set $F_\epsilon \equiv \{x \in X | d_A(x) \geq \epsilon\}$ is closed in X. Further, if V is an open set and $A = V^c$, then $F_\epsilon \uparrow V$ as $\epsilon \downarrow 0$.

Proof. Let $a \in A$ and $x, y \in X$, then

$$d(x,a) \leq d(x,y) + d(y,a).$$

Take the inf over a in the above equation shows that

$$d_A(x) \leq d(x,y) + d_A(y) \quad \forall x, y \in X.$$

Therefore, $d_A(x) - d_A(y) \leq d(x,y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \leq d(x,y)$ which implies Eq. (3.1) from which it follows that d_A is continuous on X.

Now suppose that $A = V^c$ with $V \in \tau$. It is clear that $d_A(x) = 0$ for $x \in A = V^c$ so that $F_\epsilon \subset V$ for each $\epsilon > 0$ and hence $\cup_{\epsilon>0} F_\epsilon \subset V$. Now suppose that $x \in V$, then there exists an $\epsilon > 0$ such that $B_x(\epsilon) \subset V$, that is it $y \in X$ such that $d(x,y) < \epsilon$ then $y \in V$. Therefore $d(x,y) \geq \epsilon$ for all $y \in V^c$ and hence $x \in F_\epsilon$, i.e. $V \subset \cup_{\epsilon>0} F_\epsilon$. Finally it is clear that $F_\epsilon \subset F_{\epsilon'}$ whenever $\epsilon' \leq \epsilon$. ■

Corollary 3.9. *The function d satisfies,*

$$|d(x,y) - d(x',y')| \leq d(y,y') + d(x,x')$$

and in particular $d : X \times X \to [0,\infty)$ is continuous.

Proof. By Lemma 3.8 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned} |d(x,y) - d(x',y')| &\leq |d(x,y) - d(x,y')| + |d(x,y') - d(x',y')| \\ &\leq d(y,y') + d(x,x'). \end{aligned}$$

■

Exercise 3.5. Show the closed ball $C_x(\delta) := \{y \in X : d(x,y) \leq \delta\}$ is a closed subset of X.

3.2.1. *Word of Caution.*

Example 3.10. Let (X, d) be a metric space. It is always true that $\overline{B_x(\epsilon)} \subset C_x(\epsilon)$ since $C_x(\epsilon)$ is a closed set containing $B_x(\epsilon)$. However, it is not always true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$. For example let $X = \{1, 2\}$ and $d(1, 2) = 1$, then $B_1(1) = \{1\}$, $\overline{B_1(1)} = \{1\}$ while $C_1(1) = X$. For another counter example, take

$$X = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } x = 1\right\}$$

with the usually Euclidean metric coming from the plane. Then

$$\begin{aligned} B_{(0,0)}(1) &= \left\{(0, y) \in \mathbb{R}^2 : |y| < 1\right\}, \\ \overline{B_{(0,0)}(1)} &= \left\{(0, y) \in \mathbb{R}^2 : |y| \le 1\right\}, \text{ while} \\ C_{(0,0)}(1) &= \overline{B_{(0,0)}(1)} \cup \{(0, 1)\}. \end{aligned}$$

In spite of the above examples, Lemmas 3.11 and 3.47 below shows that for certain metric spaces of interest it is true that $\overline{B_x(\epsilon)} = C_x(\epsilon)$.

Lemma 3.11. *Suppose that $(X, |\cdot|)$ is a normed vector space and d is the metric on X defined by $d(x, y) = |x - y|$. Then*

$$\begin{aligned} \overline{B_x(\epsilon)} &= C_x(\epsilon) \text{ and} \\ \partial B_x(\epsilon) &= \{y \in X : d(x, y) = \epsilon\}. \end{aligned}$$

Proof. We must show that $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. For $y \in C$, let $v = y - x$, then

$$|v| = |y - x| = d(x, y) \le \epsilon.$$

Let $\alpha_n = 1 - 1/n$ so that $\alpha_n \uparrow 1$ as $n \to \infty$. Let $y_n = x + \alpha_n v$, then $d(x, y_n) = \alpha_n d(x, y) < \epsilon$, so that $y_n \in B_x(\epsilon)$ and $d(y, y_n) = 1 - \alpha_n \to 0$ as $n \to \infty$. This shows that $y_n \to y$ as $n \to \infty$ and hence that $y \in \bar{B}$. ■

3.3. **Basic Topological Notions.** Using the metric space results above we will axiomatize the notion of being an open set to more general settings.

Definition 3.12. A collection of subsets τ of X is a **topology** if

1. $\emptyset, X \in \tau$
2. τ is closed under arbitrary unions, i.e. if $V_\alpha \in \tau$, for $\alpha \in I$ then $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
3. τ is closed under finite intersections, i.e. if $V_1, \ldots, V_n \in \tau$ then $V_1 \cap \cdots \cap V_n \in \tau$.

Notation 3.13. The subsets $V \subset X$ which are in τ are called open sets and we will abbreviate this by writing $V \subset_0 X$ and the those sets $F \subset X$ such that $F^c \in \tau$ are called closed sets. We will write $F \sqsubset X$ if F is a closed subset of X. Also if $A \subset X$, we define the closure of A to be the smallest closed set $\bar{A}$ containing A, i.e.

$$\bar{A} := \cap \{F : A \subset F \sqsubset X\}.$$

Example 3.14.
1. Let (X, d) be a metric space, we write τ_d for the collection of d – open sets in X. We have already seen that τ_d is a topology, see Exercise 3.2.
2. Let X be any set, then $\tau = \mathcal{P}(X)$ is a topology. In this topology all subsets of X are both open and closed. At the opposite extreme we have the **trivial** topology, $\tau = \{\emptyset, X\}$. In this topology only the empty set and X are open (closed).

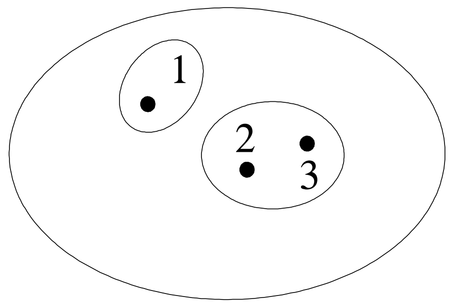

FIGURE 5. A topology

3. Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on X which does not come from a metric.
4. Again let $X = \{1, 2, 3\}$. Then $\tau = \{\{1\}, \{2, 3\}, \emptyset, X\}$. is a topology, and the sets X, $\{1\}$, $\{2, 3\}, \phi$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor closed.

