MATH 240A LECTURE NOTES: MEASURE THEORY

BRUCE K. DRIVER†

Abstract. These are lecture notes from Math 240A on Measure Theory following Folland's Book..

Contents

Date: November 29, 2000 *File:measurep.tex*.
†This research was partially supported by NSF Grant DMS 96-12651.
Department of Mathematics, 0112
University of California, San Diego
La Jolla, CA 92093-0112 .

1. What are measures and why "measurable" sets

Definition 1.1 (Preliminary)**.** Suppose that X is a set and $\mathcal{P}(X)$ denotes the collection of all subsets of X. A measure μ on X is a function $\mu : \mathcal{P}(X) \to [0,\infty]$ such that

(1) $\mu(\emptyset) = 0$
(2) If $\{A_i\}_{i=1}^N$ is a finite ($N < \infty$) or countable ($N = \infty$) collection of subsets of X which are pair-wise disjoint (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$) then

$$\mu(\cup_{i=1}^N A_i) = \sum_{i=1}^N \mu(A_i).$$

Example 1.2. Suppose that X is any set and $x \in X$ is a point. For $A \subset X$, let

$$\delta_x(A) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu = \delta_x$ is a measure on X called the at x.

Example 1.3. Suppose that μ is a measure on X and $\lambda > 0$, then $\lambda\mu$ is also a measure on X. Moreover, if $\{\mu_\alpha : \alpha \in J\}$ are all measures on X, then $\mu = \sum_{\alpha \in J} \mu_\alpha$, i.e.

$$\mu(A) = \sum_{\alpha \in J} \mu_\alpha(A) \text{ for all } A \subset X$$

is a measure on X. (See Section 2 for the meaning of this sum.) We must show that μ is countably additive. Suppose that $\{A_i\}_{i=1}^\infty$ is a collection of pair-wise disjoint subsets of X, then

$$\begin{aligned}\mu(\cup_{i=1}^{\infty} A_i) &= \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty}\sum_{\alpha\in J} \mu_\alpha(A_i) \\ &= \sum_{\alpha\in J}\sum_{i=1}^{\infty} \mu_\alpha(A_i) = \sum_{\alpha\in J} \mu_\alpha(\cup_{i=1}^{\infty} A_i) \\ &= \mu(\cup_{i=1}^{\infty} A_i)\end{aligned}$$

where in the third equality we used Theorem 2.20 below and in the fourth we used that fact that μ_α is a measure.

Example 1.4. Suppose that X is a set $\lambda : X \to [0,\infty]$ is a function. Then

$$\mu := \sum_{x\in X} \lambda(x)\delta_x$$

is a measure, explicitly

$$\mu(A) = \sum_{x\in A} \lambda(x)$$

for all $A \subset X$.

1.1. **The problem with Lebesgue "measure".**

Question 1. Does there exist a measure $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ such that

1. $\mu([a,b)) = (b-a)$ for all $a < b$ and
2. $\mu(A + x) = \mu(A)$ for all $x \in \mathbb{R}$?

The unfortunate answer is no which we demonstrate now. In fact the answer is no even if we replace 1. by the condition that $0 < \mu((0,1]) < \infty$.

Let us identify $[0,1)$ with the unit circle $S := \{z \in \mathbb{C} : |z| = 1\}$ by the map $\phi(t) = e^{i2\pi t} \in S$ for $t \in [0,1)$. Using this identification we may use μ to define a function ν on $\mathcal{P}(S)$ by $\nu(\phi(A)) = \mu(A)$ for all $A \subset [0,1)$. This new function is a measure on S with the property that $0 < \nu((0,1]) < \infty$. For $z \in S$ and $N \subset S$ let

$$zN := \{zn \in S : n \in N\}, \tag{1.1}$$

that is to say $e^{i\theta}N$ is N rotated counter clockwise by angle θ. We now claim that ν is invariant under these rotations, i.e.

$$\nu(zN) = \nu(N) \tag{1.2}$$

for all $z \in S$ and $N \subset S$. To verify this, write $N = \phi(A)$ and $z = \phi(t)$ for some $t \in [0,1)$ and $A \subset [0,1)$. Then

$$\phi(t)\phi(A) = \phi(t + A \operatorname{mod} 1)$$

where For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned}t + A \operatorname{mod} 1 &= \{a + t \operatorname{mod} 1 \in [0,1) : a \in N\} \\ &= (a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\}).\end{aligned}$$

Thus

$$\begin{aligned}\nu(\phi(t)\phi(A)) &= \mu(t + A \operatorname{mod} 1)\\ &= \mu\left((a + A \cap \{a < 1-t\}) \cup ((t-1) + A \cap \{a \geq 1-t\})\right)\\ &= \mu\left((a + A \cap \{a < 1-t\})\right) + \mu\left(((t-1) + A \cap \{a \geq 1-t\})\right)\\ &= \mu\left(A \cap \{a < 1-t\}\right) + \mu\left(A \cap \{a \geq 1-t\}\right)\\ &= \mu\left((A \cap \{a < 1-t\}) \cup (A \cap \{a \geq 1-t\})\right)\\ &= \mu(A) = \nu(\phi(A)).\end{aligned}$$

Therefore it suffices to prove that no finite measure ν on S such that Eq. (1.2) holds. To do this we will "construct" a non-measurable set $N = \phi(\ A)$ for some $A \subset [[0,1)$.

To do this let R be the countable set

$$R := \{z = e^{i2\pi t} : t \in [0,1) \cap \mathbb{Q}\}.$$

As above R acts on S by rotations and divides S up into equivalence classes, where $z, w \in S$ are equivalent if $z = rw$ for some $r \in R$. Choose (using the axiom of choice) one representative point n from each of these equivalence classes and let $N \subset S$ be the set of these representative points. Then every point $z \in S$ may be uniquely written as $z = nr$ with $n \in N$ and $r \in R$. That is to say

$$S = \coprod_{r \in R} (rN) \tag{1.3}$$

where $\coprod_\alpha A_\alpha$ is used to denote the union of pair-wise disjoint sets $\{A_\alpha\}$. By Eqs. (1.2) and (1.3) we find that

$$\nu(S) = \sum_{r \in R} \nu(rN) = \sum_{r \in R} \nu(N).$$

The right member from this equation is either 0 or ∞, 0 if $\nu(N) = 0$ and ∞ if $\nu(N) > 0$. In either case it is not equal $\nu(S) \in (0,1)$. Thus we have reached the desired contradiction.

Proof. (Second proof) For $N \subset [0,1)$ and $\alpha \in [0,1)$, let

$$\begin{aligned}N^\alpha &= N + \alpha \operatorname{mod} 1\\ &= \{a + \alpha \operatorname{mod} 1 \in [0,1) : a \in N\}\\ &= (\alpha + N \cap \{a < 1-\alpha\}) \cup ((\alpha - 1) + N \cap \{a \geq 1-\alpha\}).\end{aligned}$$

If μ is a measure satisfying the properties of the Question we would have

$$\begin{aligned}\mu(N^\alpha) &= \mu(\alpha + N \cap \{a < 1-\alpha\}) + \mu((\alpha-1) + N \cap \{a \geq 1-\alpha\})\\ &= \mu(N \cap \{a < 1-\alpha\}) + \mu(N \cap \{a \geq 1-\alpha\})\\ &= \mu(N \cap \{a < 1-\alpha\} \cup (N \cap \{a \geq 1-\alpha\}))\\ &= \mu(N). \qquad (1.4)\end{aligned}$$

We will now construct a bad set N which coupled with Eq. (1.4) will lead to a contradiction.

Set

$$Q_x \equiv \{x + r \in \mathbb{R} : r \in \mathbb{Q}\} = x + \mathbb{Q}.$$

Notice that $Q_x \cap Q_y \neq \emptyset$ implies that $Q_x = Q_y$. Let $\mathcal{O} = \{Q_x : x \in \mathbb{R}\}$ – the orbit space of the $\mathbb{Q}$ action. For all $A \in \mathcal{O}$ choose $f(A) \in [0, 1/3) \cap A$.[1] Define $N = f(\mathcal{O})$. Then observe:

(1) $f(A) = f(B)$ implies that $A \cap B \neq \emptyset$ which implies that $A = B$ so that f is injective.
(2) $\mathcal{O} = \{Q_n : n \in N\}$.

Let R be the countable set,

$$R \equiv \mathbb{Q} \cap [0, 1).$$

We now claim that

$$N^r \cap N^s = \emptyset \text{ if } r \neq s \text{ and} \tag{1.5}$$

$$[0, 1) = \cup_{r \in R} N^r. \tag{1.6}$$

Indeed, if $x \in N^r \cap N^s \neq \emptyset$ then $x = r + n \bmod 1$ and $x = s + n' \bmod 1$, then $n - n\prime \in \mathbb{Q}$, i.e. $Q_n = Q_{n'}$. That is to say, $n = f(Q_n) = f(Q_{n'}) = n'$ and hence that $s = r \bmod 1$, but $s, r \in [0, 1)$ implies that $s = r$. Furthermore, if $x \in [0, 1)$ and $n := f(Q_x)$, then $x - n = r \in \mathbb{Q}$ and $x \in N^{r \bmod 1}$.

Now that we have constructed N, we are ready for the contradiction. By Equations (1.4–1.6) we find

$$\begin{aligned} 1 = \mu([0, 1)) &= \sum_{r \in R} \mu(N^r) = \sum_{r \in R} \mu(N) \\ &= \left\{ \begin{array}{lll} \infty & \text{if} & \mu(N) > 0 \\ 0 & \text{if} & \mu(N) = 0 \end{array} \right. . \end{aligned}$$

which is certainly inconsistent. Incidentally we have just produced an example of so called "non – measurable" set. ∎

Because of this example and our desire to have a measure μ on $\mathbb{R}$ satisfying the properties in Question 1, we need to modify our definition of a measure. We will give up on trying to measure all subsets $A \subset \mathbb{R}$, i.e. we will only try to define μ on a smaller collection of "measurable" sets. Such collection will be called σ – algebras. These notion will be introduced in the nest section.

2. Limits, sums, and other basics

2.1. Set Operations. Suppose that X is a set. For $A \subset X$ let

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

and more generally if $A, B \subset X$ let

$$B \setminus A = \{x \in B : x \notin A\}.$$

We also define the symmetric difference of A and B by

$$A \triangle B = (B \setminus A) \cup (A \setminus B).$$

As usual if $\{A_\alpha\}_{\alpha \in I}$ is an indexed collection of subsets of X we define the union and the intersection of this collection by

$$\begin{aligned} \cup_{\alpha \in I} A_\alpha &:= \{x \in X : \exists \alpha \in I \ni x \in A_\alpha\} \text{ and} \\ \cap_{\alpha \in I} A_\alpha &:= \{x \in X : x \in A_\alpha \, \forall \alpha \in I\}. \end{aligned}$$

[1] We have used the Axiom of choice here, i.e. $\prod_{A \in \mathcal{F}} (A \cap [0, 1/3]) \neq \emptyset$

Notice that $\cup$ is closely related to $\exists$ and $\cap$ is closely related to $\forall$. For example let $\{A_n\}_{n=1}^\infty$ be a sequence of subsets from X and define

$$\{A_n \text{ i.o.}\} := \{x \in X : \#\{n : x \in A_n\} = \infty\} \text{ and}$$
$$\{A_n \text{ a.a.}\} := \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}.$$

Then $x \in \{A_n \text{ i.o.}\}$ iff $\forall N \in \mathbb{N}\ \exists n \geq N \ni x \in A_n$ which may be written as

$$\{A_n \text{ i.o.}\} = \cap_{N=1}^\infty \cup_{n \geq N} A_n$$

and similarly, $x \in \{A_n \text{ a.a.}\}$ iff $\exists N \in \mathbb{N} \ni \forall n \geq N,\ x \in A_n$ which may be written as

$$\{A_n \text{ a.a.}\} = \cup_{N=1}^\infty \cap_{n \geq N} A_n.$$

2.2. **Limits, Limsups, and Liminfs.**

Notation 2.1. The is the set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, i.e. it is $\mathbb{R}$ with two new points called ∞ and $-\infty$. We use the following conventions, $\pm\infty \cdot 0 = 0$, $\pm\infty + a = \pm\infty$ for any $a \in \mathbb{R}$, $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ while $\infty - \infty$ is not defined.

If $\Lambda \subset \bar{\mathbb{R}}$ we will let $\sup \Lambda$ and $\inf \Lambda$ denote the least upper bound and greatest lower bound of Λ respectively. We will also use the following convention, if $\Lambda = \emptyset$, then $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Notation 2.2. Suppose that $\{x_n\}_{n=1}^\infty \subset \bar{\mathbb{R}}$ is a sequence of numbers. Then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup_n \inf\{x_k : k \geq n\} \tag{2.1}$$

and

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf_n \sup\{x_k : k \geq n\}. \tag{2.2}$$

We will also write $\liminf$ for $\underline{\lim}$ and $\limsup$ for $\overline{\lim}$.

Remark 2.3. Notice that if $a_k := \inf\{x_k : k \geq n\}$ and $b_k := \sup\{x_k : k \geq n\}$,then $\{a_k\}$ is an increasing sequence while $\{b_k\}$ is a decreasing sequence. Therefore the limits in Eq. (2.1) and Eq. (2.2) always exist.

The following proposition contains some basic properties of liminfs and limsups.

Proposition 2.4. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two sequences of real numbers. Then*

1. *Show* $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ *and the* $\lim_{n\to\infty} a_n$ *exists in* $\bar{\mathbb{R}}$ *iff* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in \bar{\mathbb{R}}$.
2. *Suppose that* $\limsup_{n\to\infty} a_n = M \in \bar{\mathbb{R}}$, *show that there is a subsequence* $\{a_{n_k}\}_{k=1}^\infty$ *of* $\{a_n\}_{n=1}^\infty$ *such that* $\lim_{k\to\infty} a_{n_k} = M$.
3. *Suppose that* $\limsup_{n\to\infty} a_n < \infty$ *and* $\limsup_{n\to\infty} b_n > -\infty$, *then prove that*
$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \tag{2.3}$$
In other words, Eq. (2.3) holds provided the right side of the equation is well defined.
4. *Suppose that* $a_n \geq 0$ *and* $b_n \geq 0$ *for all* $n \in \mathbb{N}$. *Show*
$$\limsup_{n\to\infty} (a_n b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n, \tag{2.4}$$
provided the right hand side of (2.4) is not of the form $0 \cdot \infty$ *or* $\infty \cdot 0$.

Proof. We will only prove part 1. and leave the rest as an exercise to the reader. We begin by noticing that

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \; \forall\, n$$

so that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Now suppose that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Then for all $\epsilon > 0$, there is an integer N such that

$$a - \epsilon \leq \inf\{a_k : k \geq N\} \leq \sup\{a_k : k \geq N\} \leq a + \epsilon,$$

i.e. we have

$$a - \epsilon \leq a_k \leq a + \epsilon \text{ for all } k \geq N.$$

Hence by the definition of the limit, $\lim_{k\to\infty} a_k = a$.

If $\liminf_{n\to\infty} a_n = \infty$, then we know for all $M \in (0, \infty)$ there is an integer N such that

$$M \leq \inf\{a_k : k \geq N\}$$

and hence $\lim_{n\to\infty} a_n = \infty$. The case where $\limsup_{n\to\infty} a_n = -\infty$ is handled similarly.

Conversely, suppose that $\lim_{n\to\infty} a_n = A \in \bar{\mathbb{R}}$ exists. If $A \in \mathbb{R}$, then for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $|A - a_n| \leq \epsilon$ for all $n \geq N(\epsilon)$, i.e.

$$A - \epsilon \leq a_n \leq A + \epsilon \text{ for all } n \geq N(\epsilon).$$

From this we learn that

$$A - \epsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq A + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$A \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq A,$$

i.e. that $A = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

If $A = \infty$, then for all $M > 0$ there exist $N(M)$ such that $a_n \geq M$ for all $n \geq N(M)$. This shows that

$$\liminf_{n\to\infty} a_n \geq M$$

and since M is arbitrary it follows that

$$\infty \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

The proof is similar if $A = -\infty$ as well. ■

The following elementary lemma will often be useful.

Lemma 2.5. *Let X and Y be sets, $R \subset X \times Y$ and suppose that $a : R \to \bar{\mathbb{R}}$ is a function. Let ${}_xR := \{y \in Y : (x, y) \in R\}$ and $R_y := \{x \in X : (x, y) \in R\}$. Then*

$$\begin{aligned} \sup_{(x,y)\in R} a(x,y) &= \sup_{x\in X} \sup_{y\in {}_xR} a(x,y) = \sup_{y\in Y} \sup_{x\in R_y} a(x,y) \text{ and} \\ \inf_{(x,y)\in R} a(x,y) &= \inf_{x\in X} \inf_{y\in {}_xR} a(x,y) = \inf_{y\in Y} \inf_{x\in R_y} a(x,y). \end{aligned}$$

Proof. Let $M = \sup_{(x,y)\in R} a(x,y)$, $N_x := \sup_{y\in_x R} a(x,y)$, then $a(x,y) \le M$ for all $(x,y) \in R$. Suppose that $x \in X$, then implies that $N_x = \sup_{y\in_x R}(x,y) \le M$ and hence

$$\sup_{x\in X}\sup_{y\in_x R} a(x,y) = \sup_{x\in X} N_x \le M. \tag{2.5}$$

Similarly for any $(x,y) \in R$,

$$a(x,y) \le N_x \le \sup_{x\in X} N_x = \sup_{x\in X}\sup_{y\in_x R} a(x,y)$$

and therefore

$$\sup_{(x,y)\in R} a(x,y) \le \sup_{x\in X}\sup_{y\in_x R} a(x,y) = M \tag{2.6}$$

Equations (2.5) and (2.6) show that

$$\sup_{(x,y)\in R} a(x,y) = \sup_{x\in X}\sup_{y\in_x R} a(x,y).$$

All of the other assertions of the Lemma have a similar proof. ■

2.3. **Sums of positive functions.** In this and the next few sections, let X and Y be two sets. We will write $\alpha \subset\subset X$ to denote that α is a **finite** subset of X.