Definition 3.15. Let (X, τ) be a topological space, $A \subset X$ and $i_A : A \to X$ be the inclusion map, i.e. $i_A(a) = a$ for all $a \in A$. Define

$$\tau_A = i_A^{-1}(\tau) = \{A \cap V : V \in \tau\},$$

the so called **relative topology** on A.

Exercise 3.6. Show the relative topology is a topology on A. Also show if (X, d) is a metric space and $\tau = \tau_d$ is the topology coming from d, then $(\tau_d)_A$ is the topology induced by making A into a metric space using the metric $d|_{A \times A}$.

Definition 3.16. Let (X, τ) be a topological space and $A \subset X$. We say a subset $\mathcal{U} \subset \tau$ is an **open cover** of A if $A \subset \cup\mathcal{U}$. The set A is said to be **compact** if every open cover of A has finite a sub-cover, i.e. if $\mathcal{U}$ is an open cover of A there exists $\mathcal{U}_0 \subset\subset \mathcal{U}$ such that $\mathcal{U}_0$ is a cover of A. (We will write $A \sqsubset\sqsubset X$ to denote that $A \subset X$ and A is compact.) A subset $A \subset X$ is **precompact** if $\bar{A}$ is compact.

Exercise 3.7. Let (X, τ) be a topological space. Show that $A \subset X$ is compact iff (A, τ_A) is a compact topological space.

Definition 3.17. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** if $f^{-1}(\tau_Y) \subseteq \tau_X$. We will also say that f is τ_X/τ_Y – continuous or (τ_X, τ_Y) – continuous.

Definition 3.18 (Support)**.** Let $f : X \to Y$ be a function from a topological space (X, τ_X) to a vector space Y. Then we define the support of f by

$$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}},$$

a closed subset of X.

Notation 3.19. If X and Y are two topological spaces, let $C(X, Y)$ denote the continuous functions from X to Y. If Y is a Banach space, let

$$BC(X, Y) := \{f \in C(X, Y) : \sup_{x \in X} \|f(x)\|_Y < \infty\}$$

and

$$C_c(X,Y) := \{f \in C(X,Y) : \text{supp}(f) \text{ is compact}\}.$$

If $Y = \mathbb{R}$ or $\mathbb{C}$ we will simply write $C(X)$, $BC(X)$ and $C_c(X)$ for $C(X,Y)$, $BC(X,Y)$ and $C_c(X,Y)$ respectively.

3.4. **Completeness.**

Definition 3.20 (Cauchy sequences)**.** A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X,d) is **Cauchy** provided that

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Exercise 3.8. Show that convergent sequences are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x,y) = |x-y|$. Choose a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^\infty$ is $(\mathbb{Q},d)$ – Cauchy but not $(\mathbb{Q},d)$ – convergent. The sequence does converge in $\mathbb{R}$ however.

Definition 3.21. A metric space (X,d) is **complete** if all Cauchy sequences are convergent sequences.

Exercise 3.9. Let (X,d) be a complete metric space. Let $A \subset X$ be a subset of X viewed as a metric space using $d|_{A\times A}$. Show that $(A, d|_{A\times A})$ is complete iff A is a closed subset of X.

Definition 3.22. If $(X, \|\cdot\|)$ is a normed vector space, then we say $\{x_n\}_{n=1}^\infty \subset X$ is a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. The normed vector space is a **Banach space** if it is complete, i.e. if every $\{x_n\}_{n=1}^\infty \subset X$ which is Cauchy is convergent where $\{x_n\}_{n=1}^\infty \subset X$ is convergent iff there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. As usual we will abbreviate this last statement by writing $\lim_{n\to\infty} x_n = x$.

Lemma 3.23. *Suppose that X is a set then the bounded functions $\ell^\infty(X)$ on X is a Banach space with the norm*

$$\|f\| = \|f\|_\infty = \sup_{x\in X} |f(x)|.$$

Moreover if X is a topological space the set $BC(X) \subset \ell^\infty(X)$ is closed subspace of $\ell^\infty(X)$ and hence is also a Banach space.

Proof. Let $\{f_n\}_{n=1}^\infty \subset \ell^\infty(X)$ be a Cauchy sequence. Since for any $x \in X$, we have

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \tag{3.2}$$

which shows that $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{F}$ is a Cauchy sequence of numbers. Because $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is complete, $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Passing to the limit $n \to \infty$ in Eq. (3.2) implies

$$|f(x) - f_m(x)| \le \limsup_{n\to\infty} \|f_n - f_m\|_\infty$$

and taking the supremum over $x \in X$ of this inequality implies

$$\|f - f_m\|_\infty \le \limsup_{n\to\infty} \|f_n - f_m\|_\infty \to 0 \text{ as } m \to \infty$$

showing $f_m \to f$ in $\ell^\infty(X)$.

For the second assertion, suppose that $\{f_n\}_{n=1}^\infty \subset BC(X) \subset \ell^\infty(X)$ and $f_n \to f \in \ell^\infty(X)$. We must show that $f \in BC(X)$, i.e. that f is continuous. To this end let $x, y \in X$, then

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\le 2\|f - f_n\|_\infty + |f_n(x) - f_n(y)|.\end{aligned}$$

Thus if $\epsilon > 0$, we may choose n large so that $2\|f - f_n\|_\infty < \epsilon/2$ and then for this n there exists an open neighborhood V_x of $x \in X$ such that $|f_n(x) - f_n(y)| < \epsilon/2$ for $y \in V_x$. Thus $|f(x) - f(y)| < \epsilon$ for $y \in V_x$ showing the limiting function f is continuous. ■

Remark 3.24. Let X be a set, Y be a Banach space and $\ell^\infty(X, Y)$ denote the bounded functions $f : X \to Y$ equipped with the norm $\|f\| = \|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y$. If X is a topological space, let $BC(X, Y)$ denote those $f \in \ell^\infty(X, Y)$ which are continuous. The same proof used in Lemma 3.23 shows that $\ell^\infty(X, Y)$ is a Banach space and that $BC(X, Y)$ is a closed subspace of $\ell^\infty(X, Y)$.