Definition 2.6. Suppose that $a : X \to [0,\infty]$ is a function and $F \subset X$ is a subset, then

$$\sum_F a = \sum_{x\in F} a(x) = \sup\left\{\sum_{x\in\alpha} a(x) : \alpha \subset\subset F\right\}.$$

Remark 2.7. Suppose that $X = \mathbb{N} = \{1,2,3,\dots\}$, then

$$\sum_{\mathbb{N}} a = \sum_{n=1}^{\infty} a(n) := \lim_{N\to\infty}\sum_{n=1}^{N} a(n).$$

Indeed for all N, $\sum_{n=1}^N a(n) \le \sum_{\mathbb{N}} a$, and thus passing to the limit we learn that

$$\sum_{n=1}^{\infty} a(n) \le \sum_{\mathbb{N}} a.$$

Conversely, if $\alpha \subset\subset \mathbb{N}$, then for all N large enough so that $\alpha \subset \{1,2,\dots,N\}$, we have $\sum_\alpha a \le \sum_{n=1}^N a(n)$ which upon passing to the limit implies that

$$\sum_\alpha a \le \sum_{n=1}^{\infty} a(n)$$

and hence by taking the supremum over α we learn that

$$\sum_{\mathbb{N}} a \le \sum_{n=1}^{\infty} a(n).$$

Remark 2.8. Suppose that $\sum_X a < \infty$, then $\{x \in X : a(x) > 0\}$ is at most countable. To see this first notice that for any $\epsilon > 0$, the set $\{x : a(x) \ge \epsilon\}$ must be finite for otherwise $\sum_X a = \infty$. Thus

$$\{x \in X : a(x) > 0\} = \textstyle\bigcup_{k=1}^{\infty}\{x : a(x) \ge 1/k\}$$

which shows that $\{x \in X : a(x) > 0\}$ is a countable union of finite sets and thus countable.

Lemma 2.9. *Suppose that $a, b : X \to [0, \infty]$ are two functions, then*

$$\sum_X (a+b) = \sum_X a + \sum_X b \text{ and}$$
$$\sum_X \lambda a = \lambda \sum_X a$$

for all $\lambda \geq 0$.

I will only prove the first assertion, the second being easy. Let $\alpha \subset\subset X$ be a finite set, then

$$\sum_\alpha (a+b) = \sum_\alpha a + \sum_\alpha b \leq \sum_X a + \sum_X b$$

which after taking sups over α shows that

$$\sum_X (a+b) \leq \sum_X a + \sum_X b.$$

Similarly, if $\alpha, \beta \subset\subset X$, then

$$\sum_\alpha a + \sum_\beta b \leq \sum_{\alpha\cup\beta} a + \sum_{\alpha\cup\beta} b = \sum_{\alpha\cup\beta}(a+b) \leq \sum_X (a+b).$$

Taking sups over α and β then shows that

$$\sum_X a + \sum_X b \leq \sum_X (a+b).$$

Theorem 2.10 (Monotone Convergence Theorem)**.** *Suppose that $f_n : X \to [0, \infty]$ is an increasing sequence of functions and*

$$f(x) := \lim_{n\to\infty} f_n(x) = \sup_n f_n(x).$$

Then

$$\lim_{n\to\infty} \sum_X f_n = \sum_X f$$

Proof. Let $S_n = \sum_X f_n$ and $S = \sum_X f$. Since $f_n \leq f_m \leq f$ for all $n \leq m$, it follows that

$$S_n \leq S_m \leq S$$

which shows that $\lim_{n\to\infty} S_n$ exists and is less that S, i.e.

$$A := \lim_{n\to\infty} \sum_X f_n \leq \sum_X f. \tag{2.7}$$

Noting that $\sum_\alpha f_n \leq \sum_X f_n = S_n \leq A$ for all $\alpha \subset\subset X$ and in particular,

$$\sum_\alpha f_n \leq A \text{ for all } n \text{ and } \alpha \subset\subset X.$$

Letting n tend to infinity in this equation shows that

$$\sum_\alpha f \leq A \text{ for all } \alpha \subset\subset X$$

and then taking the sup over all $\alpha \subset\subset X$ gives

$$\sum_X f \le A = \lim_{n\to\infty} \sum_X f_n \tag{2.8}$$

which combined with Eq. (2.8) shows proves the theorem. ■

Lemma 2.11 (Fatou's Lemma)**.** *Suppose that $f_n : X \to [0,\infty]$ is a sequence of functions, then*

$$\sum_X \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \sum_X f_n.$$

Proof. Define $g_k \equiv \inf_{n\ge k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k \to \infty$. Since $g_k \le f_n$ for all $k \le n$ we have

$$\sum_X g_k \le \sum_X f_n \text{ for all } n \ge k$$

and therefore

$$\sum_X g_k \le \liminf_{n\to\infty} \sum_X f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\sum_X \liminf_{n\to\infty} f_n = \sum_X \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \sum_X g_k \le \liminf_{n\to\infty} \sum_X f_n.$$

■

Remark 2.12. If $A = \sum_X a < \infty$, then for all $\epsilon > 0$ there exists $\alpha_\epsilon \subset\subset X$ such that

$$A \ge \sum_\alpha a \ge A - \epsilon$$

for all $\alpha \subset\subset X$ containing α_ϵ or equivalently,

$$\left| A - \sum_\alpha a \right| \le \epsilon \tag{2.9}$$

for all $\alpha \subset\subset X$ containing α_ϵ. Indeed, choose α_ϵ so that $\sum_{\alpha_\epsilon} a \ge A - \epsilon$.

2.4. Sums of complex functions.

Definition 2.13. Suppose that $a : X \to \mathbb{C}$ is a function, we say that

$$\sum_X a = \sum_{x\in X} a(x)$$

exists and is equal to $A \in \mathbb{C}$, if for all $\epsilon > 0$ there is a finite subset $\alpha_\epsilon \subset X$ such that for all $\alpha \subset\subset X$ containing α_ϵ we have

$$\left| A - \sum_\alpha a \right| \le \epsilon.$$

Definition 2.14 (Summable)**.** We call a function $a : X \to \mathbb{C}$ **summable** if

$$\sum_X |a| < \infty.$$

Proposition 2.15. *Let $a : X \to \mathbb{C}$ be a function, then $\sum_X a$ exists iff $\sum_X |a| < \infty$, i.e. iff a is summable.*

Proof. If $\sum_X |a| < \infty$, using Remarks 2.12 we may choose an increasing sequence of finite subsets α_n of X, such that

$$\sum_X |a| \geq \sum_{\alpha_n} |a| \geq \sum_X |a| - 1/n \ \forall \ n.$$

Letting $S_n := \sum_{\alpha_n} a$ we have for $m > n$,

$$\begin{aligned} |S_m - S_n| &= \left| \sum_{\alpha_m \setminus \alpha_n} a \right| \leq \sum_{\alpha_m \setminus \alpha_n} |a| \\ &= \sum_{\alpha_m} |a| - \sum_{\alpha_n} |a| \\ &\leq \left| A - \sum_{\alpha_m} |a| \right| + \left| A - \sum_{\alpha_n} |a| \right| \\ &\leq \frac{1}{m} + \frac{1}{n} \end{aligned}$$

which tends to 0 as $m, n \to \infty$. Thus $\{S_m\}$ is a Cauchy sequence and we let $A := \lim_{m\to\infty} S_m$.[2] Letting $m \to \infty$ in the previous equation also shows that

$$|A - S_n| \leq \frac{1}{n}.$$

If α is a finite subset containing α_n, then

$$\begin{aligned} \left| A - \sum_{\alpha} a \right| &= \left| A - \sum_{\alpha_n} a + \sum_{\alpha \setminus \alpha_n} a \right| \\ &\leq \left| A - \sum_{\alpha_n} a \right| + \sum_{\alpha \setminus \alpha_n} |a| \\ &\leq |A - S_n| + 1/n \leq 2/n. \end{aligned}$$

This shows that $\sum_X a$ exists and is equal to A.

Conversely, if $\sum_X |a| = \infty$ then, because $|a| \leq |\mathrm{Re}\, a| + |\mathrm{Im}\, a|$, we must have

$$\sum_X |\mathrm{Re}\, a| = \infty \text{ or } \sum_X |\mathrm{Im}\, a| = \infty.$$

Thus it suffices to consider the case where $a : X \to \mathbb{R}$ is a real function. Write $a = a^+ - a^-$ where

$$a^+(x) = \max(a(x), 0) \text{ and } a^-(x) = \max(-a(x), 0). \tag{2.10}$$

Then $|a| = a^+ + a^-$ and

$$\infty = \sum_X |a| = \sum_X a^+ + \sum_X a^-$$

which shows that either $\sum_X a^+ = \infty$ or $\sum_X a^- = \infty$. Suppose with out loss of generality that $\sum_X a^+ = \infty$. Let $X' := \{x \in X : a(x) \geq 0\}$, then we know

[2](Alternative construction of A.) Using Remarks 2.8 and 2.7 there is a countable set $\Gamma = \{x_n\}_{n=1}^\infty \subset X$ such that $a(x) = 0$ if $x \notin \Gamma$ and $\sum_X |a| = \sum_{n=1}^\infty |a(x_n)| < \infty$ and hence $A := \sum_{n=1}^\infty a(x_n)$ exists.

that $\sum_{X'} a = \infty$ which means there are finite subsets $\alpha_n \subset X' \subset X$ such that $\sum_{\alpha_n} a \geq n$ for all n. This shows that $\sum_X a$ can not exist. ■

If $a : X \to \mathbb{R}$ is a summable function, let $a^{\pm}(x)$ be defined as in Eq. (2.10), then

$$\sum_X a = \sum_X a^+ - \sum_X a^-$$

and if $a : X \to \mathbb{C}$ is a summable function then

$$\sum_X a = \sum_X \operatorname{Re} a + i \sum_X \operatorname{Im} a.$$

Using these two remarks, many theorems about summable functions $a : X \to \mathbb{C}$ may be reduced to theorems about summable functions $a : X \to [0, \infty)$.

Remark 2.16. Suppose that $X = \mathbb{N}$ and $a : \mathbb{N} \to \mathbb{C}$ is a sequence, then it is not necessarily true that

$$\sum_{n=1}^{\infty} a(n) = \sum_{n \in \mathbb{N}} a(n). \tag{2.11}$$

This is because

$$\sum_{n=1}^{\infty} a(n) = \lim_{N \to \infty} \sum_{n=1}^{N} a(n)$$

depends on the ordering of the sequence a where as $\sum_{n \in \mathbb{N}} a(n)$ does not. For example, take $a(n) = (-1)^n/n$ then $\sum_{n \in \mathbb{N}} |a(n)| = \infty$ i.e. $\sum_{n \in \mathbb{N}} a(n)$ does **not** exist while $\sum_{n=1}^{\infty} a(n)$ does exist. On the other hand, if

$$\sum_{n \in \mathbb{N}} |a(n)| = \sum_{n=1}^{\infty} |a(n)| < \infty$$

then Eq. (2.11) is valid.

Exercise 2.17. Suppose that $a, b : X \to \mathbb{C}$ are two summable functions and $\lambda \in \mathbb{C}$. Show

$$\sum_X (a + \lambda b) = \sum_X a + \lambda \sum_X b.$$

Theorem 2.18 (Dominated Convergence Theorem)**.** *Suppose that $f_n : X \to \mathbb{C}$ is a sequence of functions on X such that $f(x) = \lim_{n \to \infty} f_n(x) \in \mathbb{C}$ exists for all $x \in X$. Further assume there is a **dominating function** $g : X \to [0, \infty)$ such that*

$$|f_n(x)| \leq g(x) \text{ for all } x \in X \text{ and } n \in \mathbb{N} \tag{2.12}$$

and that g is summable. Then

$$\lim_{n \to \infty} \sum_{x \in X} f_n(x) = \sum_{x \in X} f(x). \tag{2.13}$$

Proof. Notice that $|f| = \lim |f_n| \leq g$ so that f is summable. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned} \sum_X (g \pm f) &= \sum_X \liminf_{n \to \infty} (g \pm f_n) \leq \liminf_{n \to \infty} \sum_X (g \pm f_n) \\ &= \sum_X g + \liminf_{n \to \infty} \left(\pm \sum_X f_n \right). \end{aligned}$$

Since $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$, we have shown,

$$\sum_X g \pm \sum_X f \le \sum_X g + \begin{cases} \liminf_{n\to\infty} \sum_X f_n \\ -\limsup_{n\to\infty} \sum_X f_n \end{cases}$$

and therefore

$$\limsup_{n\to\infty} \sum_X f_n \le \sum_X f \le \liminf_{n\to\infty} \sum_X f_n.$$

This shows that $\lim_{n\to\infty} \sum_X f_n$exists and is equal to $\sum_X f$. ■

Proof. (Second Proof.) Passing to the limit in Eq. (2.12) shows that $|f| \le g$ and in particular that f is summable. Given $\epsilon > 0$, let $\alpha \subset\subset X$ such that

$$\sum_{X\setminus\alpha} g \le \epsilon.$$

Then for $\beta \subset\subset X$ such that $\alpha \subset \beta$, we have

$$\begin{aligned}\left|\sum_\beta f - \sum_\beta f_n\right| &\le \left|\sum_\beta (f - f_n)\right| \\ &\le \sum_\beta |f - f_n| = \sum_\alpha |f - f_n| + \sum_{\beta\setminus\alpha} |f - f_n| \\ &\le \sum_\alpha |f - f_n| + 2\sum_{\beta\setminus\alpha} g \\ &\le \sum_\alpha |f - f_n| + 2\epsilon.\end{aligned}$$

and hence that

$$\left|\sum_\beta f - \sum_\beta f_n\right| \le \sum_\alpha |f - f_n| + 2\epsilon.$$

Since this last equation is true for all such $\beta \subset\subset X$, we learn that

$$\left|\sum_X f - \sum_X f_n\right| \le \sum_\alpha |f - f_n| + 2\epsilon$$

which then implies that

$$\begin{aligned}\limsup_{n\to\infty}\left|\sum_X f - \sum_X f_n\right| &\le \limsup_{n\to\infty} \sum_\alpha |f - f_n| + 2\epsilon \\ &= 2\epsilon.\end{aligned}$$

Because $\epsilon > 0$ is arbitrary we conclude that

$$\limsup_{n\to\infty}\left|\sum_X f - \sum_X f_n\right| = 0.$$

which is the same as Eq. (2.13). ■

2.5. **Iterated sums.** Let X and Y be two sets.

Exercise 2.19. Suppose that $a : X \to \mathbb{C}$ is function and $F \subset X$ is a subset such that $a(x) = 0$ for all $x \notin F$. Show that $\sum_F a$ exists iff $\sum_X a$ exists, and if the sums exist then

$$\sum_X a = \sum_F a.$$

Theorem 2.20. *Suppose that $a : X \times Y \to [0, \infty]$, then*

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. It suffices to show, by symmetry, that

$$\sum_{X\times Y} a = \sum_X \sum_Y a$$

Let $\Lambda \subset\subset X \times Y$. The for any $\alpha \subset\subset X$ and $\beta \subset\subset Y$ such that $\Lambda \subset \alpha \times \beta$, we have

$$\sum_\Lambda a \le \sum_{\alpha\times\beta} a = \sum_\alpha \sum_\beta a \le \sum_\alpha \sum_Y a \le \sum_X \sum_Y a,$$

i.e. $\sum_\Lambda a \le \sum_X \sum_Y a$. Taking the sup over Λ in this last equation shows

$$\sum_{X\times Y} a \le \sum_X \sum_Y a.$$

We must now show the opposite inequality. If $\sum_{X\times Y} a = \infty$ we are done so we now assume that a is summable. By Remark 2.8, there is a countable set $\{(x'_n, y'_n)\}_{n=1}^\infty \subset X \times Y$ off of which a is identically 0. Let $A = \{x'_n : n \in \mathbb{N}\}$ and $B = \{y'_n : n \in \mathbb{N}\}$ and let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be an enumeration of A and B respectively. With this notation we know that $a(x, y) = 0$ for all $(x, y) \notin A \times B$, so that

$$\sum_{X\times Y} a = \sum_{(m,n)\in\mathbb{N}^2} a(x_m, y_n)$$

It is also easy to verify that

$$\sum_X \sum_Y a = \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} a(x_m, y_n).$$

Hence we have now reduced the problem to the special case where $X = Y = \mathbb{N}$, so we need to prove: if $a : \mathbb{N}^2 \to [0, \infty)$ is summable, then

$$\sum_{\mathbb{N}^2} a \ge \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} a(m, n).$$

Let $M, N \in \mathbb{N}$, then

$$\begin{aligned}
\sum_{m=1}^M \sum_{n=1}^N a(m, n) &\le \sum_{\mathbb{N}^2} a, \\
\sum_{m=1}^M \sum_{n=1}^\infty a(m, n) &= \sum_{m=1}^M \lim_{N\to\infty} \sum_{n=1}^N a(m, n) \\
&= \lim_{N\to\infty} \sum_{m=1}^M \sum_{n=1}^N a(m, n) \le \sum_{\mathbb{N}^2} a.
\end{aligned}$$

Similarly

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a(m,n) = \lim_{M\to\infty}\sum_{m=1}^{M}\sum_{n=1}^{\infty} a(m,n) \le \sum_{\mathbb{N}^2} a$$

which was to be proved. ■

Theorem 2.21. *Now suppose that $a : X \times Y \to \mathbb{C}$ is a summable function, i.e. by Theorem 2.20 any one of the following equivalent conditions hold*

(1) $\sum_{X\times Y} |a| < \infty$,
(2) $\sum_X \sum_Y |a| < \infty$ *or*
(3) $\sum_Y \sum_X |a| < \infty$.

Then

$$\sum_{X\times Y} a = \sum_X \sum_Y a = \sum_Y \sum_X a.$$

Proof. Working as in the proof of Theorem 2.20, we can reduce the proof to the case where $X = Y = \mathbb{N}$.[3] Hence we need to show

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a(m,n) = \sum_{\mathbb{N}^2} a.$$

Let us first note that for all $m \in \mathbb{N}$

$$\sum_{n=1}^{\infty} |a(m,n)| \le \sum_{\mathbb{N}^2} |a| < \infty$$

so that $\sum_{n=1}^{\infty} a(m,n)$ is well defined. Furthermore,

$$\sum_{m=1}^{\infty}\left|\sum_{n=1}^{\infty} a(m,n)\right| \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a(m,n)| < \infty$$

so that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a(m,n)$ is well defined.

Let $\epsilon > 0$, then by the definition of $A := \sum_{\mathbb{N}^2} a$, for all $M, N \in \mathbb{N}$ sufficiently large,

$$\left|\sum_{m=1}^{M}\sum_{n=1}^{N} a(m,n) - A\right| \le \epsilon.$$

Letting $N \to \infty$ in this equation shows that

$$\left|\sum_{m=1}^{M}\sum_{n=1}^{\infty} a(m,n) - A\right| \le \epsilon$$

and then letting $M \to \infty$ shows

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a(m,n) - A\right| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done. ■

[3] The short proof of this theorem is to first prove the result for real valued functions a by applying Theorem 2.20 to $a^{\pm}$ – the positive and negative parts of a. The theorem then follows for complex valued functions a by applying the real version just proved to the real and imaginary parts of a.

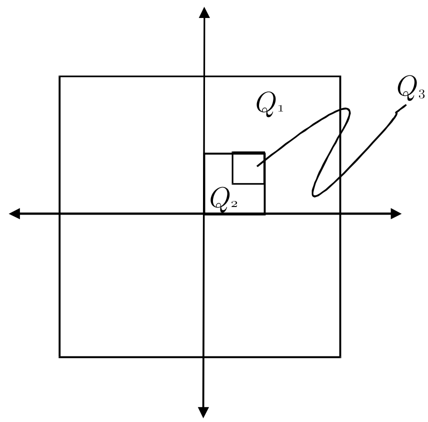

FIGURE 1. Nested Sequence of cubes

2.6. Compact Sets in $\mathbb{R}^n$.

Theorem 2.22. *Suppose that K is a closed and bounded subset of $\mathbb{R}^n$ and that $\mathcal{V}$ is a collection of open subsets of $\mathbb{R}^n$ which covers K in the sense that $K \subset \cup_{V\in\mathcal{V}} V$, then there is finite subcover $\mathcal{F} \subset \mathcal{V}$, i.e. $\mathcal{F}$ is a finite subset of $\mathcal{V}$ such that $K \subset \cup_{V\in\mathcal{F}} V$.*

Proof. Let us first assume that $K = [-M, M]^n$ for some $M \in \mathbb{N}$. Let $C_\epsilon := [0, \epsilon]^n$ and for an integer $k > 0$, let

$$\tilde{\mathcal{Q}}_k := \{x + C_{2^{-k}} : x \in 2^{-k}\mathbb{N} \text{ and } x + C_{1/k} \subset K\}.$$

Let $\mathcal{Q}_1 = \tilde{\mathcal{Q}}_1$. Since $K = \cup_{Q\in\mathcal{Q}_1} Q$, there must be some $Q_1 \in \mathcal{Q}_1$ for which no finite subset of $\mathcal{V}$ covers Q_1. Let $\mathcal{Q}_2 = \{Q \in \tilde{\mathcal{Q}}_2 : Q \subset Q_1\}$, then since $Q_1 = \cup_{Q\in\mathcal{Q}_2} Q$, there must be some $Q_2 \in \mathcal{Q}_2$ for which no finite subset of $\mathcal{V}$ covers Q_2. Then let $\mathcal{Q}_3 = \{Q \in \tilde{\mathcal{Q}}_3 : Q \subset Q_2\}$ and choose $Q_3 \in \mathcal{Q}_3$ or which no finite subset of $\mathcal{V}$ covers Q_3. Continue this way by induction, we may construct cubes $Q_i \in \tilde{\mathcal{Q}}_i$ such that

$$K \supset Q_1 \supset Q_2 \supset Q_3 \supset \dots$$

and no Q_i may be covered by a finite subcollection of $\mathcal{V}$.

Now choose $x_i \in Q_i$ and notice that since the diameter of the Q_i 's tends to 0 as $i \to \infty$, it follows that $\{x_i\}_{i=1}^\infty$ is a Cauchy sequence and hence $x := \lim_{i\to\infty} x_i$ exists. Since all of the Q_i 's are closed we further know that

$$x \in \cap_i Q_i.$$

Since $\mathcal{V}$ is a cover, there is an open set $V \in \mathcal{V}$ such that $x \in V$ and since V is open and $\lim_{i\to\infty} diam(Q_i) = 0$, it follows that $Q_i \subset V$ for all large i. But this contradicts the property that no Q_i may be covered by a finite collection of sets from $\mathcal{V}$. This proves the case where K is a cube.

For general K, choose $M \in \mathbb{N}$ so large that $K \subset [-M, M]^n$. Let $\tilde{\mathcal{V}} = \mathcal{V} \cup \{K^c\}$, then $\tilde{\mathcal{V}}$ is an open cover of $[-M, M]^n$ and hence there is a finite subset $\mathcal{F} \subset \mathcal{V}$ such that $\mathcal{F} \cup \{K^c\}$ covers $[-M, M]^n$ and therefore $\mathcal{F}$ is a finite cover of K. ■

Theorem 2.23. *Suppose that K is a closed and bounded subset of $\mathbb{R}^n$ and $x = \{x_n\}_{n=1}^\infty \subset K$ is a sequence. Then x has a subsequence $y = \{x_{n_k}\}_{k=1}^\infty$ which is convergent and $\lim_{k\to\infty} y_k \in K$.*

Proof. Choose $M \in \mathbb{N}$ so large that $K \subset [-M, M]^n$. Continuing the notation and the method of the proof of Theorem 2.22, we may find a cube $Q_1 \in \mathcal{Q}_1$ such that $\{n : x_n \in Q_1\}$ is an infinite set. Similarly we may find $Q_2 \in \mathcal{Q}_2$ such that $\{n : x_n \in Q_2\}$ is an infinite set. Continuing inductively, we may construct cubes Q_i such that $Q_1 \supset Q_2 \supset Q_3 \supset \dots$ and $\{n : x_n \in Q_i\}$ is an infinite set for all i. Now let $n_1 := \min\{n : x_n \in Q_1\}$, $n_2 = \min\{n > n_1 : x_n \in Q_2\}$ and so on, so that

$$n_{k+1} := \min\{n > n_k : x_n \in Q_{k+1}\},$$

then $y = \{x_{n_k}\}_{k=1}^\infty$ is a subsequence of x such that $y_i \in Q_i$ for all i. As in the previous proof, we conclude that y is a Cauchy sequence and hence $\lim_{k\to\infty} y_k$ exists and is necessarily in K since K is closed. ■

2.7. **Basic metric space notions.**

Definition 2.24. A function $d : X \times X \to [0, \infty)$ is called a metric if

(1) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$
(2) (Non-degenerate) $d(x, y) = 0$ if and only if $x = y \in X$
(3) (Triangle inequality) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.25. Let (X, d) be a metric space. The open ball $B(x, \delta) \subset X$ centered at $x \in X$ with radius $\delta > 0$ is the set

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

Definition 2.26. A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X, d) is said to be convergent if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$.

Exercise 2.27. Show that x in Definition 2.26 is necessarily unique.

Definition 2.28. A set $F \subset X$ is closed iff every convergent sequence $\{x_n\}_{n=1}^\infty$ which is contained in F has its limit back in F. A set $V \subset X$ is open iff V^c is closed.

Exercise 2.29. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B(x, \delta) \subset V$.

Definition 2.30 (Cauchy sequences)**.** A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X, d) is said Cauchy provided that

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Exercise 2.31. Show that convergent sequences are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x, y) = |x - y|$. Choose a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^\infty$ is $(\mathbb{Q}, d)$ – Cauchy but not $(\mathbb{Q}, d)$ – convergent. The sequence does converge in $\mathbb{R}$ however.

Definition 2.32. A metric space (X, d) is **complete** if all Cauchy sequences are convergent sequences..

Example 2.33. Here are a couple of examples of complete metric spaces.

(1) $X = \mathbb{R}$ and $d(x, y) = |x - y|$.

(2) $X = \mathbb{R}^n$ and $d(x,y) = \|x-y\|$.
(3) $X = C([0,1],\mathbb{R})$ – the space of continuous functions from $[0,1]$ to $\mathbb{R}$ and $d(f,g) := \max_{t\in[0,1]} |f(t)-g(t)|$.
(4) Here is a typical example of a non-complete metric space. Let $X = C([0,1],\mathbb{R})$ and

$$d(f,g) := \int_0^1 |f(t)-g(t)|\,dt.$$

Let us verify Item 3. in the previous example. Suppose that $\{f_n\} \subset X$ is a Cauchy sequence, then for each $x \in [0,1]$, $\{f_n(x)\} \subset \mathbb{R}$ is Cauchy and hence convergent. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for each $x \in [0,1]$. We will now shows that $f \in X$ and that $\lim_{n\to\infty} d(f,f_n) = 0$. First off, given $\epsilon > 0$ we have

$$\begin{aligned}|f(x)-f_n(x)| &\le |f(x)-f_m(x)| + |f_m(x)-f_n(x)| \\ &\le |f(x)-f_m(x)| + \epsilon\end{aligned}$$

provided that $m,n \ge N$ for some $N = N(\epsilon)$ sufficiently large (and not depending on x). Letting $m \to \infty$ in this last equation shows that

$$|f(x)-f_n(x)| \le \epsilon \text{ for all } n \ge N$$

so that $f_n(x) \to f(x)$ uniformly in x as $n \to \infty$. So to finish the argument, we need only shows that f is continuous. For this let $\epsilon > 0$ and N be as above, then for $x, y \in [0,1]$,

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f(y)| \\ &\le 2\epsilon + |f_N(x)-f_N(y)|.\end{aligned}$$

Now for x fixed, by the continuity of f_N, we may choose $\delta > 0$ so that $|f_N(x) - f_N(y)| < \epsilon$ provided that $|x-y| < \delta$. For this δ we then have

$$|f(x)-f(y)| < 2\epsilon + \epsilon = 3\epsilon$$

which shows that f is continuous as well.

Suppose that (X,d) and (Y,ρ) are two metric spaces and $f : X \to Y$ is a function.

Definition 2.34. A function $f : X \to Y$ is continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(f(x), f(x')) < \epsilon \text{ provided that } d(x,x') < \delta.$$

The function f is said to be continuous if f is continuous at all points $x \in X$.

Exercise 2.35. Show that f is continuous at x iff for all sequences $\{x_n\}_{n=1}^\infty \subset X$ converging to x, the sequence $\{f(x_n)\}_{n=1}^\infty \subset Y$ should converge to $f(x)$. Put briefly,

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

Exercise 2.36. Suppose that $f : X \to Y$ is a continuous function and that $C \subset Y$ is closed set, show that $f^{-1}(C)$ is then closed in X.

The next lemma supplies some examples of continuous functions on metric spaces.

Lemma 2.37. *For any non empty subset $A \subset X$, let $d_A(x) \equiv \inf\{d(x,a)|a \in A\}$, then*

$$|d_A(x) - d_A(y)| \le d(x,y) \; \forall \, x, y \in X. \tag{2.14}$$

In particular, d_A is a continuous function on X. Moreover, by Exercise 2.36, for all $\epsilon > 0$ the set $F_\epsilon \equiv \{x \in X | d_A(x) \ge \epsilon\}$ is closed in X. Further, if V is an open set and $A = V^c$, then $F_\epsilon \uparrow V$ as $\epsilon \downarrow 0$.

Proof. Let $a \in A$ and $x, y \in X$, then

$$d(x,a) \le d(x,y) + d(y,a).$$

Take the inf over a in the above equation shows that

$$d_A(x) \le d(x,y) + d_A(y) \quad \forall x, y \in X.$$

Therefore, $d_A(x) - d_A(y) \le d(x,y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \le d(x,y)$ which implies Eq. (2.14) from which it follows that d_A is continuous on X.

Now suppose that $A = V^c$ with $V \in \tau$. It is clear that $d_A(x) = 0$ for $x \in A = V^c$ so that $F_\epsilon \subset V$ for each $\epsilon > 0$ and hence $\cup_{\epsilon>0} F_\epsilon \subset V$. Now suppose that $x \in V$, then there exists an $\epsilon > 0$ such that $B_x(\epsilon) \subset V$, that is it $y \in X$ such that $d(x,y) < \epsilon$ then $y \in V$. Therefore $d(x,y) \ge \epsilon$ for all $y \in V^c$ and hence $x \in F_\epsilon$, i.e. $V \subset \cup_{\epsilon>0} F_\epsilon$. Finally it is clear that $F_\epsilon \subset F_{\epsilon'}$ whenever $\epsilon' \le \epsilon$. ■

Corollary 2.38. *The function d satisfies,*

$$|d(x,y) - d(x',y')| \le d(y,y') + d(x,x')$$

and in particular $d : X \times X \to [0,\infty)$ is continuous.

Proof. By Lemma 2.37 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned} |d(x,y) - d(x',y')| &\le |d(x,y) - d(x,y')| + |d(x,y') - d(x',y')| \\ &\le d(y,y') + d(x,x'). \end{aligned}$$

■

3. Topologies, Algebras and σ – Algebras

Definition 3.1. A collection of subsets τ of X is a **topology** if

(1) $\emptyset, X \in \tau$
(2) τ is closed under arbitrary unions, i.e. if $V_\alpha \in \tau$, for $\alpha \in I$ then $\bigcup_{\alpha \in I} V_\alpha \in \tau$.
(3) τ is closed under finite intersections, i.e. if $V_1, \ldots, V_n \in \tau$ then $V_1 \cap \cdots \cap V_n \in \tau$.

Notation 3.2. The subsets $V \subset X$ which are in τ are called open sets and we will abbreviate this by writing $V \subset_0 X$ and the those sets $F \subset X$ such that $F^c \in \tau$ are called closed sets. We will write $F \sqsubset X$ if F is a closed subset of X.

Definition 3.3. A collection of subsets $\mathcal{A}$ of X is an **Algebra** if

(1) $\emptyset, X \in \mathcal{A}$
(2) $A \in \mathcal{A}$ implies that $A^c \in \mathcal{A}$
(3) $\mathcal{A}$ is closed under finite unions, i.e. if $A_1, \ldots, A_n \in \mathcal{A}$ then $A_1 \cap \cdots \cap A_n \in \mathcal{A}$.
(4) $\mathcal{A}$ is closed under finite intersections.

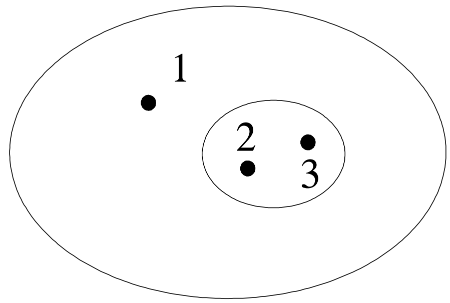

FIGURE 2. A topology

Definition 3.4. A collection of subsets $\mathcal{M}$ of X is a σ – algebra (σ – field) if $\mathcal{M}$ is an algebra which also closed under countable unions, i.e. if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Notice that since $\mathcal{M}$ is also closed under taking compliments, $\mathcal{M}$ is also closed under taking countable intersections.

Example 3.5. Here are a number of examples.

(1) $\tau = \mathcal{M} = \mathcal{P}(X)$ in which case all subsets of X are open, closed, and measurable.
(2) Let $X = \{1, 2, 3\}$, then $\tau = \{\emptyset, X, \{2, 3\}\}$ is a topology on X which is not an algebra.
(3) $\tau = \mathcal{A} = \{\{1\}, \{2, 3\}, \emptyset, X\}$. is a topology, an algebra, and a σ – algebra on X. The sets X, $\{1\}$, $\{2, 3\}, \phi$ are open and closed. The sets $\{1, 2\}$ and $\{1, 3\}$ are neither open nor **closed** and are not measurable..

Proposition 3.6. *Let $\mathcal{E}$ be any collection of subsets of X. Then there exists a unique smallest topology $\tau(\mathcal{E})$, algebra $\mathcal{A}(\mathcal{E})$ and σ-algebra $\mathcal{M}(\mathcal{E})$ which contains $\mathcal{E}$. I will also tend to write $\sigma(\mathcal{E})$ for $\mathcal{M}(\mathcal{E})$, i.e. $\sigma(\mathcal{E}) = \mathcal{M}(\mathcal{E})$. The notation $\mathcal{M}(\mathcal{E})$ is used in Folland, but $\sigma(\mathcal{E})$ is the more standard notation.*

Proof. Note $\mathcal{P}(X)$ is a topology and an algebra and a σ-algebra and $\mathcal{E} \subseteq \mathcal{P}(X)$, so that $\mathcal{E}$ is always a subset of a topology, algebra, and σ – algebra. One may now easily check that

$$\tau(\mathcal{E}) \equiv \bigcap \{\tau : \tau \text{ is a topology and } \mathcal{E} \subset \tau\}$$

is a topology which is clearly the smallest topology containing $\mathcal{E}$. The analogous construction works for the other cases as well. ■

We may give explicit descriptions of $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$.