Theorem 3.25 (Completeness of $\ell^p(\mu)$)**.** *Let X be a set and $\mu : X \to (0, \infty]$ be a given function. Then for any $p \in [1, \infty]$, $(\ell^p(\mu), \|\cdot\|_p)$ is a Banach space.*

Proof. We have already proved this for $p = \infty$ in Lemma 3.23 so we now assume that $p \in [1, \infty)$ and write $\|\cdot\|$ for $\|\cdot\|_p$. Let $\{f_n\}_{n=1}^\infty \subset \ell^p(\mu)$ be a Cauchy sequence. Since for any $x \in X$,

$$|f_n(x) - f_m(x)| \le \frac{1}{\mu(x)} \|f_n - f_m\|_p \to 0 \text{ as } m, n \to \infty$$

it follows that $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence of numbers and $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. By Fatou's Lemma,

$$\begin{aligned}\|f_n - f\|_p^p &= \sum_X \mu \cdot \lim_{m\to\infty} \inf |f_n - f_m|^p \le \lim_{m\to\infty} \inf \sum_X \mu \cdot |f_n - f_m|^p \\ &= \lim_{m\to\infty} \inf \|f_n - f_m\|_p^p \to 0 \text{ as } n \to \infty.\end{aligned}$$

This then shows that $f = (f - f_n) + f_n \in \ell^p(\mu)$ (being is the sum of two ℓ^p – functions) and that $f_n \xrightarrow{\ell^p} f$. ■

Example 3.26. Here are a couple of examples of complete metric spaces.

1. $X = \mathbb{R}$ and $d(x, y) = |x - y|$.
2. $X = \mathbb{R}^n$ and $d(x, y) = \|x - y\|_2 = \sum_{i=1}^n (x_i - y_i)^2$.
3. $X = \ell^p(\mu)$ for $p \in [1, \infty]$ and any weight function μ.
4. $X = C([0, 1], \mathbb{R})$ – the space of continuous functions from $[0, 1]$ to $\mathbb{R}$ and $d(f, g) := \max_{t\in[0,1]} |f(t) - g(t)|$. This is a special case of Lemma 3.23.
5. Here is a typical example of a non-complete metric space. Let $X = C([0, 1], \mathbb{R})$ and
$$d(f, g) := \int_0^1 |f(t) - g(t)|\, dt.$$

3.5. **Compactness in Metric Spaces.** Let (X, ρ) be a metric space and let $B'_x(\epsilon) = B_x(\epsilon) \setminus \{x\}$.

Definition 3.27. A point $x \in X$ is an accumulation point of a subset $E \subset X$ if $\emptyset \ne E \cap V \setminus \{x\}$ for all $V \subset_o X$ containing x.

Let us start with the following elementary lemma which is left as an exercise to the reader.

Lemma 3.28. *Let $E \subset X$ be a subset of a metric space (X, ρ). Then the following are equivalent:*

1. *$x \in X$ is an accumulation point of E.*
2. *$B'_x(\epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$.*
3. *$B_x(\epsilon) \cap E$ is an infinite set for all $\epsilon > 0$.*
4. *There exists $\{x_n\}_{n=1}^{\infty} \subset E \setminus \{x\}$ with $\lim_{n\to\infty} x_n = x$.*

Definition 3.29. A metric space (X, ρ) is said to be $\epsilon -$ **bounded** $(\epsilon > 0)$ provided there exists a finite cover of X by balls of radius ϵ. The metric space is **totally bounded** if it is $\epsilon -$ bounded for all $\epsilon > 0$.

Theorem 3.30. *Let X be a metric space. The following are equivalent.*

(a) *X is compact.*
(b) *Every infinite subset of X has an accumulation point.*
(c) *X is totally bounded and complete.*

Proof. The proof will consist of showing that $a \Rightarrow b \Rightarrow c \Rightarrow a$.

$(a \Rightarrow b)$ We will show that **not** $b \Rightarrow$ **not** a. Suppose there exists $E \subset X$, such that $\#(E) = \infty$ and E has no accumulation points. Then for all $x \in X$ there exists $V_x \in \tau_x$ such that $(V_x \setminus \{x\}) \cap E = \emptyset$. Clearly $\mathcal{V} = \{V_x\}_{x\in X}$ is a cover of X, yet $\mathcal{V}$ has no finite sub cover. Indeed, for each $x \in X$, $V_x \cap E$ consists of at most one point, therefore if $\Lambda \subset\subset X$, $\cup_{x\in\Lambda} V_x$ can only contain a finite number of points from E, in particular $X \neq \cup_{x\in\Lambda} V_x$.

$(b \Rightarrow c)$ To show X is complete, let $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence and $E := \{x_n : n \in \mathbb{N}\}$. If $\#(E) < \infty$, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}$ which is constant and hence convergent. If E is an infinite set it has an accumulation point by assumption and hence Lemma 3.28 implies that $\{x_n\}$ has a convergence subsequence.

We now show that X is totally bounded. Let $\epsilon > 0$ be given and choose $x_1 \in X$. If possible choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$, then if possible choose $x_3 \in X$ such that $d(x_3, \{x_1, x_2\}) \geq \epsilon$ and continue inductively choosing points $\{x_j\}_{j=1}^{n} \subset X$ such that $d(x_n, \{x_1, \ldots, x_{n-1}\}) \geq \epsilon$. This process must terminate, for otherwise we could choose $E = \{x_j\}_{j=1}^{\infty}$ and infinite number of distinct points such that $d(x_j, \{x_1, \ldots, x_{j-1}\}) \geq \epsilon$ for all $j = 2, 3, 4, \ldots$. Since for all $x \in X$ the $B_x(\epsilon/3) \cap E$ can contain at most one point, no point $x \in X$ is an accumulation point of E.