Proposition 3.7. *Let X and $\mathcal{E} \subset \mathcal{P}(X)$. For simplicity of notation, assume that $X, \emptyset \in \mathcal{E}$ (otherwise adjoin them to $\mathcal{E}$ if necessary) and let $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$ Then*

$$\tau(\mathcal{E}) = \{\textit{arbitrary unions of finite intersections of elements from } \mathcal{E}\} \tag{3.1}$$

and

$$\mathcal{A}(\mathcal{E}) = \{\textit{finite unions of finite intersections of elements from } \mathcal{E}_c\}. \tag{3.2}$$

Proof. From the definition of a topology and an algebra, it is clear that $\tau(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$ contain those sets in the right side of Eqs. (3.1) and (3.2) respectively. Hence it suffices to show that the right members of Eqs. (3.1) and (3.2) form a topology and an algebra respectively. The proof of these assertions are routine except for possibly showing that

$$\mathcal{A} := \{\text{finite unions of finite intersections of element from } \mathcal{E}_c\}$$

is closed under complementation. To check this, we notice that the typical element $Z \in \mathcal{A}$ is of the form

$$Z = \bigcup_{i=1}^{N} \bigcap_{j=1}^{K} A_{ij}$$

where $A_{ij} \in \mathcal{E}_c$. Therefore, writing $B_{ij} = A_{ij}^c \in \mathcal{E}_c$, we find that

$$Z^c = \bigcap_{i=1}^{N} \bigcup_{j=1}^{K} B_{ij} = \bigcup_{j_1,\ldots,j_N=1}^{K} (B_{1j_1} \cap B_{2j_2} \cap \cdots \cap B_{Nj_N}) \in \mathcal{A}(\mathcal{E})$$

wherein we have used the fact that $B_{1j_1} \cap B_{2j_2} \cap \cdots \cap B_{Nj_N}$ is a finite intersection of sets from $\mathcal{E}_c$. ■

Remark 3.8. One might think that in general $\mathcal{M}(\mathcal{E})$ may be described as the countable unions of countable intersections of sets in $\mathcal{E}^c$ However this is **false,** since if

$$Z = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} A_{ij}$$

with $A_{ij} \in \mathcal{E}_c$, then

$$Z^c = \bigcup_{j_1=1, j_2=1, \ldots j_N=1, \ldots}^{\infty} \left(\bigcap_{\ell=1}^{\infty} B_{1 j_\ell} \right)$$

which is now an **uncountable** union. Thus the above description is not correct. In general it is fairly complicated to explicitly describe $\mathcal{M}(\mathcal{E})$, see Proposition 1.23 on page 39 for details.

The following notion will be useful in the sequel.

Definition 3.9. A set $\mathcal{E} \subset \mathcal{P}(X)$ is said to be an **elementary family or elementary class** provided that

- $\emptyset \in \mathcal{E}$
- $\mathcal{E}$ is closed under finite intersections
- if $E \in \mathcal{E}$, then E^c is a finite disjoint union of sets from $\mathcal{E}$.

Proposition 3.10. *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family, then $\mathcal{A} = \mathcal{A}(\mathcal{E})$ consists of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$.*

Proof. (First Proof.) By Proposition 3.7

$$\mathcal{A}(\mathcal{E}) = \{\text{finite unions of finite intersections of elements from } \mathcal{E}_c\},$$

where $\mathcal{E}^c \equiv \{A^c : A \in \mathcal{E}\}$ and $\mathcal{E}_c = \mathcal{E} \cup \{X, \emptyset\} \cup \mathcal{E}^c$. Using the definition of an elementary family we see that $\mathcal{A}(\mathcal{E})$ may be described more simply as

$$\mathcal{A}(\mathcal{E}) = \{\text{finite unions of elements from } \mathcal{E}\}.$$

Let $A = \cup_{i=1}^n E_i \in \mathcal{A}(\mathcal{E})$ with $E_i \in \mathcal{E}$. To finish the proof we need to show that A may be written disjoint union of elements from $\mathcal{E}$. We prove this by induction on n. For $n = 1$ and $A = E_1$ there is nothing to prove. If $n = 2$ and $A = E_1 \cup E_2$, let $E_2^c = \coprod_{i=1}^k F_i$ with $F_i \in \mathcal{E}$. Then

$$E_2 \backslash E_1 = E_2 \cap E_1^c = \coprod_{i=1}^k E_2 \cap F_i$$

so that

$$A = E_1 \cup \left(\coprod_{i=1}^k E_2 \cap F_i \right)$$

is the desired decomposition. Now for the induction step, suppose that

$$A = \cup_{i=1}^n E_i = B \cup E_n = (B \backslash E_n) \cup E_n$$

where $B = \coprod_{j=1}^N E_j'$ where $\left\{E_j'\right\} \subset \mathcal{E}$ are pairwise disjoint. Write $E_n^c = \coprod_{i=1}^k F_i$ with $F_i \in \mathcal{E}$, then

$$B \backslash E_n = B \cap E_n^c = \coprod_{i=1}^k B \cap F_i = \coprod_{i=1}^k \coprod_{j=1}^N E_j' \cap F_i$$

and hence

$$A = \left(\coprod_{i=1}^k \coprod_{j=1}^N E_j' \cap F_i \right) \coprod E_n$$

is the desired decomposition.

(**Second more direct proof.**) Let $\mathcal{A}$ denote the collection of sets which may be written as finite disjoint unions of sets from $\mathcal{E}$. Clearly $\mathcal{A} \subset \mathcal{A}(\mathcal{E})$ so it suffices to show that $\mathcal{A}$ is an algebra. By the properties of $\mathcal{E}$, we know that $\emptyset, X \in \mathcal{A}$. Further, if $A = \coprod_{i=1}^n E_i$ with $E_i \in \mathcal{A}$, then

$$A^c = \cap_{i=1}^n E_i^c.$$

Since $\mathcal{E}$ is an elementary class, for each i there exists a collection of disjoint sets $\{F_{ij}\}_j \subset \mathcal{E}$ such that $E_i^c = \coprod_j F_{ij}$. Therefore,

$$A^c = \cap_{i=1}^n (\cup_j F_{ij}) = \bigcup_{j_1, j_2, \dots, j_n} (F_{1j_1} \cap F_{2j_2} \cap \dots \cap F_{n,j_n})$$

and this is a disjoint union. Hence $A^c \in \mathcal{A}$, i.e. $\mathcal{A}$ is closed under complementation.

Now suppose that $A_i = \coprod_j F_{ij} \in \mathcal{A}$ for $i = 1, 2, \dots, n$, then

$$\cap_i A_i = \bigcup_{j_1, j_2, \dots, j_n} (F_{1j_1} \cap F_{2j_2} \cap \dots \cap F_{n,j_n})$$

which is again a disjoint unison of sets from $\mathcal{E}$ so that $\mathcal{A}$ is closed under finite intersections. ∎

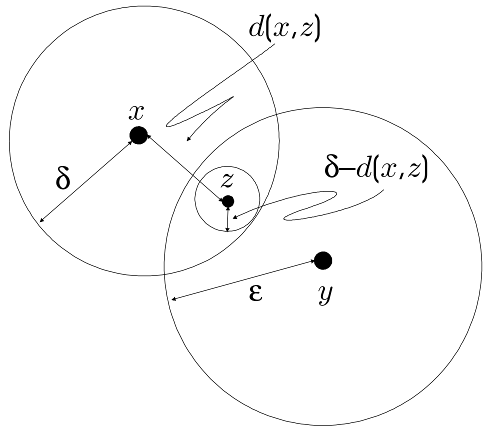

FIGURE 3. Fitting balls in the intersection.

Exercise 3.11. Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be elementary families. Show the collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is also an elementary family.

Definition 3.12. Let (X, ρ) be a metric space. We associate a topology, τ_ρ, to ρ by setting $\tau_\rho := \tau(\mathcal{E})$ where $\mathcal{E} = \{B(x,\delta) : x \in X \text{ and } \delta > 0\}$.

Proposition 3.13. *A set $V \subset X$ is in τ_ρ, i.e. V is open, iff V is a union of open balls. So τ_ρ may also be described as*

$$\tau_\rho := \{V \subseteq X : \forall \; x \in V \; \exists \; r > 0 \ni B(x,r) \subseteq V\} \cup \emptyset. \tag{3.3}$$

Proof. Let us first notice that $B(x,\delta)$ and $B(y,\epsilon)$ are two open ball in X and $z \in B(x,\delta) \cap B(y,\epsilon)$, then

$$B(z,\alpha) \subset B(x,\delta) \cap B(y,\epsilon) \tag{3.4}$$

where $\alpha = \min\{\delta - d(x,z), \epsilon - d(y,z)\}$, see Figure 3. This is a formal consequence of the triangle inequality. For example let us show that $B(z,\alpha) \subset B(x,\delta)$. By the definition of α, we have that $\alpha \le \delta - d(x,z)$ or that $d(x,z) \le \delta - \alpha$. Hence if $w \in B(z,\alpha)$, then

$$d(x,w) \le d(x,z) + d(z,w) \le \delta - \alpha + d(z,w) < \delta - \alpha + \alpha = \delta$$

which shows that $w \in B(x,\delta)$. Similarly we show that $w \in B(y,\epsilon)$ as well.

Equation (3.4) may be generalized to finite intersection of balls, namely if $x_i \in X$, $\delta_i > 0$ and $z \in \cap_{i=1}^n B(x_i,\delta_i)$, then

$$B(z,\alpha) \subset \cap_{i=1}^n B(x_i,\delta_i) \tag{3.5}$$

where now $\alpha := \min\{\delta_i - d(x_i,z) : i = 1,2,\ldots,n\}$. By Eq. (3.5) it follows that any finite intersection of open balls may be written as a union of open balls, thus we see by Proposition 3.7 that $\tau_\rho := \tau(\mathcal{E})$ is given as the right hand side of Eq. (3.3). ■

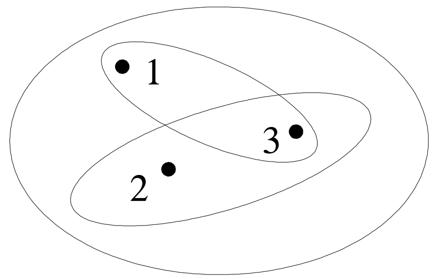

FIGURE 4. A collection of subsets.

Proposition 3.14. *If $\mathcal{E} \subseteq \mathcal{P}(X)$ is countable then $\tau(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E}) = \sigma(\mathcal{E})$. In particular $\sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E})$.*

Proof. Let $\mathcal{E}_f$ denote the collection of subsets of X which are finite intersection of elements from $\mathcal{E}$ along with X and $\emptyset$. Notice that $\mathcal{E}_f$ is still countable (you prove). A set Z is in $\tau(\mathcal{E})$ iff Z is an arbitrary union of sets from $\mathcal{E}_f$. Therefore $Z = \bigcup_{A\in\mathcal{F}} A$ for some subset $\mathcal{F} \subseteq \mathcal{E}_f$ which is necessarily countable. Since $\mathcal{E}_f \subseteq \mathcal{M}(\mathcal{E})$ and $\mathcal{M}(\mathcal{E})$ is closed under countable unions it follows that $Z \in \mathcal{M}(\mathcal{E})$ and hence that $\tau(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E})$. ■

Example 3.15. Suppose that $\mathcal{E} = \{\emptyset, X, \{1,2\}, \{1,3\}\}$,
Then

$$\begin{aligned}\tau(\mathcal{E}) &= \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\} \\ \mathcal{A}(\mathcal{E}) &= \mathcal{M}(\mathcal{E}) = \mathcal{P}(X).\end{aligned}$$

Example 3.16. Let X be a set and $\mathcal{E} = \{A_1, \ldots, A_n\} \cup \{X, \emptyset\}$ where $A_1, \ldots, A_n$ is a partition of X, i.e. $X = \cup_{j=1}^n A_j$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. In this case

$$\mathcal{A}(\mathcal{E}) = \mathcal{M}(\mathcal{E}) = \tau(\mathcal{E}) = \{\cup_{i\in\Lambda} A_i : \Lambda \subset \{1,2,\ldots,n\}\}$$

where $\cup_{i\in\Lambda} A_i := \emptyset$ when $\Lambda = \emptyset$. Notice that

$$\#\mathcal{A}(\mathcal{E}) = \#(\mathcal{P}(\{1,2,\ldots,n\})) = 2^n.$$

Proposition 3.17. *Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra and $\mathcal{M}$ is at most a countable set. Then there exists a unique **finite** partition $\mathcal{F}$ of X such that $\mathcal{F} \subset \mathcal{M}$ and every element $A \in \mathcal{M}$ is of the form*

$$A = \cup_{\alpha\in\mathcal{F}\ni\alpha\subset A}\,\alpha. \tag{3.6}$$

In particular $\mathcal{M}$ is actually a finite set.

Proof. For each $x \in X$ let

$$A_x = (\cap_{x\in A\in\mathcal{A}} A) \in \mathcal{A}.$$

That is, A_x is the smallest set in $\mathcal{A}$ which contains x. Suppose that $C = A_x \cap A_y$ is non-empty. If $x \notin C$ then $x \in A_x \setminus C \in \mathcal{A}$ and hence $A_x \subset A_x \setminus C$ which shows that $A_x \cap C = \emptyset$ which is a contradiction. Hence $x \in C$ and similarly $y \in C$, therefore $A_x \subset C = A_x \cap A_y$ and $A_y \subset C = A_x \cap A_y$ which shows that $A_x = A_y$.

Therefore, $\mathcal{F} = \{A_x : x \in X\}$ is a partition of X (which is necessarily countable) and Eq. (3.6) holds for all $A \in \mathcal{M}$. Let $\mathcal{F} = \{P_n\}_{n=1}^N$ where for the moment we allow $N = \infty$. If $N = \infty$, then $\mathcal{M}$ is one to one correspondence with $\{0,1\}^{\mathbb{N}}$. Indeed to each $a \in \{0,1\}^{\mathbb{N}}$, let $A_a \in \mathcal{M}$ be defined by

$$A_a = \cup\{P_n : a_n = 1\}.$$

This shows that $\mathcal{M}$ is uncountable since $\{0,1\}^{\mathbb{N}}$ is uncountable, think of the base two expansion of numbers in $[0,1]$ for example. Thus any countable σ – algebra is necessarily finite. This finishes the proof modulo the uniqueness assertion which is left as an exercise to the reader. ■

Unfortunately, as already mentioned the structure of general σ – algebras is not so simple.

Example 3.18. Let $X = \mathbb{R}$ and $\mathcal{E} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\} \subseteq \mathcal{P}(\mathbb{R})$. Notice that $\mathcal{E}_f = \mathcal{E}$ and that $\mathcal{E}$ is closed under unions, which shows that $\tau(\mathcal{E}) = \mathcal{E}$, i.e. $\mathcal{E}$ is already a topology. Since $(a,\infty)^c = (-\infty, a]$ we find that $\mathcal{E}_c = \{(a,\infty), (-\infty,a], -\infty \le a < \infty\} \cup \{\mathbb{R},\emptyset\}$. Noting that

$$(a,\infty) \cap (-\infty, b] = (a,b]$$

it is easy to verify that the algebra $\mathcal{A}(\mathcal{E})$ generated by $\mathcal{E}$ may be described as being those sets which are finite disjoint unions of sets from the following list

$$\{(a,\infty), (-\infty,a], (a,b] : a,b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

The σ – algebra, $\sigma(\mathcal{E})$, generated by $\mathcal{E}$ is **very complicated**. Here are some sets in $\sigma(\mathcal{E})$.

(a) $(a,b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \in \sigma(\mathcal{E})$.
(b) All of the standard open subsets of $\mathbb{R}$ are in $\sigma(\mathcal{E})$.
(c) $\{x\} = \bigcap_n \left(x - \frac{1}{n}, x\right] \in \sigma(\mathcal{E})$
(d) $[a,b] = \{a\} \cup (a,b] \in \sigma(\mathcal{E})$
(e) Any countable subset of $\mathbb{R}$ is in $\sigma(\mathcal{E})$.

Exercise 3.19. Show that the following σ – algebras on $\mathbb{R}$ are all the same:

$$\begin{aligned}
&\sigma(\text{standard open sets}) \text{ – the Borel } \sigma \text{ – algebra}\\
&\sigma(\{(a,\infty) : a \in \mathbb{R}\})\\
&\sigma(\{(a,\infty) : a \in \mathbb{Q}\})\\
&\sigma(\{[a,\infty) : a \in \mathbb{Q}\}).
\end{aligned}$$

4. Measures

Definition 4.1. A set X equipped with a σ – algebra $\mathcal{M}$ is called a **measurable space**.

Definition 4.2. A measure μ on a measurable space $(X,\mathcal{M})$ is a function $\mu : \mathcal{M} \to [0,\infty]$ such that

a: $\mu(\emptyset) = 0$ and
b: $\mu(\bigcup_i A_i) = \sum \mu(A_i)$ if $A_i \in \mathcal{M}$ and $A_i \cap A_j = \emptyset$ when $i \ne j$.

Proposition 4.3 (Basic properties of measures)**.** *Suppose that $(X,\mathcal{M},\mu)$ is a measure space and $E,F\in\mathcal{M}$ and $\{E_j\}_{j=1}^\infty\subset\mathcal{M}$, then :*

1. *If $E\subseteq F$ then $\mu(E)\le\mu(F)$.*
2. *$\mu(\cup E_j)\le\sum\mu(E_j)$.*
3. *If $E_j\uparrow E$, i.e. $E_1\subset E_2\subset E_3\subset\dots$ and $E=\cup_j E_j$, then $\mu(E_j)\uparrow\mu(E)$ as $j\to\infty$.*
4. *If $\mu(E_1)<\infty$ and $E_j\searrow E$, i.e. $E_1\supset E_2\supset E_3\supset\dots$ and $E=\cap_j E_j$, then $\mu(E_j)\searrow\mu(E)$ as $j\to\infty$.*

Proof. (1) Define $F=E\cup(F\setminus E)$, then $\mu(F)=\mu(E)+\mu(F\setminus E)\ge\mu(E)$.

(2) Let $\widetilde{E}_j=E_j\setminus(E_1\cup\dots\cup E_{j-1})$ so that the $\tilde{E}_j$'s are pair-wise disjoint and $E=\cup\widetilde{E}_j$. Since $\tilde{E}_j\subset E_j$ it follows from Definition 4.2 and part (1), that

$$\mu(E)=\sum\mu(\widetilde{E}_j)\le\sum\mu(E_j).$$

(3) Again let $\widetilde{E}_j=E_j\setminus(E_1\cup\dots\cup E_{j-1})=E_j\setminus E_{j-1}$, then $E=\cup E_i=\cup\widetilde{E}_i$ and thus

$$\mu(E)=\sum_{i=1}^{\infty}\mu(\widetilde{E}_i)=\lim_{N\to\infty}\sum_{i=1}^{N}\mu(\widetilde{E}_i)=\lim_{N\to\infty}\mu(E_N).$$

(4) Define $D_i\equiv E_1\setminus E_i$ then $D_i\uparrow E_1\setminus E$ which implies that

$$\mu(E_1)-\mu(E)=\lim_{i\to\infty}\mu(D_i)=\mu(E_1)-\lim_{i\to\infty}\mu(E_i)$$

which shows that $\lim_{i\to\infty}\mu(E_i)=\mu(E)$. ■

Definition 4.4. A set $E\in\mathcal{M}$ is a **null** set if $\mu(E)=0$.

Definition 4.5. A measure space $(X,\mathcal{M},\mu)$ is **complete** if every subset of a null set is in $\mathcal{M}$, i.e. $\forall F\subset X$ such that $F\subseteq E\in\mathcal{M}$ with $\mu(E)=0$ implies that $F\in\mathcal{M}$.

Proposition 4.6. *Let $(X,\mathcal{M},\mu)$ be a measure space. Set $\mathcal{N}\equiv\{N\subseteq X$:there exists $F\in\mathcal{M}$ such that $N\subseteq F$ and $\mu(F)=0\}$.*

$$\bar{\mathcal{M}}=\{A\cup N:A\in\mathcal{M},N\in\mathcal{M}\},$$

see Fig. 5. Then $\bar{\mathcal{M}}$ is a σ-algebra. Define $\bar{\mu}(A\cup N)=\mu(A)$, then $\bar{\mu}$ is the unique measure on $\bar{\mathcal{M}}$ which extends μ.

Proof. Clearly $X,\emptyset\in\bar{\mathcal{M}}$. Let $A\in\mathcal{M}$ and $N\in\mathcal{N}$ and choose $F\in\mathcal{M}$ such that $N\subseteq F$ and $\mu(F)=0$. Since $N^c=(F\setminus N)\cup F^c$,

$$(A\cup N)^c=A^c\cap N^c=A^c\cap(F\setminus N\cup F^c)=[A^c\cap(F\setminus N)]\cup[A^c\cap F^c]$$

where $[A^c\cap(F\setminus N)]\in\mathcal{N}$ and $[A^c\cap F^c]\in\mathcal{M}$. Thus $\bar{\mathcal{M}}$ is closed under compliments.

If $A_i\in\mathcal{M}$ and $N_i\subseteq F_i\in\mathcal{M}$ such that $\mu(F_i)=0$ then $\cup(A_i\cup N_i)=(\cup A_i)\cup(\cup N_i)\in\bar{\mathcal{M}}$ since $\cup A_i\in\mathcal{M}$ and $\cup N_i\subseteq\cup F_i$ and $\mu(\cup F_i)\le\sum\mu(F_i)=0$. Therefore, $\bar{\mathcal{M}}$ is a σ-algebra.

Suppose $A\cup N_1=B\cup N_2$ with $A,B\in\mathcal{M}$ and $N_1,N_2,\in\mathcal{N}$. Then $A\subseteq A\cup N_1\subseteq A\cup N_1\cup F_2=B\cup F_2$ which shows that

$$\mu(A)\le\mu(B)+\mu(F_2)=\mu(B).$$

Similarly, we show that $\mu(B)\le\mu(A)$ so that $\mu(A)=\mu(B)$ and hence $\bar{\mu}(A\cup N):=\mu(A)$ is well defined. It is left as an exercise to show that $\bar{\mu}$ is a measure, i.e. that it is countable additive. ■

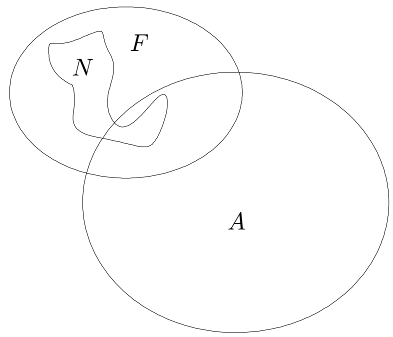

FIGURE 5. Completing a σ – algebra.

4.1. **Regularity Results.** The question now is how do we describe measures on general σ – algebras. This is tricky because there tends to be no "explicit" description of the general element of they typical σ – algebras. On the other hand, we do know how to explicitly describe algebras which are generated by some class of set $\mathcal{E} \subset \mathcal{P}(X)$. Therefore, we might try to define measures on $\sigma(\mathcal{E})$ by there restrictions to $\mathcal{A}(\mathcal{E})$. The next theorem shows that this is a plausible method in many situations.

Theorem 4.7 (Uniqueness)**.** *Suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra of sets and $\mathcal{M} = \sigma(\mathcal{A})$ is the σ – algebra generated by $\mathcal{A}$. If μ and ν are two finite measures on $\mathcal{M}$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$, then $\mu = \nu$ on $\mathcal{M}$. More generally, the theorem holds when μ and ν are σ – finite measures on $\mathcal{A}$, that is there are set $X_n \in \mathcal{A}$ such that $\mu(X_n) = \nu(X_n) < \infty$ and $\cup X_n = X$.*

The following definition and technical lemma will be useful in the proof.

Definition 4.8 (Monotone Class)**.** $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **monotone class** if it is closed under increasing unions and decreasing intersections.

Example 4.9. Let μ, ν be two measure on $(X, \mathcal{M})$. Let

$$\mathcal{C} = \{A \subseteq X : \mu(A) = \nu(A)\}.$$

Assume $X \in \mathcal{C}$ and $\mu(X) = \nu(X) < \infty$. Then $\mathcal{C}$ is a monotone class.

Lemma 4.10. *Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and $\mathcal{C}$ is the smallest monotone class containing $\mathcal{A}$. Then $\mathcal{C} = \sigma(\mathcal{A})$.*

Proof. For $A \in \mathcal{C}$ let $\mathcal{C}(A) = \{B \in \mathcal{C} | A \cap B, A \cap B^c, B \cap A^c$ are in $\mathcal{C}\}$. Then $\mathcal{C}(A)$ is a monotone class contained in $\mathcal{C}$.

Moreover if $A \in \mathcal{A}$ then $\mathcal{A} \subseteq \mathcal{C}(A)$ implies $\mathcal{C}(A) = \mathcal{C}$ for all $A \in \mathcal{A}$. Now $B \in \mathcal{C}(A) \Leftrightarrow A \in \mathcal{C}(B)$. Since $\mathcal{C}(A) = \mathcal{C}$ for all $A \in \mathcal{A}$, $A \in \mathcal{C}(B)$ for all $B \in \mathcal{C}$ and $A \in \mathcal{C}$. Thus $\mathcal{C}(B) = \mathcal{C}$ for all $B \in \mathcal{C}$. So we have shown, if $A, B \in \mathcal{C}$ then $A \cap B$, $A \cap B^c$, $A^c \cap B$ are back in $\mathcal{C}$. In particular, $\mathcal{C}$ is closed under compliments and finite intersections. So $\mathcal{C}$ is an algebra which is also closed under increasing unions and hence it is a σ-algebra. ■

Proof. (Proof of Theorem 4.7.) Assume that $\mu(X) = \nu(X) < \infty$. Let

$$\mathcal{D} = \{A \in \mathcal{M} : \mu(A) = \nu(A)\}$$

The reader may easily check that $\mathcal{D}$ is a monotone class. Since $\mathcal{A} \subset \mathcal{D}$, the monotone class lemma asserts that $\mathcal{M} = \sigma(\mathcal{A}) \subset \mathcal{D} \subset \mathcal{M}$ showing that $\mathcal{D} = \mathcal{M}$ and hence that $\mu = \nu$ on $\mathcal{M}$.

For the σ – finite case, by replacing X_n by $\cup_{j=1}^n X_j$ if necessary, we may assume that $X_n \uparrow X$. For $n \in \mathbb{N}$, let $\mu_n(A) := \mu(A \cap X_n)$ and $\nu_n(A) = \nu(A \cap X_n)$ for all $A \in \mathcal{M}$. Then one easily checks that μ_n and ν_n are finite measure on $\mathcal{M}$ such that $\mu_n = \nu_n$ on $\mathcal{A}$. Therefore, by what we have just proved, $\mu_n = \nu_n$ on $\mathcal{M}$. Hence we have

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap X_n) = \lim_{n\to\infty} \nu(A \cap X_n) = \nu(A)$$

for all $A \in \mathcal{M}$. ■

Definition 4.11. Suppose that $\mathcal{E}$ is a collection of subsets of X, let $\mathcal{E}_\sigma$ denote the collection of subsets of X which are finite or countable unions of sets from $\mathcal{E}$. Similarly let $\mathcal{E}_\delta$ denote the collection of subsets of X which are finite or countable intersections of sets from $\mathcal{E}$. We also write $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$ and $\mathcal{E}_{\delta\sigma} = (\mathcal{E}_\delta)_\sigma$, etc.

Remark 4.12. Notice that if $C = \cup C_i$ and $D = \cup D_j$ with $C_i, D_j \in \mathcal{A}_\sigma$, then

$$C \cap D = \cup_{i,j} (C_i \cap D_j) \in \mathcal{A}_\sigma$$

so that $\mathcal{A}_\sigma$ is closed under finite intersections.

The following theorem shows how recover a measure μ on $\sigma(\mathcal{A})$ from its values on $\mathcal{A}$.

Theorem 4.13 (Regularity Theorem)**.** *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets, $\mathcal{M} = \sigma(\mathcal{A})$ and $\mu : \mathcal{M} \to [0, \infty]$ be a measure on $\mathcal{M}$ which is σ – finite on $\mathcal{A}$. Then for all $A \in \mathcal{M}$,*

$$\mu(A) = \inf \{\mu(B) : A \subseteq B \in \mathcal{A}_\sigma\} . \tag{4.1}$$

Moreover, if $A \in \mathcal{M}$ and $\epsilon > 0$ are given, then there exists $B \in \mathcal{A}_\sigma$ such that $A \subset B$ and $\mu(B \setminus A) \le \epsilon$.

Proof. For $A \subset X$, define

$$\mu^*(A) = \inf \{\mu(B) : A \subseteq B \in \mathcal{B}_\sigma\} .$$

We are trying to show that $\mu^* = \mu$ on $\mathcal{M}$. We will begin by first assuming that μ is a finite measure, i.e. $\mu(X) < \infty$.

Let

$$\mathcal{F} = \{B \in \mathcal{M} : \mu^*(B) = \mu(B)\} = \{B \in \mathcal{M} : \mu^*(B) \le \mu(B)\}.$$

It is clear that $\mathcal{A} \subset \mathcal{F}$, so the finite case will be finished by showing the $\mathcal{F}$ is a monotone class. Suppose that $B_n \in \mathcal{F}$ and $B_n \uparrow B$ as $n \to \infty$ and let $\epsilon > 0$ be given. Since $\mu^*(B_n) = \mu(B_n)$ there exists $A_n \in \mathcal{A}_\sigma$ such that $B_n \subset A_n$ and $\mu(A_n) \le \mu(B_n) + \epsilon 2^{-n}$ i.e.

$$\mu(A_n \setminus B_n) \le \epsilon 2^{-n}.$$

Let $A = \cup_n A_n \in \mathcal{A}_\sigma$, then $B \subset A$ and

$$\begin{aligned}\mu(A \setminus B) = \mu(\cup_n (A_n \setminus B)) &\le \sum_{n=1}^\infty \mu((A_n \setminus B)) \\ &\le \sum_{n=1}^\infty \mu((A_n \setminus B_n)) \le \sum_{n=1}^\infty \epsilon 2^{-n} = \epsilon.\end{aligned}$$

Therefore,

$$\mu^*(B) \le \mu(A) \le \mu(B) + \epsilon$$

and since $\epsilon > 0$ was arbitrary it follows that $B \in \mathcal{F}$.

Now suppose that $B_n \in \mathcal{F}$ and $B_n \downarrow B$ as $n \to \infty$ so that

$$\mu(B_n) \downarrow \mu(B) \text{ as } n \to \infty.$$

As above choose $A_n \in \mathcal{A}_\sigma$ such that $B_n \subset A_n$ and

$$0 \le \mu(A_n) - \mu(B_n) = \mu(A_n \setminus B_n) \le 2^{-n}.$$

Combining the previous two equations shows that $\lim_{n\to\infty} \mu(A_n) = \mu(B)$. Since $\mu^*(B) \le \mu(A_n)$ for all n, we conclude that $\mu^*(B) \le \mu(B)$, i.e. that $B \in \mathcal{F}$.

Since $\mathcal{F}$ is a monotone class containing the algebra $\mathcal{A}$, the monotone class theorem asserts that

$$\mathcal{M} = \sigma(\mathcal{A}) \subset \mathcal{F} \subset \mathcal{M}$$

showing the $\mathcal{F} = \mathcal{M}$ and hence that $\mu^* = \mu$ on $\mathcal{M}$.

For the σ – finite case, by replacing X_n by $\cup_{j=1}^n X_j$ if necessary, we may assume that $X_n \uparrow X$. Let μ_n be the finite measure on $\mathcal{M}$ defined by $\mu_n(A) := \mu(A \cap X_n)$ for all $A \in \mathcal{M}$. Suppose that $\epsilon > 0$ and $A \in \mathcal{M}$ are given. By what we have just proved, for all $A \in \mathcal{M}$, there exists $B_n \in \mathcal{A}_\sigma$ such that $A \subset B_n$ and

$$\mu\left((B_n \cap X_n) \setminus (A \cap X_n)\right) = \mu_n(B_n \setminus A) \le \epsilon 2^{-n}.$$

Notice that since $X_n \in \mathcal{A}_\sigma$, $B_n \cap X_n \in \mathcal{A}_\sigma$ and

$$B := \cup_{n=1}^\infty \left(B_n \cap X_n\right) \in \mathcal{A}_\sigma.$$

Moreover, $A \subset B$ and

$$\begin{aligned}
\mu(B \setminus A) &\le \sum_{n=1}^\infty \mu((B_n \cap X_n) \setminus A) \le \sum_{n=1}^\infty \mu((B_n \cap X_n) \setminus (A \cap X_n)) \\
&\le \sum_{n=1}^\infty \epsilon 2^{-n} = \epsilon.
\end{aligned}$$

Since this implies that

$$\mu(A) \le \mu(B) \le \mu(A) + \epsilon$$

and $\epsilon > 0$ is arbitrary, this equation shows that Eq. (4.1) holds. ∎

Corollary 4.14. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets, $\mathcal{M} = \sigma(\mathcal{A})$ and $\mu : \mathcal{M} \to [0, \infty]$ be a measure on $\mathcal{M}$ which is σ – finite on $\mathcal{A}$. Then for all $A \in \mathcal{M}$ and $\epsilon > 0$ there exists $B \in \mathcal{A}_\delta$ such that $B \subset A$ and*

$$\mu(A \setminus B) < \epsilon.$$

Furthermore, for any $B \in \mathcal{M}$ there exists $A \in \mathcal{A}_{\delta\sigma}$ and $C \in \mathcal{A}_{\sigma\delta}$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.

Proof. By Theorem 4.13, there exist $C \in \mathcal{A}_\sigma$ such that $A^c \subset C$ and $\mu(C \setminus A^c) \le \epsilon$. Let $B = C^c \subset A$ and notice that $B \in \mathcal{A}_\delta$ and that $C \setminus A^c = B^c \cap A = A \setminus B$, so that

$$\mu(A \setminus B) = \mu(C \setminus A^c) \le \epsilon.$$

Finally, given $B \in \mathcal{M}$, we may choose $A_n \in \mathcal{A}_\delta$ and $C_n \in \mathcal{A}_\sigma$ such that $A_n \subset B \subset C_n$ and $\mu(C_n \setminus B) \le 1/n$ and $\mu(B \setminus A_n) \le 1/n$. By replacing A_N by $\cup_{n=1}^N A_n$ and C_N

by $\cap_{n=1}^N C_n$, we may assume that $A_n \uparrow$ and $C_n \downarrow$ as n increases. Let $A = \cup A_n \in \mathcal{A}_{\delta\sigma}$ and $C = \cap C_n \in \mathcal{A}_{\sigma\delta}$, then $A \subset B \subset C$ and

$$\begin{aligned}\mu(C \setminus A) &= \mu(C \setminus B) + \mu(B \setminus A) \leq \mu(C_n \setminus B) + \mu(B \setminus A_n) \\ &\leq 2/n \to 0 \text{ as } n \to \infty.\end{aligned}$$

■

Corollary 4.15. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets, $\mathcal{M} = \sigma(\mathcal{A})$ and $\mu : \mathcal{M} \to [0,\infty]$ be a measure on $\mathcal{M}$ which is σ – finite on $\mathcal{A}$. Then for every $B \in \mathcal{M}$ such that $\mu(B) < \infty$ and $\epsilon > 0$ there exists $D \in \mathcal{A}$ such that $\mu(B \triangle D) < \epsilon$.*

Proof. By Corollary 4.14, there exists $C \in \mathcal{A}_\sigma$ such $B \subset C$ and $\mu(C \setminus B) < \epsilon$. Now write $C = \cup_{n=1}^\infty C_n$ with $C_n \in \mathcal{A}$ for each n. By replacing C_n by $\cup_{k=1}^n C_k \in \mathcal{A}$ if necessary, we may assume that $C_n \uparrow C$ as $n \to \infty$. Since $C_n \setminus B \uparrow C \setminus B$ and $B \setminus C_n \downarrow B \setminus C = \emptyset$ as $n \to \infty$ and $\mu(B \setminus C_1) \leq \mu(B) < \infty$, we know that

$$\lim_{n\to\infty} \mu(C_n \setminus B) = \mu(C \setminus B) < \epsilon \text{ and } \lim_{n\to\infty} \mu(B \setminus C_n) = \mu(B \setminus C) = 0$$

Hence for n sufficiently large,

$$\mu(B \triangle C_n) = (\mu(C_n \setminus B) + \mu(B \setminus C_n) < \epsilon.$$

Hence we are done by taking $D = C_n \in \mathcal{A}$ for an n sufficiently large. ■

For Exercises 4.17 – 4.20 let $\tau \subseteq \mathcal{P}(X)$ be a topology, $\mathcal{M} = \sigma(\tau)$ and $\mu : \mathcal{M} \to [0,\infty)$ be a finite measure ($\mu(X) < \infty$).