$(c \Rightarrow a)$ For sake of contradiction, assume there exists a cover an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ of X with no finite subcover. Since X is totally bounded for each $n \in \mathbb{N}$ there exists $\Lambda_n \subset\subset X$ such that

$$X = \bigcup_{x\in\Lambda_n} B_x(1/n) = \bigcup_{x\in\Lambda_n} C_x(1/n).$$

Choose $x_1 \in \Lambda_1$ such that no finite subset of $\mathcal{V}$ covers $K_1 := C_{x_1}(1)$. Since $K_1 = \cup_{x\in\Lambda_2} K_1 \cap C_x(1/2)$, there exists $x_2 \in \Lambda_2$ such that $K_2 := K_1 \cap C_{x_2}(1/2)$ can not be covered by a finite subset of $\mathcal{V}$. Continuing this way inductively, we construct sets $K_n = K_{n-1} \cap C_{x_n}(1/n)$ with $x_n \in \Lambda_n$ such no K_n can be covered by a finite subset of $\mathcal{V}$. Now choose $y_n \in K_n$ for each n. Since $\{K_n\}_{n=1}^{\infty}$ is a decreasing sequence of closed sets such that $\mathrm{diam}(K_n) \leq 2/n$, it follows that $\{y_n\}$ is a Cauchy and hence convergent with

$$y = \lim_{n\to\infty} y_n \in \cap_{m=1}^\infty K_m.$$

Since $\mathcal{V}$ is a cover of X, there exists $V \in \mathcal{V}$ such that $x \in V$. Since $K_n \downarrow \{y\}$ and $\operatorname{diam}(K_n) \to 0$, it now follows that $K_n \subset V$ for some n large. But this violates the assertion that K_n can not be covered by a finite subset of $\mathcal{V}$. ■

Corollary 3.31. *Let X be a metric space then X is compact iff **all** sequences $\{x_n\} \subset X$ have convergent subsequences.*

Proof. If X is compact and $\{x_n\} \subset X$

1. If $\#(\{x_n : n = 1, 2, \dots\}) < \infty$ then choose $x \in X$ such that $x_n = x$ i.o. let $\{n_k\} \subset \{n\}$ such that $x_{n_k} = x$ for all k. Then $x_{n_k} \to x$
2. If $\#(\{x_n : n = 1, 2, \dots\}) = \infty$. We know $E = \{x_n\}$ has an accumulation point $\{x\}$, hence there exists $x_{n_k} \to x$.

Conversely if E is an infinite set let $\{x_n\}_{n=1}^\infty \subset E$ be a sequence of distinct elements of E. We may, by passing to a subsequence, assume $x_n \to x \in X$ as $n \to \infty$. Now $x \in X$ is an accumulation point of E by Theorem 3.30 and hence X is compact. ■

Corollary 3.32. *Compact subsets of $\mathbb{R}^n$ are the closed and bounded sets.*

Proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M, M]^n$ for some positive integer M. For $\delta > 0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M, M]^n := \{\delta x : x \in \mathbb{Z}^n \text{ and } \delta|x_i| \le M \text{ for } i = 1, 2, \dots, n\}.$$

We will shows that by choosing $\delta > 0$ sufficiently small, that

$$K \subset [-M, M]^n \subset \cup_{x\in\Lambda_\delta} B(x, \epsilon) \tag{3.3}$$

which shows that K is totally bounded. Hence by Theorem 64.8, K will be compact.

Suppose that $y \in [-M, M]^n$, then there exists $x \in \Lambda_\delta$ such that $|y_i - x_i| \le \delta$ for $i = 1, 2, \dots, n$. Hence

$$d^2(x, y) = \sum_{i=1}^n (y_i - x_i)^2 \le n\delta^2$$

which shows that $d(x, y) \le \sqrt{n}\delta$. Hence if choose $\delta < \epsilon/\sqrt{n}$ we have shows that $d(x, y) < \epsilon$, i.e. Eq. (3.3) holds. ■

For Exercises 3.10 – 3.12, let (X, d) be a compact metric space.

Exercise 3.10 (Extreme value theorem)**.** Let $f : X \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf f \le \sup f < \infty$ and there exists $a, b \in X$ such that $f(a) = \inf f$ and $f(b) = \sup f$.

Exercise 3.11 (Uniform Continuity)**.** Let $f : X \to \mathbb{R}$ be a continuous function. Show that f is uniformly continuous, i.e. if $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ if $x, y \in X$ with $d(x, y) < \delta$.

Exercise 3.12 (Dini's Theorem)**.** Let $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x\in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\epsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \epsilon\}$.

Definition 3.33. Let L be a vector space. We say that two norms, $|\cdot|$ and $\|\cdot\|$, on L are equivalent if there exists constants $\alpha, \beta \in (0, \infty)$ such that

$$\|f\| \leq \alpha |f| \text{ and } |f| \leq \beta \|f\| \text{ for all } f \in L.$$

Lemma 3.34. *Let L be a finite dimensional vector space. Then any two norms $|\cdot|$ and $\|\cdot\|$ on L are equivalent. (This is typically not true for norms on infinite dimensional spaces.)*

Proof. Let $\{f_i\}_{i=1}^n$ be a basis for L and define a new norm on L by

$$\left\|\sum_{i=1}^n a_i f_i\right\|_1 \equiv \sum_{i=1}^n |a_i| \text{ for } a_i \in \mathbb{F}.$$

By the triangle inequality of the norm $|\cdot|$, we find

$$\left|\sum_{i=1}^n a_i f_i\right| \leq \sum_{i=1}^n |a_i| |f_i| \leq M \sum_{i=1}^n |a_i| = M \left\|\sum_{i=1}^n a_i f_i\right\|_1$$

where $M = \max_i |f_i|$. Thus we have

$$|f| \leq M \|f\|_1$$

for all $f \in L$. This inequality shows that $|\cdot|$ is continuous relative to $\|\cdot\|_1$. Now let $S := \{f \in L : \|f\|_1 = 1\}$, a compact subset of L relative to $\|\cdot\|_1$. Therefore my Exercise 3.10 there exists $f_0 \in S$ such that

$$m = \inf\{|f| : f \in S\} = |f_0| > 0.$$

Hence given $0 \neq f \in L$, then $\frac{f}{\|f\|_1} \in S$ so that

$$m \leq \left|\frac{f}{\|f\|_1}\right| = |f| \frac{1}{\|f\|_1}$$

or equivalently

$$\|f\|_1 \leq \frac{1}{m} |f|.$$

This shows that $|\cdot|$ and $\|\cdot\|_1$ are equivalent norms. Similarly one shows that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent and hence so are $|\cdot|$ and $\|\cdot\|$. ■

Definition 3.35. A subset D of a topological space X is **dense** if $\bar{D} = X$. A topological space is said to be **separable** if it contains a countable dense subset, D.