Remark 4.16. We have to assume that $\mu(B) < \infty$ as the following example shows. Let $X = \mathbb{R}$, $\mathcal{M} = \mathcal{B}$, $\mu = m$, $\mathcal{A}$ be the algebra generated by half open intervals of the form $(a,b]$, and $B = \cup_{n=1}^\infty (2n, 2n+1]$. It is easily checked that for every $D \in \mathcal{A}$, that $m(B \Delta D) = \infty$.

Exercise 4.17. Let

$$\mathcal{F} := \{A \in \mathcal{M} : \mu(A) = \inf\{\mu(V) : A \subseteq V \in \tau\}\}. \tag{4.2}$$

Show $\mathcal{F}$ is a monotone class.

Exercise 4.18. Give an example of a topology τ on $X = \{1,2,3\}$ and a measure μ on $\mathcal{M} = \sigma(\tau)$ such that $\mathcal{F}$ defined in Eq. (4.2) is **not** $\mathcal{M}$.

Exercise 4.19. Let τ be a topology on a set X and $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by τ. Show $\mathcal{A}$ is the collection of subsets of X which may be written as finite union of sets of the form $F \cap V$ where F is closed and V is open.

Exercise 4.20. Suppose now $\tau \subseteq \mathcal{P}(X)$ is a topology with the property that to every closed set $C \subset X$, there exists $V_n \in \tau$ such that $V_n \downarrow C$ as $n \to \infty$. Let $\mathcal{A} = \mathcal{A}(\tau)$ be the algebra generated by τ. With the aid of Exercise 4.19, show that $\mathcal{A} \subset \mathcal{F}$ and use this along with exercise 4.17 to conclude that

$$\mu(A) = \inf\{\mu(V) : A \subseteq V \in \tau\}$$

for all $A \in \mathcal{M}$.

The next exercise is the generalization of Exercise 4.20 to case where μ is σ – finite.

Exercise 4.21 (Generalization to the σ – finite case)**.** Let $\tau \subseteq \mathcal{P}(X)$ be a topology with the property that to every closed set $C \subset X$, there exists $V_n \in \tau$ such that $V_n \downarrow C$ as $n \to \infty$. Also let $\mathcal{M} = \sigma(\tau)$ and $\mu : \mathcal{M} \to [0, \infty)$ be a measure which is σ – finite on τ.

(1) Show that for all $\epsilon > 0$ and $A \in \mathcal{M}$ there exists an open set $V \in \tau$ and a closed set F such that $F \subset A \subset V$ and $\mu(V \setminus F) \leq \epsilon$.
(2) Let F_σ denote the collection of subsets of X which may be written as a countable union of closed sets. Use item 1. to show for all $B \in \mathcal{M}$, there exists $C \in \tau_\delta$ (τ_δ is customarily written as G_δ) and $A \in F_\sigma$ such that $A \subset B \subset C$ and $\mu(C \setminus A) = 0$.

Exercise 4.22 (Metric Space Examples)**.** Suppose that (X, d) is a metric space. A set $V \subset X$ is said to be open if for all $x \in V$ there exists an $\epsilon = \epsilon(x) > 0$ such that the ball

$$B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \tag{4.3}$$

is contained in V. Let τ_d denote the collection of open sets. Given a set $F \subset X$ and $\epsilon > 0$ let F_ϵ be the open set

$$F_\epsilon = \cup_{x \in F} B(x, \epsilon).$$

Show that if F is closed, then $F_\epsilon \downarrow F$ as $\epsilon \downarrow 0$. Therefore the results of Exercises 4.20 and 4.21 apply to measures on metric spaces with the Borel σ – algebra, $\mathcal{B} = \sigma(\tau_d)$.

Corollary 4.23. *Let $\mathcal{B}$ be the Borel σ – algebra on $\mathbb{R}^n$ equipped with the standard topology induced by open balls with respect to the Euclidean distance. Suppose that $\mu : \mathcal{B} \to [0, \infty]$ is a measure such that $\mu(A) < \infty$ whenever A is a bounded set.*

(1) *Then for all $A \in \mathcal{B}$ and $\epsilon > 0$ there exist a closed set F and an open set V such that $F \subset A \subset V$ and $\mu(V \setminus F) < \epsilon$.*
(2) *If $\mu(A) < \infty$, the set F in item 1. may be chosen to be compact.*
(3) *For all $A \in \mathcal{B}$ we may compute $\mu(A)$ using*

$$\mu(A) = \inf\{\mu(V) : A \subset V \text{ and } V \text{ is open}\} \tag{4.4}$$

$$= \sup\{\mu(K) : K \subset A \text{ and } K \text{ is compact}\}. \tag{4.5}$$

Proof. Item 1. follows from Exercises 4.21 and 4.22. If $\mu(A) < \infty$ and $F \subset A \subset V$ as in item 1. Let

$$K_n := \{x \in F : |x| \leq n\}. \tag{4.6}$$

Then K_n is closed and bounded in $\mathbb{R}^n$ and hence compact and $K_n \uparrow F$ as $n \to \infty$. Since $\mu(A) < \infty$ and $\mu(V \setminus A) < \epsilon$ we know that $\mu(V) < \infty$. Using this fact and the fact that $V \setminus K_n \downarrow V \setminus F$, we conclude that $\mu(V \setminus K_n) \downarrow \mu(V \setminus F) < \epsilon$ as $n \to \infty$. Thus for sufficiently large n we have $K = K_n$ is a compact set such that $K \subset A \subset V$ and $\mu(V \setminus K) < \epsilon$.

Item 1. easily implies that Eq. (4.4) holds and item 2. implies Eq. (4.5) holds when $\mu(A) < \infty$. So we need only check Eq. (4.5) when $\mu(A) = \infty$. By Item 1. there is a closed set $F \subset A$ such that $\mu(A \setminus F) < 1$ and in particular $\mu(F) = \infty$. Letting $K_n \subset F \subset A$ be the compact set as in Eq. (4.6), we have $\mu(K_n) \uparrow \mu(F) = \infty = \mu(A)$ which shows that Eq. (4.5) also holds when $\mu(A) = \infty$. ■

5. Constructing examples of measures

Most σ − algebras and σ -additive measures are somewhat difficult to describe and define. However, one special case is fairly easy to understand. Namely suppose that $\mathcal{F} \subset \mathcal{P}(X)$ is a countable or finite partition of X and $\mathcal{M} \subset \mathcal{P}(X)$ is the σ − algebra which consists of the collection of set $A \subset X$ such that

$$A = \bigcup_{\alpha\in\mathcal{F}\ni\alpha\subset A} \alpha. \tag{5.1}$$

It is easily seen that $\mathcal{M}$ is a σ − algebra.

Any measure $\mu : \mathcal{M} \to [0, \infty]$ is determined uniquely by its values on $\mathcal{F}$. Conversely, if we are given any function $\lambda : \mathcal{F} \to [0, \infty]$ we may define, for $A \in \mathcal{M}$,

$$\mu(A) = \sum_{\alpha\in\mathcal{F}\ni\alpha\subset A} \lambda(\alpha) = \sum_{\alpha\in\mathcal{F}} \lambda(\alpha)1_{\alpha\subset A}$$

where $1_{\alpha\subset A}$ is one if $\alpha \subset A$ and zero otherwise. We may check that μ is a measure on $\mathcal{M}$. Indeed, if $A = \coprod_{i=1}^{\infty} A_i$ and $\alpha \in \mathcal{F}$, then $\alpha \subset A$ iff $\alpha \subset A_i$ for one and hence exactly one A_i. Therefore,

$$1_{\alpha\subset A} = \sum_{i=1}^{\infty} 1_{\alpha\subset A_i}$$

and hence

$$\begin{aligned}\mu(A) &= \sum_{\alpha\in\mathcal{F}} \lambda(\alpha)1_{\alpha\subset A} = \sum_{\alpha\in\mathcal{F}} \lambda(\alpha)\sum_{i=1}^{\infty} 1_{\alpha\subset A_i} = \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{F}} \lambda(\alpha)1_{\alpha\subset A_i} \\ &= \sum_{i=1}^{\infty} \mu(A_i)\end{aligned}$$

as desired. Thus we have shown that there is a one to one correspondence between measures μ on $\mathcal{M}$ and functions $\lambda : \mathcal{F} \to [0, \infty]$.

The construction of measures on more general σ − algebras will be motivated by the regularity results in Section 4.1 above.

5.1. Constructing Measures from Premeasures.

Definition 5.1. An additive function μ_0 on an algebra $\mathcal{A}$ of subsets of X is called a premeasure when μ_0 is also countably additive on $\mathcal{A}$. This means that if $A \in \mathcal{A}$ and $A_i \in \mathcal{A}$ such that $A = \coprod_{i=1}^{\infty} A_i$, then

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

The premeasure is said to σ − **finite** if there exists $X_n \in \mathcal{A}$ such that $X_n \uparrow X$ as $n \uparrow \infty$ and $\mu_0(X_n) < \infty$.

Definition 5.2. e that $\mathcal{E}$ is a collection of subsets of X, let $\mathcal{E}_\sigma$ denote the collection of subsets of X which are finite or countable unions of sets from $\mathcal{E}$. Similarly let $\mathcal{E}_\delta$ denote the collection of subsets of X which are finite or countable intersections of sets from $\mathcal{E}$. We also write $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$ and $\mathcal{E}_{\delta\sigma} = (\mathcal{E}_\delta)_\sigma$, etc.

Proposition 5.3. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$. For $A \in \mathcal{A}_\sigma$ such that $A = \coprod_{n=1}^{\infty} A_n$, with $A_n \in \mathcal{A}$, define*

$$\mu_0(A) \equiv \sum_{n=1}^{\infty} \mu_0(A_n).$$

Then $\mu_0 : \mathcal{A}_\sigma \to [0,\infty]$ is well defined.

Proof. Suppose that A may also be written as $A = \coprod_{n=1}^{\infty} B_n$, then

$$A_n = \coprod_{k=1}^{\infty} (A_n \cap B_k) \Rightarrow \mu_0(A_n) = \sum_{k=1}^{\infty} \mu_0(A_n \cap B_k).$$

Similarly one shows that $\sum_{n=1}^{\infty} \mu_0(A_n \cap B_k) = \mu_0(B_k)$ and thus hence

$$\begin{aligned}\sum \mu_0(A_n) &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu_0(A_n \cap B_k) \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mu_0(A_n \cap B_k) = \sum_{k=1}^{\infty} \mu_0(B_k)\end{aligned}$$

■

The key to constructing measures is the following theorem which will be proved in Section 6.

Theorem 5.4 (Key Construction Theorem)**.** *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mathcal{M} = \sigma(\mathcal{A})$ and μ_0 be a σ – finite premeasure on $\mathcal{A}$ which we extend to $\mathcal{A}_\sigma$ using the previous Proposition 5.3. Then there exists a unique measure μ on $\mathcal{M}$ such that $\mu|_{\mathcal{A}} = \mu_0$. Moreover, μ is given by the formula we have*

$$\begin{aligned}\mu(A) &= \inf\{\mu_0(B) : A \subset B \in \mathcal{A}_\sigma\} \\ &= \inf\{\mu_0(\cup_{n=1}^{\infty} A_n) : A \subset \cup_{n=1}^{\infty} A_n \text{ with } A_n \in \mathcal{A}\}.\end{aligned} \tag{5.2}$$

By replacing A_n by $A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$ if necessary, Eq. (5.2) is equivalent to writing

$$\mu(A) = \inf\{\sum_{n=1}^{\infty} \mu_0(A_n) : A \subset \coprod_{n=1}^{\infty} A_n \text{ with } A_n \in \mathcal{A}\}. \tag{5.3}$$

This theorem is a consequence of Theorem 6.7 of Section 6 below.

Remark 5.5. we drop the σ – finite assumption on μ_0 we may loose uniqueness in the above theorem. We will give some examples of this phenomena in the subsection 5.4

In order to use this theorem it is necessary to first construct premeasures on algebras. The main result that we are heading for is contained in Theorem 5.7 below.

Definition 5.6. Suppose that $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family.

(1) A function $\mu : \mathcal{E} \to [0,\infty]$ is **additive** if $\mu(E) = \sum_{i=1}^{n} \mu(E_i)$ whenever $E = \coprod_{i=1}^{n} E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$.

(2) A function $\mu : \mathcal{E} \to [0,\infty]$ is sub**additive** if $\mu(E) \le \sum_{i=1}^\infty \mu(E_i)$ whenever $E = \coprod_{i=1}^\infty E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$.
(3) A function $\mu : \mathcal{E} \to [0,\infty]$ is $\sigma -$ **finite** on $\mathcal{E}$ if there exist $E_n \in \mathcal{E}$ such that $X = \cup_n E_n$ and $\mu(E_n) < \infty$,

Theorem 5.7. *Suppose that $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family and $\mu^0 : \mathcal{E} \to [0,\infty]$ is a function.*

(1) *If μ^0 is additive on $\mathcal{E}$, then μ^0 has a unique extension to a finitely additive measure μ_0 on $\mathcal{A} = \mathcal{A}(\mathcal{E})$.*
(2) *If we further assume that μ^0 is countably subadditive on $\mathcal{E}$, then μ_0 is a preameasure on $\mathcal{A}$.*
(3) *If we further assume that μ^0 is σ – finite on $\mathcal{E}$, then there exists a* ***unique*** *measure μ on $\sigma(\mathcal{E})$ such that $\mu|_{\mathcal{E}} = \mu^0$. Moreover, for $A \in \sigma(\mathcal{E})$,*

$$\begin{aligned} \mu(A) &= \inf\{\mu_0(B) : A \subset B \in \mathcal{A}_\sigma\} \\ &= \inf\{\sum_{n=1}^\infty \mu^0(E_n) : A \subset \coprod_{n=1}^\infty E_n \text{ with } E_n \in \mathcal{E}\}. \end{aligned}$$

This theorem is proved in the next subsection. Item 1. is proved in Proposition 5.8, Item 2. in Proposition 5.10 and Item 3. is a consequence of Items 1. and 2. and Theorem 5.4 above.

5.2. Proof of Theorem 5.7.

Proposition 5.8. *Suppose $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family and $\mathcal{A} = \mathcal{A}(\mathcal{E})$ is the algebra generated by $\mathcal{E}$. Then every additive function $\mu : \mathcal{E} \to [0,\infty]$ extends uniquely to an additive measure (which we still denote by μ) on $\mathcal{A}$.*

Proof. Since by Proposition 3.10, every element $A \in \mathcal{A}$ is of the form $A = \coprod_i E_i$ with $E_i \in \mathcal{E}$, it is clear that if μ extends to a measure the extension is unique and must be given by

$$\mu(A) = \sum_i \mu(E_i). \tag{5.4}$$

To prove the existence of the extension, the main point is to show that defining $\mu(A)$ by Eq. (5.4) is well defined. That is to say if we also have $A = \coprod_j F_j$ with $F_j \in \mathcal{E}$, we must show that

$$\sum_i \mu(E_i) = \sum_j \mu(F_j). \tag{5.5}$$

To prove this, we make use of the fact that

$$E_i = \cup_j (E_i \cap F_j)$$

and the property that μ is additive on $\mathcal{E}$ to conclude that

$$\mu(E_i) = \sum_j \mu(E_i \cap F_j) \text{ and hence}$$

$$\sum_i \mu(E_i) = \sum_i \sum_j \mu(E_i \cap F_j) = \sum_{i,j} \mu(E_i \cap F_j).$$

Similarly, we show that

$$\sum_j \mu(F_j) = \sum_{i,j} \mu(E_i \cap F_j)$$

which combined with the previous equation shows that Eq. (5.5) holds. It is now easy to verify that μ extended to $\mathcal{A}$ as in Eq. (5.4) is an additive measure on $\mathcal{A}$. ■

Proposition 5.9. *Suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra of sets and $\mu : \mathcal{A} \to [0, \infty]$ is a **finitely** additive measure on $\mathcal{A}$. Then if $A \in \mathcal{A}$ and $A = \coprod_{i=1}^{\infty} A_i$ with each $A_i \in \mathcal{A}$, we have*

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A). \tag{5.6}$$

Proof. Since

$$A = \left(\coprod_{i=1}^{N} A_i \right) \cup \left(A \setminus \bigcup_{i=1}^{N} A_i \right)$$

we find using the finite additivity of μ that

$$\mu(A) = \sum_{i=1}^{N} \mu(A_i) + \mu\left(A \setminus \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \mu(A_i).$$

Letting $N \to \infty$ in this last expression shows that $\sum_{i=1}^{\infty} \mu(A_i) \leq \mu(A)$. ■

So in order to prove that μ is a premeasure on $\mathcal{A}$, it suffices to show

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i) \tag{5.7}$$

whenever $A = \coprod_{i=1}^{\infty} A_i$ with $A \in \mathcal{A}$ and each $A_i \in \mathcal{A}$.