Example 3.36. Let $\mu : \mathbb{N} \to (0, \infty)$ be a function, then $\ell^p(\mu)$ is separable for all $1 \leq p < \infty$. For example, let $\Gamma \subset \mathbb{F}$ be a countable dense set, then

$$D := \{x \in \ell^p(\mu) : x_i \in \Gamma \text{ for all } i \text{ and } \#\{j : x_j \neq 0\} < \infty\}.$$

The set Γ can be taken to be $\mathbb{Q}$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q} + i\mathbb{Q}$ if $\mathbb{F} = \mathbb{C}$.

Lemma 3.37. *Any compact metric space (X, d) is separable.*

Proof. To each integer n, there exists $\Lambda_n \subset\subset X$ such that $X = \cup_{x \in \Lambda_n} B(x, 1/n)$. Let $D := \cup_{n=1}^\infty \Lambda_n$ – a countable subset of X. Moreover, it is clear by construction that $\bar{D} = X$. ■

3.6. Bounded Linear Operators Basics.

Definition 3.38. Let X and Y be normed spaces and $T : X \to Y$ be a linear map. Then T is said to be bounded provided there exists $C < \infty$ such that $\|T(x)\| \leq C\|x\|_X$ for all $x \in X$. We denote the best constant by $\|T\|$, i.e.

$$\|T\| = \sup_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{x\neq 0} \{\|T(x)\| : \|x\| = 1\}.$$

The number $\|T\|$ is called the operator norm of T.

Proposition 3.39. *Suppose that X and Y are normed spaces and $T : X \to Y$ is a linear map. The the following are equivalent:*

(a) *T is continuous.*
(b) *T is continuous at 0.*
(c) *T is bounded.*

Proof. (a) $\Rightarrow$ (b) trivial. (b) $\Rightarrow$ (c) If T continuous at 0 then there exist $\delta > 0$ such that $\|T(x)\| \leq 1$ if $\|x\| \leq \delta$. Therefore for any $x \in X$, $\|T(\delta x/\|x\|)\| \leq 1$ which implies that $\|T(x)\| \leq \frac{1}{\delta}\|x\|$ and hence $\|T\| \leq \frac{1}{\delta} < \infty$. (c) $\Rightarrow$ (a) Let $x \in X$ and $\epsilon > 0$ be given. Then

$$\|T(y) - T(x)\| = \|T(y - x)\| \leq \|T\| \ \|y - x\| < \epsilon$$

provided $\|y - x\| < \epsilon/\|T\| \equiv \delta$. ■

Example 3.40. Suppose that $K : [0,1] \times [0,1] \to \mathbb{C}$ is a continuous function and let (for now) $L^1([0,1])$ denote $C([0,1])$ with the norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Let $T : L^1([0,1], dm) \to C([0,1])$ be defined by

$$(Tf)(x) = \int_0^1 K(x,y)f(y)dy.$$

It is easily checked that this map is linear and maps to $C([0,1])$ as advertised. (To prove this use the fact the K is uniformly continuous.) If $M = \sup\{|K(x,y)| : x,y \in [0,1]\}$, then

$$|(Tf)(x)| \leq \int_0^1 |K(x,y)f(y)|\, dy \leq M \|f\|_1$$

which shows that $\|Tf\|_\infty \leq M \|f\|_1$ and hence,

$$\|T\|_{L^1\to C} \leq \max\{|K(x,y)| : x,y \in [0,1]\} < \infty.$$

We can in fact show that $\|T\| = M$ as follows. Let $(x_0, y_0) \in [0,1]^2$ such that $|K(x_0,y_0)| = M$. Then given $\epsilon > 0$, there exists a neighborhood $U = I \times J$ of (x_0,y_0) such that $|K(x,y) - K(x_0,y_0)| < \epsilon$ for all $(x,y) \in U$. Let $f \in C_c(I,[0,\infty))$ such that $\int_0^1 f(x)dx = 1$. Choose $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha K(x_0,y_0) = M$, then

$$\begin{aligned}|(T\alpha f)(x_0)| &= \left|\int_0^1 K(x_0,y)\alpha f(y)dy\right| = \left|\int_I K(x_0,y)\alpha f(y)dy\right| \\ &\geq \operatorname{Re}\int_I \alpha K(x_0,y)f(y)dy \geq \int_I (M-\epsilon)\, f(y)dy = (M-\epsilon)\,\|\alpha f\|_{L^1}\end{aligned}$$

and hence

$$\|T\alpha f\|_C \geq (M-\epsilon)\,\|\alpha f\|_{L^1}$$

showing that $\|T\| \geq M-\epsilon$. Since $\epsilon > 0$ is arbitrary, we learn that $\|T\| \geq M$ and hence $\|T\| = M$.

Similarly one easily shows that $T|_{C([0,1])} : C([0,1]) \to C([0,1])$ is bounded and

$$\|T\|_{C\to C} \leq \sup\left\{\int_0^1 |K(x,y)|\,dy : x \in [0,1]\right\} < \infty.$$

One may also view T as a map from $T : C([0,1]) \to L^1([0,1])$ in which case it can be seen that

$$\|T\|_{L^1\to C} \leq \int_0^1 \max_y |K(x,y)|\,dx < \infty.$$

For the next three exercises, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and $T : X \to Y$ be a linear transformation so that T is given by matrix multiplication by an $m \times n$ matrix. Let us identify the linear transformation T with this matrix.

Exercise 3.13. Assume the norms on X and Y are the ℓ^1 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^n |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq j\leq n} \sum_{i=1}^m |T_{ij}|.$$

Exercise 3.14. Assume the norms on X and Y are the ℓ^∞ – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \max_{1\leq j\leq n} |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq i\leq m} \sum_{j=1}^n |T_{ij}|.$$

Exercise 3.15. Assume the norms on X and Y are the ℓ^2 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\|^2 = \sum_{j=1}^n x_j^2$. Show $\|T\|^2$ is the largest eigenvalue of the matrix $T^{tr}T : \mathbb{R}^n \to \mathbb{R}^n$.

Exercise 3.16. If X is finite dimensional normed space then all linear maps are bounded.

Notation 3.41. Let $L(X,Y)$ denote the bounded linear operators from X to Y.

Lemma 3.42. *Let X, Y be normed spaces, then the operator norm $\|\cdot\|$ on $L(X,Y)$ is a norm. Moreover if Z is another normed space and $T : X \to Y$ and $S : Y \to Z$ are linear maps, then $\|ST\| \leq \|S\|\|T\|$, where $ST := S \circ T$.*

Proof. As usual, the main point in checking the operator norm is a norm is to verify the triangle inequality, the other axioms being easy to check. If $A, B \in L(X,Y)$ then the triangle inequality is verified as follows:

$$\|A+B\| = \sup_{x\neq 0} \frac{\|Ax+Bx\|}{\|x\|} \leq \sup_{x\neq 0} \frac{\|Ax\|+\|Bx\|}{\|x\|}$$
$$\leq \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x\neq 0} \frac{\|Bx\|}{\|x\|} = \|A\| + \|B\|.$$

For the second assertion, we have for $x \in X$, that

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\|.$$

From this inequality and the definition of $\|ST\|$, it follows that $\|ST\| \leq \|S\|\|T\|$. ∎

Proposition 3.43. *Suppose that X is a normed vector space and Y is a Banach space. Then $(L(X,Y), \|\cdot\|_{op})$ is a Banach space.*

We will use the following characterization of a Banach space in the proof of this proposition.

Theorem 3.44. *A normed space $(X, \|\cdot\|)$ is a Banach space iff for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} \|x_n\| < \infty$ then $\lim_{N\to\infty} \sum\limits_{n=1}^{N} x_n = S$ exists in X (that is to say every absolutely convergent series is a convergent series in X). As usual we will denote S by $\sum\limits_{n=1}^{\infty} x_n$.*

Proof. ($\Rightarrow$)If X is complete and $\sum\limits_{n=1}^{\infty} \|x_n\| < \infty$ then sequence $S_N \equiv \sum\limits_{n=1}^{N} x_n$ for $N \in \mathbb{N}$ is Cauchy because (for $N > M$)

$$\|S_N - S_M\| \le \sum_{n=M+1}^{N} \|x_n\| \to 0 \text{ as } M, N \to \infty.$$

Therefore$S = \sum\limits_{n=1}^{\infty} x_n := \lim_{N\to\infty} \sum\limits_{n=1}^{N} x_n$ exists in X.

($\Longleftarrow$) Suppose that $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence and let $\{y_k = x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} \|y_{n+1} - y_n\| < \infty$. By assumption

$$y_{N+1} - y_1 = \sum_{n=1}^{N} (y_{n+1} - y_n) \to S = \sum_{n=1}^{\infty} (y_{n+1} - y_n) \in X \text{ as } N \to \infty.$$

This shows that $\lim_{N\to\infty} y_N$ exists and is equal to $x := y_1 + S$. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy,

$$\|x - x_n\| \le \|x - y_k\| + \|y_k - x_n\| \to 0 \text{ as } k, n \to \infty$$

showing that $\lim_{n\to\infty} x_n$ exists and is equal to x. ■

Proof. (Proof of Proposition 3.43.) We must show $(L(X,Y), \|\cdot\|_{op})$ is complete. Suppose that $T_n \in L(X,Y)$ is a sequence of operators such that $\sum\limits_{n=1}^{\infty} \|T_n\| < \infty$. Then

$$\sum_{n=1}^{\infty} \|T_n x\| \le \sum_{n=1}^{\infty} \|T_n\| \|x\| < \infty$$

and therefore by the completeness of Y, $Sx := \sum\limits_{n=1}^{\infty} T_n x = \lim_{N\to\infty} S_N x$ exists in Y, where $S_N := \sum\limits_{n=1}^{N} T_n$. The reader should check that $S : X \to Y$ so defined in linear. Since,

$$\|Sx\| = \lim_{N\to\infty} \|S_N x\| \le \lim_{N\to\infty} \sum_{n=1}^{N} \|T_n x\| \le \sum_{n=1}^{\infty} \|T_n\| \|x\|,$$

S is bounded and

$$\|S\| \le \sum_{n=1}^{\infty} \|T_n\|. \tag{3.4}$$

Similarly,

$$\|Sx - S_M x\| = \lim_{N\to\infty} \|S_N x - S_M x\| \le \lim_{N\to\infty} \sum_{n=M+1}^{N} \|T_n\| \, \|x\| = \sum_{n=M+1}^{\infty} \|T_n\| \, \|x\|$$

and therefore,

$$\|S - S_M\| \le \sum_{n=M}^{\infty} \|T_n\| \to 0 \text{ as } M \to \infty.$$

■

3.7. Appendix: Sums in Banach spaces.

Definition 3.45. Suppose that X is a Normed space and $\{v_\alpha \in X : \alpha \in A\}$ is a given collection of vectors in X. We say that $s = \sum_{\alpha\in A} v_\alpha \in X$ if for all $\epsilon > 0$ there exists a finite set $\Gamma_\epsilon \subset A$ such that $\left\|s - \sum_{\alpha\in\Lambda} v_\alpha\right\| < \epsilon$ for all $\Lambda \subset\subset A$ such that $\Gamma_\epsilon \subset \Lambda$. (Unlike the case of real valued sums, this does not imply that $\sum_{\alpha\in\Lambda} \|v_\alpha\| < \infty$. See Proposition 18.16, from which one may manufacture counter-examples to this false premise.)

Lemma 3.46. *(1) When X is a Banach space, $\sum_{\alpha\in A} v_\alpha$ exists in X iff for all $\epsilon > 0$ there exists $\Gamma_\epsilon \subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| < \epsilon$ for all $\Lambda \subset\subset A \setminus \Gamma_\epsilon$. Also if $\sum_{\alpha\in A} v_\alpha$ exists in X then $\{\alpha \in A : v_a \neq 0\}$ is at most countable. (2) If $s = \sum_{\alpha\in A} v_\alpha \in X$ exists and $T : X \to Y$ is a bounded linear map between normed spaces, then $\sum_{\alpha\in A} T v_\alpha$ exists in Y and*

$$Ts = T \sum_{\alpha\in A} v_\alpha = \sum_{\alpha\in A} T v_\alpha.$$

Proof. (1) Suppose that $s = \sum_{\alpha\in A} v_\alpha$ exists and $\epsilon > 0$. Let $\Gamma_\epsilon \subset\subset A$ be as in Definition 3.45. Then for $\Lambda \subset\subset A \setminus \Gamma_\epsilon$,

$$\begin{aligned} \left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| &\le \left\|\sum_{\alpha\in\Lambda} v_\alpha + \sum_{\alpha\in\Gamma_\epsilon} v_\alpha - s\right\| + \left\|\sum_{\alpha\in\Gamma_\epsilon} v_\alpha - s\right\| \\ &= \left\|\sum_{\alpha\in\Gamma_\epsilon\cup\Lambda} v_\alpha - s\right\| + \epsilon < 2\epsilon. \end{aligned}$$