Proposition 5.10. *Suppose that $\mathcal{E} \subset \mathcal{P}(X)$ is an elementary family, $\mathcal{A} = \mathcal{A}(\mathcal{E})$ and $\mu : \mathcal{A} \to [0, \infty]$ is an additive measure. Then the following are equivalent:*

1. *μ is a premeasure on $\mathcal{A}$.*
2. *μ has the subadditivity property: whenever $E \in \mathcal{E}$ is of the form $E = \coprod_{i=1}^{\infty} E_i \in \mathcal{E}$ with $E_i \in \mathcal{E}$ then*

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i). \tag{5.8}$$

Proof. It is clear that 1. implies 2. since if μ is a premeasure, then $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. For the converse, it suffices to show by Proposition 5.9 that if $A = \coprod_{n=1}^{\infty} A_n$ with $A \in \mathcal{A}$ and each $A_n \in \mathcal{A}$ then Eq. (5.7) holds. To prove this, write $A = \coprod_{j=1}^{n} E_j$ with $E_j \in \mathcal{E}$ and $A_n = \coprod_{i=1}^{N_n} E_{n,i}$ with $E_{n,i} \in \mathcal{E}$. Then

$$E_j = A \cap E_j = \coprod_{n=1}^{\infty} A_n \cap E_j = \coprod_{n=1}^{\infty} \coprod_{i=1}^{N_n} E_{n,i} \cap E_j$$

which is a countable union and hence by assumption,

$$\mu(E_j) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \mu\left(E_{n,i} \cap E_j\right).$$

Summing this equation on j and using the additivity of μ shows that

$$\begin{aligned}\mu(A) &= \sum_{j=1}^{n}\mu(E_j) \le \sum_{j=1}^{n}\sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\mu\left(E_{n,i}\cap E_j\right) = \sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\sum_{j=1}^{n}\mu\left(E_{n,i}\cap E_j\right)\\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{N_n}\mu\left(E_{n,i}\right) = \sum_{n=1}^{\infty}\mu\left(A_n\right)\end{aligned}$$

as desired. ■

5.3. Examples of finitely additive measures.

Example 5.11. Let $X = \mathbb{R}$ and $\mathcal{E}$ be the elementary class

$$\mathcal{E} = \{(a,b]\cap\mathbb{R} : -\infty \le a \le b \le \infty\},$$

and $\mathcal{A} = \mathcal{A}(\mathcal{E})$ be the algebra of disjoint union of elements from $\mathcal{E}$. Suppose that $\mu : \mathcal{A} \to [0,\infty]$ is an additive measure such that $\mu((a,b]) < \infty$ for all $a < b$. Then there is a unique increasing function $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F(0) = 0$, $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$ and

$$\mu((a,b]\cap\mathbb{R}) = F(b) - F(a)\ \forall\ a \le b \text{ in } \bar{\mathbb{R}}. \tag{5.9}$$

Conversely, given an increasing function $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$ there is a unique measure μ on $\mathcal{A}$ such that the relation in Eq. (5.9) holds. (So the finitely additive measures μ on $\mathcal{A}$ which are finite on bounded sets are in one to one correspondence with increasing functions $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ such that $F(0) = 0$, $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$.)

Proof. If F is going to exist, we must have

$$\begin{aligned}\mu((0,b]\cap\mathbb{R}) &= F(b) - F(0) = F(b) \text{ if } b \in [0,\infty],\\ \mu((a,0]) &= F(0) - F(a) = -F(a) \text{ if } a \in [-\infty,0]\end{aligned}$$

from which we learn

$$F(x) = \begin{cases} -\mu((x,0]) & \text{if} \quad x \le 0 \\ \mu((0,x]\cap\mathbb{R}) & \text{if} \quad x \ge 0. \end{cases}$$

Moreover, one easily checks using the additivity of μ that Eq. (5.9) holds for this F.

Conversely, suppose $F : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is an increasing function such that $F^{-1}(\{-\infty\}) \subset \{-\infty\}$, $F^{-1}(\{\infty\}) \subset \{\infty\}$. Define μ on $\mathcal{E}$ using the formula in Eq. (5.9). I claim that μ is additive on $\mathcal{E}$ and hence has a unique extension to $\mathcal{A}$ which will finish the argument. Suppose that

$$(a,b] = \coprod_{i=1}^{n}(a_i,b_i].$$

By reordering $(a_i,b_i]$ if necessary, we may assume that

$$a = a_1 > b_1 = a_2 < b_2 = a_3 < \cdots < a_n < b_n = b.$$

Therefore,

$$\mu((a,b]) = F(b) - F(a) = \sum_{i=1}^{n}\left[F(b_i) - F(a_i)\right] = \sum_{i=1}^{n}\mu((a_i,b_i])$$

as desired. ■

Theorem 5.12. *Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be algebras. Suppose that*

$$\mu : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$$

is a function such that for each $A \in \mathcal{A}$, the function

$$B \in \mathcal{B} \to \mu(A \times B) \in \mathbb{C}$$

is an additive measure on $\mathcal{B}$ and for each $B \in \mathcal{B}$, the function

$$A \in \mathcal{A} \to \mu(A \times B) \in \mathbb{C}$$

is an additive measure on $\mathcal{A}$. Then μ extends uniquely to a measure on the product algebra $\mathcal{C}$ generated by $\mathcal{A} \times \mathcal{B}$.

Proof. The collection

$$\mathcal{E} = \mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

is an elementary family (you check). Therefore, it suffices to shows that μ is additive on $\mathcal{E}$. To check this suppose that $A \times B \in \mathcal{E}$ and

$$A \times B = \coprod_k (A_k \times B_k)$$

with $A_k \times B_k \in \mathcal{E}$. We wish to shows

$$\mu(A \times B) = \sum_k \mu(A_k \times B_k).$$

For this consider the finite algebras $\mathcal{A}' \subset \mathcal{P}(A)$ and $\mathcal{B}' \subset \mathcal{P}(B)$ generated by $\{A_k\}$ and $\{B_k\}$ respectively. Let $\mathcal{F} \subset \mathcal{A}'$ and $\mathcal{G} \subset \mathcal{B}'$ be partition of A and B respectively as found Proposition 3.17. Then for each k we may write

$$A_k = \coprod_{\alpha \in \mathcal{F}, \alpha \subset A_k} \alpha \text{ and } B_k = \coprod_{\beta \in \mathcal{G}, \beta \subset B_k} \beta.$$

Therefore,

$$\begin{aligned}
\mu(A_k \times B_k) &= \mu(A_k \times \bigcup_{\beta \subset B_k} \beta) = \sum_{\beta \subset B_k} \mu(A_k \times \beta) \\
&= \sum_{\beta \subset B_k} \mu(\left(\bigcup_{\alpha \subset A_k} \alpha\right) \times \beta) = \sum_{\alpha \subset A_k, \beta \subset B_k} \mu(\alpha \times \beta)
\end{aligned}$$

so that

$$\begin{aligned}
\sum_k \mu(A_k \times B_k) &= \sum_k \sum_{\alpha \subset A_k, \beta \subset B_k} \mu(\alpha \times \beta) = \sum_{\alpha \subset A, \beta \subset B} \mu(\alpha \times \beta) \\
&= \sum_{\beta \subset B} \mu(A \times \beta) = \mu(A \times B)
\end{aligned}$$

as desired. ■

5.4. **"Radon" measures on** $(\mathbb{R}, \mathcal{B})$**.** Let us continue with Example 5.11. So $X = \mathbb{R}$ and $\mathcal{E}$ be the elementary class

$$\mathcal{E} = \{(a,b] \cap \mathbb{R} : -\infty \le a \le b \le \infty\},$$

and $\mathcal{A} = \mathcal{A}(\mathcal{E})$ be the algebra of disjoint union of elements from $\mathcal{E}$. Also let $\mathcal{B} = \sigma(\mathcal{E})$ be the Borel σ -algebra and suppose $\mu : \mathcal{B} \to [0,\infty]$ is a $\sigma -$ additive measure such that $\mu((a,b]) < \infty$ for all $-\infty < a < b < \infty$. Then as in Example 5.11, let F be the increasing function defined for $x \in \bar{\mathbb{R}}$ by

$$F(x) = \begin{cases} -\mu((x,0]) & \text{if} \quad x \le 0 \\ \mu((0,x] \cap \mathbb{R}) & \text{if} \quad x \ge 0 \end{cases} \tag{5.10}$$

Since for $0 \le a < \infty$ and $-\infty < b < 0$,

$$\begin{aligned} \lim_{x \downarrow a} F(x) &= \lim_{x \downarrow a} \mu((0,x]) = \mu((0,a]) = F(a) \text{ and} \\ \lim_{x \downarrow b} F(x) &= -\lim_{x \downarrow b} \mu((x,0]) = \mu((b,0]) = F(b) \end{aligned}$$

(where this is checked using sequences) we see that F is necessarily right continuous on $\mathbb{R}$. Moreover, since

$$\begin{aligned} \lim_{x \to \infty} F(x) &= \lim_{x \to \infty} \mu((0,x]) = \mu((0,\infty)) = F(\infty) \text{ and} \\ \lim_{x \to -\infty} F(x) &= -\lim_{x \to -\infty} \mu((x,0]) = -\mu((-\infty,0]) = F(-\infty) \end{aligned}$$

we learn that F is continuous at $\pm\infty$ as well. Conversely we have the following Theorem.

Notation 5.13. If $F : \mathbb{R} \to \mathbb{R}$ is an increasing function, let $F(\pm\infty) := \lim_{x \to \pm\infty} F(x)$.

Theorem 5.14. *To every right continuous non-decreasing function* $F : \mathbb{R} \to \mathbb{R}$ *there exists a unique measure* μ_F *on* $\mathcal{B}$ *such that*

$$\mu_F((a,b] \cap \mathbb{R}) = F(b) - F(a) \ \forall \ -\infty \le a \le b \le \infty \tag{5.11}$$

Moreover, if $A \in \mathcal{B}$ *then*

$$\mu_F(A) = \inf\left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) : A \subseteq \cup_{i=1}^{\infty} (a_i, b_i] \right\} \tag{5.12}$$

$$= \inf\left\{ \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) : A \subseteq \coprod_{i=1}^{\infty} (a_i, b_i] \right\}. \tag{5.13}$$

In fact the relations (5.10) and (5.11) give a one to one correspondence between right continuous functions F *with* $F(0) = 0$ *on one hand and radon measure* μ *on* $\mathcal{B}$ *on the other.*

Proof. Extend F to a function from $\bar{\mathbb{R}} \to \bar{\mathbb{R}}$ by defining $F(\pm\infty) := \lim_{x \to \pm\infty} F(x)$. As above let

$$\mathcal{E} = \{(a,b] \cap \mathbb{R} : -\infty \le a \le b \le \infty\}$$

and use Example 5.11 to show there is a unique additive measure μ_0 on $\mathcal{A}(\mathcal{E})$ such that $\mu_0((a,b]) = F(b) - F(a)$ for all $a, b \in \bar{\mathbb{R}}$ with $a \le b$. The proof will be finished by Theorem 5.7 if we can show that μ_0 is sub-additive on $\mathcal{E}$.

First suppose that $-\infty < a < b < \infty$, $J = (a, b]$, $J_n = (a_n, b_n]$ such that $J = \coprod_{n=1}^{\infty} J_n$. We wish to shows that

$$\mu_0(J) \leq \sum_{i=1}^{\infty} \mu_0(J_i). \tag{5.14}$$

To do this choose numbers $\tilde{a} > a$, $\tilde{b}_n > b_n$ and set

$$\begin{aligned} I &= (\tilde{a}, b] \subset J, \\ \tilde{J}_n &= (a_n, \tilde{b}_n] \supset J_n \text{ and} \\ \tilde{J}_n^0 &= (a_n, \tilde{b}_n). \end{aligned}$$

Since $\bar{I}$ is compact and $\bar{I} \subseteq J \subset \bigcup_{n=1}^{\infty} \tilde{J}_n^0$ there exists $N < \infty$ such that

$$I \subset \bar{I} \subseteq \bigcup_{n=1}^{N} \tilde{J}_n^0 \subseteq \bigcup_{n=1}^{N} \tilde{J}_n.$$

Hence finite sub-additivity of μ_0,

$$F(b) - F(\tilde{a}) = \mu_0(I) \leq \sum_{n=1}^{N} \mu_0\left(\tilde{J}_n\right) \leq \sum_{n=1}^{\infty} \mu_0\left(\tilde{J}_n\right).$$

Using the right continuity of F and letting $\tilde{a} \downarrow a$ in the above inequality shows that

$$\begin{aligned} \mu_0((a, b]) = F(b) - F(a) &\leq \sum_{n=1}^{\infty} \mu_0\left(\tilde{J}_n\right) \\ &= \sum_{n=1}^{\infty} \mu_0\left(J_n\right) + \sum_{n=1}^{\infty} \mu_0(\tilde{J}_n \setminus J_n) \end{aligned} \tag{5.15}$$

Given $\epsilon > 0$ we may use the right continuity of F to choose $\tilde{b}_n$ so that

$$\mu_0(\tilde{J}_n \setminus J_n) = F(\tilde{b}_n) - F(b_n) \leq \epsilon 2^{-n} \ \forall n.$$

Using this in Eq. (5.15) shows that

$$\mu_0(J) = \mu_0((a, b]) \leq \sum_{n=1}^{\infty} \mu_0\left(J_n\right) + \epsilon$$

and since $\epsilon > 0$ we have verified Eq. (5.14).

We have now done the hard work. We still have to check the cases where $a = -\infty$ or $b = \infty$ or both. For example, suppose that $b = \infty$ so that

$$J = (a, \infty) = \coprod_{n=1}^{\infty} J_n$$

with $J_n = (a_n, b_n] \cap \mathbb{R}$. Then let $I_M := (a, M]$, and notice that

$$I_M = J \cap I_M = \coprod_{n=1}^{\infty} J_n \cap I_M$$

So by what we have already proved,

$$F(M)-F(a)=\mu_0(I_M)\leq\sum_{n=1}^\infty\mu_0(J_n\cap I_M)\leq\sum_{n=1}^\infty\mu_0(J_n)$$

Now let $M\to\infty$ in this last inequality to find that

$$\mu_0((a,\infty))=F(\infty)-F(a)\leq\sum_{n=1}^\infty\mu_0(J_n).$$

The other cases where $a=-\infty$ and $b\in\mathbb{R}$ and $a=-\infty$ and $b=\infty$ are handled similarly. ■

Remark 5.15. It is definitely not true that all measures μ on $(\mathbb{R},\mathcal{B})$ may be described uniquely by their restriction to $\mathcal{A}$. For example, let $D\subset\mathbb{R}$ be a countable dense set and define $\mu_D(A):=\#(D\cap A)$. Then $\mu_D(A)=\infty$ for all $A\in\mathcal{A}$ which are not empty. On the other hand it is clear that the measures μ_D with D a countable dense set in $\mathbb{R}$ are different for different D 's. Also notice that μ_D is σ – finite on $\mathcal{B}$ but **not** on $\mathcal{A}$.

Example 5.16. The most important special case of Theorem 5.14 is when $F(x)=x$, in which case we write m for μ_F. The measure m is called Lebesgue measure.

Theorem 5.17. *Lebesgue measure m is invariant under translations, i.e. for $A\in\mathcal{B}$ and $x\in\mathbb{R}$,*

$$m(x+B)=m(B). \tag{5.16}$$

Moreover, m is the unique measure on $\mathcal{B}$ such that $m((0,1])=1$ and Eq. (5.16) holds for $A\in\mathcal{B}$ and $x\in\mathbb{R}$. Moreover, m has the scaling property

$$m(\lambda B)=|\lambda|\,m(B) \tag{5.17}$$

where $\lambda\in\mathbb{R}$, $B\in\mathcal{B}$ and $\lambda B:=\{\lambda x:x\in B\}$.

Proof. Let $m_x(B):=m(x+B)$, then one easily shows that m_x is a measure on $\mathcal{B}$ such that $m_x((a,b])=b-a$ for all $a<b$. Therefore, $m_x=m$ by the uniqueness assertion in Theorem 5.14.

For the converse, suppose that m is translation invariant and $m((0,1])=1$. Given $n\in\mathbb{N}$, we have

$$(0,1]=\cup_{k=1}^n(\frac{k-1}{n},\frac{k}{n}]=\cup_{k=1}^n\left(\frac{k-1}{n}+(0,\frac{1}{n}]\right).$$

Therefore,

$$\begin{aligned}1&=m((0,1])=\sum_{k=1}^n m\left(\frac{k-1}{n}+(0,\frac{1}{n}]\right)\\&=\sum_{k=1}^n m((0,\frac{1}{n}])=n\cdot m((0,\frac{1}{n}]).\end{aligned}$$

That is to say

$$m((0,\frac{1}{n}])=1/n.$$

Similarly we show that $m((0,\frac{l}{n}])=l/n$ for all $l,n\in\mathbb{N}$. Using the translation invariance of m, we then learn that

$$m((a,b])=b-a$$

for all $a, b \in \mathbb{Q}$ such that $a < b$. Finally for $a, b \in \mathbb{R}$ such that $a < b$, choose $a_n, b_n \in \mathbb{Q}$ such that $b_n \downarrow b$ and $a_n \uparrow a$, then $(a_n, b_n] \downarrow (a, b]$ and thus

$$m((a,b]) = \lim_{n\to\infty} m((a_n, b_n]) = \lim_{n\to\infty} (b_n - a_n) = b - a,$$

i.e. m is Lebesgue measure.

To prove Eq. (5.17) we may assume that $\lambda \neq 0$ since this case is trivial to prove. Now let $m_\lambda(B) := |\lambda|^{-1} m(\lambda B)$. It is easily checked that m_λ is again a measure on $\mathcal{B}$ which satisfies

$$m_\lambda((a,b]) = \lambda^{-1} m((\lambda a, \lambda b]) = \lambda^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda > 0$ and

$$m_\lambda((a,b]) = |\lambda|^{-1} m([\lambda b, \lambda a)) = -|\lambda|^{-1}(\lambda b - \lambda a) = b - a$$

if $\lambda < 0$. Hence $m_\lambda = m$. ■

6. Appendix: Construction of Measures

6.1. **Outer Measures.**

Definition 6.1. A function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an **outer measure** if

(1) $\mu^*(\emptyset) = 0$
(2) $\mu^*(\cup A_i) \leq \sum \mu^*(A_i)$
(3) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.

Proposition 6.2 (Example of an outer measure.)**.** *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be arbitrary collection of subsets of X which contains both $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ be a function such that $\rho(\emptyset) = 0$. For any $A \subseteq X$, define*

$$\mu^*(A) = \inf\{\sum_i \rho(E_i) : A \subseteq \cup E_i, E_i \in \mathcal{E}\}. \tag{6.1}$$

Then μ^* is an outer measure.