Conversely, suppose for all $\epsilon > 0$ there exists $\Gamma_\epsilon \subset\subset A$ such that $\left\|\sum_{\alpha\in\Lambda} v_\alpha\right\| < \epsilon$ for all $\Lambda \subset\subset A \setminus \Gamma_\epsilon$. Let $\gamma_n := \cup_{k=1}^n \Gamma_{1/k} \subset A$ and set $s_n := \sum_{\alpha\in\gamma_n} v_\alpha$. Then for $m > n$,

$$\|s_m - s_n\| = \left\|\sum_{\alpha\in\gamma_m\setminus\gamma_n} v_\alpha\right\| \le 1/n \to 0 \text{ as } m, n \to \infty.$$

Therefore $\{s_n\}_{n=1}^{\infty}$ is Cauchy and hence convergent in X. Let $s := \lim_{n\to\infty} s_n$, then for $\Lambda \subset\subset A$ such that $\gamma_n \subset \Lambda$, we have

$$\left\| s - \sum_{\alpha\in\Lambda} v_\alpha \right\| \le \|s - s_n\| + \left\| \sum_{\alpha\in\Lambda\setminus\gamma_n} v_\alpha \right\| \le \|s - s_n\| + \frac{1}{n}.$$

Since the right member of this equation goes to zero as $n \to \infty$, it follows that $\sum_{\alpha\in A} v_\alpha$ exists and is equal to s.

Let $\gamma := \cup_{n=1}^{\infty}\gamma_n$ – a countable subset of A. Then for $\alpha \notin \gamma$, $\{\alpha\} \subset A \setminus \gamma_n$ for all n and hence

$$\|v_a\| = \left\| \sum_{\beta\in\{\alpha\}} v_\beta \right\| \le 1/n \to 0 \text{ as } n \to \infty.$$

Therefore $v_\alpha = 0$ for all $\alpha \in A \setminus \gamma$.

(2) Let Γ_ϵ be as in Definition 3.45 and $\Lambda \subset\subset A$ such that $\Gamma_\epsilon \subset \Lambda$. Then

$$\left\| Ts - \sum_{\alpha\in\Lambda} Tv_\alpha \right\| \le \|T\| \left\| s - \sum_{\alpha\in\Lambda} v_\alpha \right\| < \|T\|\, \epsilon$$

which shows that $\sum_{\alpha\in\Lambda} Tv_\alpha$ exists and is equal to Ts. ■

3.8. **Appendix on Riemannian Metrics.** This subsection is not completely self contained and may safely be skipped.

Lemma 3.47. *Suppose that X is a Riemannian (or sub-Riemannian) manifold and d is the metric on X defined by*

$$d(x,y) = \inf\{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}$$

where $\ell(\sigma)$ is the length of the curve σ. We define $\ell(\sigma) = \infty$ if σ is not piecewise smooth.

Then

$$\overline{B_x(\epsilon)} = C_x(\epsilon) \text{ and}$$
$$\partial B_x(\epsilon) = \{y \in X : d(x,y) = \epsilon\}.$$

Proof. Let $C := C_x(\epsilon) \subset \overline{B_x(\epsilon)} =: \bar{B}$. We will show that $C \subset \bar{B}$ by showing $\bar{B}^c \subset C^c$. Suppose that $y \in \bar{B}^c$ and choose $\delta > 0$ such that $B_y(\delta) \cap \bar{B} = \emptyset$. In particular this implies that

$$B_y(\delta) \cap B_x(\epsilon) = \emptyset.$$

We will finish the proof by showing that $d(x,y) \ge \epsilon + \delta > \epsilon$ and hence that $y \in C^c$. This will be accomplished by showing: if $d(x,y) < \epsilon + \delta$ then $B_y(\delta) \cap B_x(\epsilon) \ne \emptyset$.

If $d(x,y) < \max(\epsilon,\delta)$ then either $x \in B_y(\delta)$ or $y \in B_x(\epsilon)$. In either case $B_y(\delta) \cap B_x(\epsilon) \ne \emptyset$. Hence we may assume that $\max(\epsilon,\delta) \le d(x,y) < \epsilon + \delta$. Let $\alpha > 0$ be a number such that

$$\max(\epsilon,\delta) \le d(x,y) < \alpha < \epsilon + \delta$$

and choose a curve σ from x to y such that $\ell(\sigma) < \alpha$. Also choose $0 < \delta' < \delta$ such that $0 < \alpha - \delta' < \epsilon$ which can be done since $\alpha - \delta < \epsilon$. Let $k(t) = d(y, \sigma(t))$ a continuous function on $[0,1]$ and therefore $k([0,1]) \subset \mathbb{R}$ is a connected set which

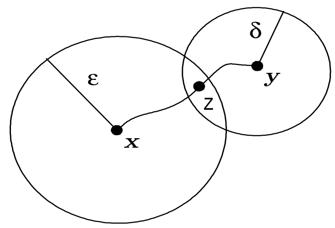

FIGURE 6. An almost length minimizing curve joining x to y.

contains 0 and $d(x, y)$. Therefore there exists $t_0 \in [0, 1]$ such that $d(y, \sigma(t_0)) = k(t_0) = \delta'$. Let $z = \sigma(t_0) \in B_y(\delta)$ then

$$d(x, z) \le \ell(\sigma|_{[0,t_0]}) = \ell(\sigma) - \ell(\sigma|_{[t_0,1]}) < \alpha - d(z, y) = \alpha - \delta' < \epsilon$$

and therefore $z \in B_x(\epsilon) \cap B_x(\delta) \neq \emptyset$. ■

Remark 3.48. Suppose again that X is a Riemannian (or sub-Riemannian) manifold and

$$d(x, y) = \inf \{\ell(\sigma) : \sigma(0) = x \text{ and } \sigma(1) = y\}.$$

Let σ be a curve from x to y and let $\epsilon = \ell(\sigma) - d(x, y)$. Then for all $0 \le u < v \le 1$,

$$d(\sigma(u), \sigma(v)) \le \ell(\sigma|_{[u,v]}) + \epsilon.$$

So if σ is within ϵ of a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is within ϵ of a length minimizing curve from $\sigma(u)$ to $\sigma(v)$. In particular if $d(x, y) = \ell(\sigma)$ then $d(\sigma(u), \sigma(v)) = \ell(\sigma|_{[u,v]})$ for all $0 \le u < v \le 1$, i.e. if σ is a length minimizing curve from x to y that $\sigma|_{[u,v]}$ is a length minimizing curve from $\sigma(u)$ to $\sigma(v)$.

To prove these assertions notice that

$$\begin{aligned} d(x, y) + \epsilon = \ell(\sigma) &= \ell(\sigma|_{[0,u]}) + \ell(\sigma|_{[u,v]}) + \ell(\sigma|_{[v,1]}) \\ &\ge d(x, \sigma(u)) + \ell(\sigma|_{[u,v]}) + d(\sigma(v), y) \end{aligned}$$

and therefore

$$\begin{aligned} \ell(\sigma|_{[u,v]}) &\le d(x, y) + \epsilon - d(x, \sigma(u)) - d(\sigma(v), y) \\ &\le d(\sigma(u), \sigma(v)) + \epsilon. \end{aligned}$$

3.9. **Exercises.**

Exercise 3.17. Prove Lemma 3.28.

Exercise 3.18. Let $X = C([0, 1], \mathbb{R})$ and for $f \in X$, let

$$\|f\|_1 := \int_0^1 |f(t)|\, dt.$$

Show that $(X, \|\cdot\|_1)$ is normed space and show by example that this space is **not** complete.

Exercise 3.19. Let (X, d) be a metric space. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ is a sequence and set $\epsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} \epsilon_k \le \sum_{k=n}^{\infty} \epsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^{\infty} \epsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$ then

$$d(x, x_n) \le \sum_{k=n}^{\infty} \epsilon_k.$$

Exercise 3.20. Show that (X, d) is a complete metric space iff every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ is a convergent sequence in X. You may find it useful to prove the following statements in the course of the proof.

1. If $\{x_n\}$ is Cauchy sequence, then there is a subsequence $y_j \equiv x_{n_j}$ such that $\sum_{j=1}^{\infty} d(y_{j+1}, y_j) < \infty$.
2. If $\{x_n\}_{n=1}^{\infty}$ is Cauchy and there exists a subsequence $y_j \equiv x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j\to\infty} y_j$ exists, then $\lim_{n\to\infty} x_n$ also exists and is equal to x.

Exercise 3.21. Suppose that $f : [0, \infty) \to [0, \infty)$ is a C^2 – function such that $f(0) = 0$, $f' > 0$ and $f'' \le 0$ and (X, ρ) is a metric space. Show that $d(x, y) = f(\rho(x, y))$ is a metric on X. In particular show that

$$d(x, y) \equiv \frac{\rho(x, y)}{1 + \rho(x, y)}$$

is a metric on X. (Hint: use calculus to verify that $f(a + b) \le f(a) + f(b)$ for all $a, b \in [0, \infty)$.)

Exercise 3.22. Let $d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0, \infty)$ be defined by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n},$$

where $\|f\|_n \equiv \sup\{|f(x)| : |x| \le n\} = \max\{|f(x)| : |x| \le n\}$.

1. Show that d is a metric on $C(\mathbb{R})$.
2. Show that a sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ as $n \to \infty$ iff f_n converges to f uniformly on compact subsets of $\mathbb{R}$.
3. Show that $(C(\mathbb{R}), d)$ is a complete metric space.

Exercise 3.23 (Contraction Mapping Principle)**.** Suppose now that (X, d) is complete, $T : X \to X$ is a map and there exists $\alpha \in (0, 1)$ such that $d(T(x), T(y)) \le \alpha d(x, y)$ for all $x, y \in X$. Prove that T has a fixed point, i.e. there is a unique element $x \in X$ such that $T(x) = x$. (Notice that this fixed point is unique since if $x = T(x)$ and $y = T(y)$, then $d(x, y) = d(T(x), T(y)) \le \alpha d(x, y)$ and therefore $d(x, y)(1 - \alpha) \le 0$. This shows that $d(x, y) = 0.$, i.e. that $x = y$.) **Hint:**Let $x_0 \in X$ be arbitrary and define x_n inductively by $x_{n+1} = T(x_n)$. Then show that $d(x_{n+1}, x_n) \le C\alpha^n$ where C is a finite constant. Use the above problems to conclude that $x \equiv \lim_{n\to\infty} x_n$ exists to show that

$$d(x, x_n) \le C \sum_{k=n}^{\infty} \alpha^k = C \frac{\alpha^n}{1 - \alpha}.$$

Exercise 3.24. Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be a sequence of metric spaces, $X := \prod_{n=1}^{\infty} X_n$, and for $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ in X let

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}.$$

Show: 1) (X, d) is a metric space, 2) a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges to $x \in X$ iff $x_k(n) \to x(n) \in X_n$ as $k \to \infty$ for every $n = 1, 2, \ldots$, and 3) X is complete if X_n is complete for all n.

Exercise 3.25 (Tychonoff's Theorem)**.** Let us continue the notation of the previous problem. Further assume that the spaces X_n are compact for all n. Show (X, d) is compact. **Hint:** Either use Cantor's method to show every sequence $\{x_m\}_{m=1}^{\infty} \subset X$ has a convergent subsequence or alternatively show (X, d) is complete and totally bounded.

3.9.1. *Banach Space Problems.*

Exercise 3.26. Show that all finite dimensional normed vector spaces $(L, \|\cdot\|)$ are necessarily complete. Also shows that closed and bounded sets (relative to the given norm) are compact.

Exercise 3.27. Let $p \in [1, \infty]$ and X be an infinite set. Show the unit ball in $\ell^p(X)$ is not compact.

Exercise 3.28. Let $X = \mathbb{N}$ and for $p, q \in [1, \infty)$ let $\|\cdot\|_p$ denote the $\ell^p(\mathbb{N})$ – norm. Show $\|\cdot\|_p$ and $\|\cdot\|_q$ are inequivalent norms for $p \neq q$ by showing

$$\sup_{f \neq 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.$$

Exercise 3.29. Folland Problem 5.5. Closure of subspaces are subspaces.

Exercise 3.30. Folland Problem 5.9. Showing $C^k([0,1])$ is a Banach space.

Exercise 3.31. Folland Problem 5.11. Showing Holder spaces are Banach spaces.