Proof. It is clear that $\mu^*(\emptyset) = 0$ and $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B$. Suppose that $A_i \in \mathcal{P}(X)$ and $\mu^*(A_i) < \infty$ for all i, otherwise there will be nothing to prove. Let $\epsilon > 0$ and choose $E_{ij} \in \mathcal{E}$ such that $A_i \subseteq \bigcup_{j=1}^{\infty} E_{ij}$ and

$$\mu^*(A_i) \geq \sum_{j=1}^{\infty} \rho(E_{ij}) - 2^{-i}\epsilon.$$

Then

$$\begin{aligned} \mu^*(\cup A_i) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho(E_{ij}) \\ &\leq \sum_{i=1}^{\infty} (\mu^*(A_i) + 2^{-i}\epsilon) = \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have shown property 2. in Definition 6.1. ■

Definition 6.3. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Define the μ^*-**measurable sets** to be

$$\mathcal{M}(\mu^*) \equiv \{A \subseteq X : \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \,\forall\, E \subseteq X\}.$$

Because of the subadditivity of μ^*, we may equivalently define $\mathcal{M}(\mu^*)$ by

$$\mathcal{M}(\mu^*) = \{A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \, \forall \, E \subseteq X\}.$$

The next Proposition helps to motivate this definition and the Carathéodory's construction Theorem 6.5.

Proposition 6.4. *Suppose $\mathcal{E} = \mathcal{M}$ is a σ – algebra, $\rho = \mu : \mathcal{M} \to [0, \infty]$ is a measure and μ^* is defined as in Eq. (6.1). Then*

1. *For $A \subseteq X$*
$$\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{M} \text{ and } A \subseteq B\}.$$
In particular, $\mu^* = \mu$ on $\mathcal{M}$.
2. *The σ – algebra $\mathcal{M} \subset \mathcal{M}(\mu^*)$, i.e. if $A \in \mathcal{M}$ and $E \subset X$ then*
$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^c). \tag{6.2}$$
3. *Assume further that μ is σ – finite on $\mathcal{M}$, then $\mathcal{M}(\mu^*) = \bar{\mathcal{M}} = \bar{\mathcal{M}}^\mu$ and $\mu^*|_{\mathcal{M}(\mu^*)} = \bar{\mu}$ where $(\bar{\mathcal{M}} = \bar{\mathcal{M}}^\mu, \bar{\mu})$ is the completion of $(\mathcal{M}, \mu)$.*

Proof. Item 1. If $E_i \in \mathcal{M}$ such that $A \subseteq \cup E_i = B$ and $\widetilde{E}_i = E_i \setminus (E_1 \cup \cdots \cup E_{i-1})$ then

$$\sum \mu(E_i) \geq \sum \mu(\widetilde{E}_i) = \mu(B)$$

so

$$\mu^*(A) \leq \sum \mu(\widetilde{E}_i) = \mu(B) \leq \sum \mu(E_i).$$

Therefore, $\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{M} \text{ and } A \subseteq B\}$.

Item 2. If $\mu^*(E) = \infty$ Eq. (6.2) holds trivially. So assume that $\mu^*(E) < \infty$. Let $\epsilon > 0$ be given and choose, by Item 1., $B \in \mathcal{M}$ such that $E \subseteq B$ and $\mu(B) \leq \mu^*(E) + \epsilon$. Then

$$\begin{aligned} \mu^*(E) + \epsilon \geq \mu(B) &= \mu(B \cap A) + \mu(B \cap A^c) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we are done.

Item 3. Let us begin by assuming the $\mu(X) < \infty$. We have already seen that $\mathcal{M} \subseteq \mathcal{M}(\mu^*)$. Suppose that $A \in \mathcal{P}(X)$ satisfies,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \, \forall \, E \in \mathcal{P}(X). \tag{6.3}$$

By Item 1., there exists $B_n \in \mathcal{M}$ such that $A \subseteq B_n$ and $\mu^*(B_n) \leq \mu^*(A) + \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore $B = \cap B_n \supset A$ and $\mu(B) \leq \mu^*(A) + \frac{1}{n}$ for all n which implies that $\mu(B) \leq \mu^*(A)$ which implies that $\mu(B) = \mu^*(A)$. Similarly there exists $C \in \mathcal{M}$ such that $A^c \subseteq C$ and $\mu^*(A^c) = \mu(C)$. Taking $E = X$ in Eq. (6.3) shows

$$\mu(X) = \mu^*(A) + \mu^*(A^c) = \mu(B) + \mu(C)$$

so

$$\mu(C^c) = \mu(X) - \mu(C) = \mu(B).$$

Thus letting $D = C^c$, we have

$$D \subseteq A \subseteq B \text{ and } \mu(D) = \mu^*(A) = \mu(B)$$

so $\mu(B \setminus D) = 0$ and hence

$$A = D \cup [(B \setminus D) \cap A]$$

where $D \in \mathcal{M}$ and $(B \setminus D) \cap A \in \mathcal{N}$ showing that $A \in \bar{\mathcal{M}}$and $\mu^*(A) = \bar{\mu}(A)$.

Now if μ is $\sigma-$finite, choose $X_n \in \mathcal{M}$ such that $\mu(X_n) < \infty$ and $X_n \uparrow X$. Given $A \in \mathcal{M}(\mu^*)$ set $A_n = X_n \cap A$. Therefore

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \forall E \in \mathcal{P}(X).$$

Replace E by X_n to learn,

$$\mu^*(X_n) = \mu^*(A_n) + \mu^*(X_n \setminus A) = \mu^*(A_n) + \mu^*(X_n \setminus A_n).$$

The same argument as above produce sets $D_n \subseteq A_n \subseteq B_n$ such that $\mu(D_n) = \mu^*(A_n) = \mu(B_n)$. Hence $A_n = D_n \cup N_n$ and $N_n \equiv (B_n \setminus D_n) \cap A_n \in \mathcal{N}$. So we learn that

$$A = D \cup N := (\cup D_n) \cup (\cup N_n) \in \mathcal{M} \cup \mathcal{N} = \bar{\mathcal{M}}.$$

We also see that $\mu^*(A) = \mu(D)$ since $D \subseteq A \subseteq D \cup F$ where $F \in \mathcal{M}$ such that $N \subseteq F$ and

$$\mu(D) = \mu^*(D) \leq \mu^*(A) \leq \mu(D \cup F) = \mu(D).$$

■

6.2. **Carathéodory's Construction Theorem.**

Theorem 6.5 (Carathéodory's Construction Theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{M} = \mathcal{M}(\mu^*)$ then $\mathcal{M}$ is a σ-algebra and $\mu \equiv \mu^*|_{\mathcal{M}}$ is a complete measure.*

Proof. Clearly $\emptyset, X \in \mathcal{M}$ and if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$. So to show that $\mathcal{M}$ is an algebra we must show that $\mathcal{M}$ is closed under finite unions, i.e if $A, B \in \mathcal{M}$ and $E \in \mathcal{P}(X)$ then

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu(E \setminus (A \cup B)).$$

Using the definition of $\mathcal{M}$ we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \tag{6.4}$$

$$= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B) + \mu^*((E \setminus A) \cap B) + \mu^*(E \setminus (A \cup B)). \tag{6.5}$$

We will now make use of the identity:

$$E \cap (A \cup B) = (E \cap A) \cup (E \cap B) = [(E \cap A) \setminus B] \cup [(E \setminus A) \cap B] \cup (E \cap A \cap B)$$

which in words states that $E \cap A \cap B \cup (E \cap A \setminus B) \cup ((E \setminus A) \cap B)$ is equal to the points common to E, A and B union the points in (E and A but not B) union the points in (E and B but not A) which is equal to those points in E which are also in either A or B. Using this identity in Eq. (6.5) along with the subadditivity of μ^* shows

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)) \tag{6.6}$$

which implies that $A \cup B \in \mathcal{M}$ and therefore shows that $\mathcal{M}$ is an algebra.

Now suppose $A, B \in \mathcal{M}$ are disjoint, then taking $E = A \cup B$ in Eq. (6.4) implies

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

So $\mathcal{M}$ is an algebra and $\mu = \mu^*|_{\mathcal{M}}$ is finitely additive. We now must show that $\mathcal{M}$ is a $\sigma-$algebra and the μ is $\sigma-$additive.

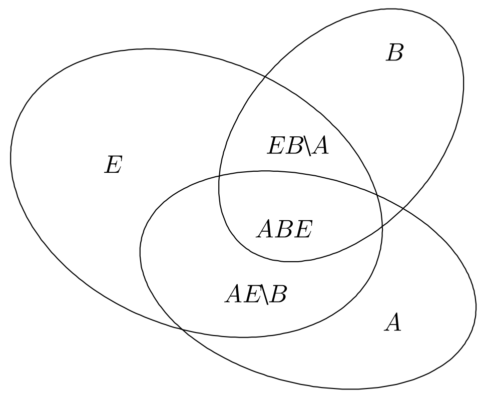

FIGURE 6. A useful set identity.

Let $A_i \in \mathcal{M}$ (without loss of generality assume $A_i \cap A_j = \emptyset$ if $i \neq j$) $B_n = \bigcup_{i=1}^n A_i$, and $B = \bigcup\limits_{j=1}^{\infty} A_j$, then for $E \subset X$ we have

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap A_{n-1}) + \mu^*(E \cap B_{n-2}) \\ &\vdots \\ &= \sum_{k=1}^n \mu^*(E \cap A_k). \end{aligned} \tag{6.7}$$

Therefore we find that

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &= \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{k=1}^n \mu^*(E \cap A_k) + \mu^*(E \cap B^c). \end{aligned}$$

Letting $n \to \infty$ in this equation shows that

$$\begin{aligned} \mu^*(E) &\geq \sum_{k=1}^\infty \mu^*(E \cap A_k) + \mu^*(E \cap B^c) \\ &\geq \mu^*(\cup_k (E \cap A_k)) + \mu^*(E \setminus B) \text{ (subadditivity)} \\ &= \mu^*(E \cup B) + \mu^*(E \setminus B) \\ &\geq \mu^*(E). \text{ (subadditivity)} \end{aligned}$$

This shows that $\mathcal{M}$ is a σ – algebra.

Since $\mu^*(E) \geq \mu^*(E \cap B_n)$ we may let $n \to \infty$ in Eq. (6.7) to find

$$\mu^*(E) \geq \sum_{k=1}^{\infty} \mu^*(E \cap A_k).$$

Letting $E = B = \cup A_k$ in this last expression then shows $\mu^*(B) \geq \sum_{k=1}^{\infty} \mu^*(A_k)$ and hence by the subadditivity of μ^*,

$$\mu^*(A) = \sum_{k=1}^{\infty} \mu^*(A_k).$$

Therefore, $\mu = \mu^*|_{\mathcal{M}}$ is countably additive on $\mathcal{M}$.

Finally we show μ is complete. If $N \subseteq F \in \mathcal{M}$ and $\mu(F) = 0 = \mu^*(F)$, then $\mu^*(N) = 0$ and

$$\mu^*(E) \leq \mu^*(E \cap N) + \mu^*(E \cap N^c) = \mu^*(E \cap N^c) \leq \mu^*(E).$$

which shows that $N \in \mathcal{M}$. ■

Proposition 6.6. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$ and μ^* be the associated outer measure as defined in Eq. (6.1) with $\rho = \mu_0$. Then $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$ and $\mu^*|_{\mathcal{A}} = \mu_0$.*

Proof. Let $A \in \mathcal{A}$, $A \subseteq \bigcup_1^{\infty} A_j$ with $A_j \in \mathcal{A}$ and set

$$\tilde{A}_j \equiv A \cap A_j \setminus (A_1 \cup \cdots \cup A_{j-1}) \subseteq A_j.$$

Then $A = \cup \tilde{A}_j \subseteq \bigcup_1^{\infty} A_j$ and hence

$$\mu_0(A) = \sum \mu_0(\tilde{A}_j) \leq \sum \mu_0(A_j)$$

which shows that $\mu^*(A) = \mu_0(A)$ for all $A \in \mathcal{A}$.

Now let $A \in \mathcal{A}$ and $E \subseteq X$ such that $\mu^*(E) < \infty$. Given $\epsilon > 0$ choose $B_j \in \mathcal{A}$ (which we may and do assume to satisfy $B_j \cap B_k = \emptyset$ if $k \neq j$) such that $E \subseteq \cup B_j$ and

$$\sum_{j=1}^{\infty} \mu_0(B_j) \leq \mu^*(E) + \epsilon.$$

Since $A \cap E \subseteq \cup(B_j \cap A)$ and $E \cap A^c \subseteq \cup(B_j \setminus A)$,

$$\begin{aligned}\mu^*(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu_0(B_j) &= \sum_{j=1}^{\infty} [\mu_0(B_j \cap A) + \mu_0(B_j \cap A^c)] \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)\end{aligned}$$

where in the second line we used the subadditivity of μ^*. Since $\epsilon > 0$ is arbitrary this shows that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

and therefore that $A \in \mathcal{M}(\mu^*)$. ■

Theorem 6.7. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra μ_0 be a premeasure on $\mathcal{A}$ and $\mathcal{M} = \sigma(\mathcal{A})$. Then $\mu = \mu^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$ which extends μ_0. Moreover if $\nu : \mathcal{M} \to [0, \infty]$ is another measure such that $\nu = \mu_0$ on $\mathcal{A}$ then $\nu(A) \leq \mu(A)$ and $\nu(A) = \mu(A)$ if $\mu(A) < \infty$. If μ_0 is σ-finite then in fact $\nu = \mu$.*

Proof. We already know $\mathcal{A} \subset \mathcal{M}(\mu^*)$ and that $\mathcal{M}(\mu^*)$ is a $\sigma -$ algebra, hence $\mathcal{A} \subseteq \sigma(\mathcal{A}) = \mathcal{M} \subset \mathcal{M}(\mu^*)$. Most of this theorem has already been proved, we need only shows the assertions regarding the measure ν. Let $E \in \mathcal{M}$ and $E \subseteq \cup A_j$ with $A_j \in \mathcal{A}$, then

$$\nu(E) \leq \sum \nu(A_j) = \sum \mu(A_j)$$

implies

$$\nu(E) \leq \mu^*(E) = \mu(E). \tag{6.8}$$

If $A = \cup A_j$ then

$$\nu(A) = \lim_{N\to\infty} \nu\left(\bigcup_{i=1}^N A_j\right) = \lim_{N\to\infty} \mu\left(\bigcup_1^N A_j\right) = \mu(A)$$

so $\nu = \mu$ on $\mathcal{A}_\sigma$.

Suppose $\mu(E) < \infty$ and choose $A_j \in \mathcal{A}$ such that $E \subseteq A = \cup A_j \in \mathcal{A}_\sigma$ such that and

$$\mu(A) \leq \sum_{j=1}^\infty \mu(A_j) \leq \mu(E) + \epsilon \text{ i.e. } \mu(A \setminus E) \leq \epsilon.$$

Then

$$\begin{aligned} \mu(E) &\leq \mu(A) = \nu(A) \\ &= \nu(E) + \nu(A \setminus E) \\ &\leq \nu(E) + \mu(A \setminus E) \\ &\leq \nu(E) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu(E) \leq \nu(E)$ which combined with Eq. (6.8) shows that $\mu(E) = \nu(E)$.

For the $\sigma -$ finite case, choose $X_j \in \mathcal{M}$ such that $X_j \uparrow X$ and $\mu(X_j) < \infty$ then

$$\mu(E) = \lim_{j\to\infty} \mu(E \cap X_j) = \lim_{j\to\infty} \nu(E \cap X_j) = \nu(E).$$

■

Example 6.8. For examples of lack of uniqueness of measures μ extending a premeasure μ_0 on an Algebra in the non-σ-**finite case**. see Problem #23 in Chapter 1. of Folland. There one considers the measures: $\mu_1 = \infty$ except on the empty set, μ_2 is counting measure, and $\mu_3(A) = \mu_3(A \cap D)$ where $D \subseteq \mathbb{Q}$ is any dense set.

6.3. Regularity results revisited.

Lemma 6.9. *Suppose that μ_0 is a premeasure on an algebra $\mathcal{A}$ of subsets of X. Also suppose that μ_0 is σ-finite on $\mathcal{A}$. Let μ^* be the associated outer measure, $\mathcal{M} = \mathcal{M}(\mu^*)$ and $\bar{\mu} = \mu^*|\mathcal{M}$. Then for any $B \in \mathcal{M} = \mathcal{M}(\mu^*)$ there exists $A \subseteq B \subseteq C$ such that $A \in \mathcal{A}_{\delta\sigma}$, $C \in \mathcal{A}_{\sigma\delta}$ and $\bar{\mu}(C \setminus A) = 0$.*

Proof. Choose $X_m \in \mathcal{A}$ increasing to X such that $\mu_0(X_m) < \infty$ for all m and set $B_m = X_m \cap B$. Choose C_m so that $B_m \subseteq C_m \in \mathcal{A}_\sigma$, and $C_m \subseteq X_m$ and $\bar{\mu}(C_m \setminus B_m)$ is small. Let $C \equiv \cup C_m$, then

$$\begin{aligned} \bar{\mu}(C \setminus B) &\leq \bar{\mu}(\cup_m (C_m \setminus B)) \\ &\leq \sum \bar{\mu}(C_m \setminus B) \leq \sum \bar{\mu}(C_m \setminus B_m) < \epsilon. \end{aligned}$$

Since $\mathcal{A}_\sigma \cap \mathcal{A}_\sigma = \mathcal{A}_\sigma$ we may choose $C_k \in \mathcal{A}_\sigma$ such that $B \subseteq C_k$, C_k is decreasing in k and $\lim_{k\to\infty} \bar{\mu}(C_k \setminus B) = 0$. Define $C \equiv \bigcap_{k=1}^{\infty} C_k$. Then $\bar{\mu}(C \setminus B) = 0$ and $C \in \mathcal{A}_{\sigma\delta}$.
Now: Choose $A^c \subseteq B^c \in \mathcal{A}_{\sigma\delta}$ such that $\bar{\mu}(B^c \setminus A^c) = 0$, i.e. $0 = \bar{\mu}(B^c \cap A) = \bar{\mu}(A \setminus B)$. Then $A \in \mathcal{A}_{\delta\sigma}$ and $\bar{\mu}(A \setminus B) = 0$ and hence we have $A \subseteq B \subseteq C$ and

$$\bar{\mu}(C \setminus A) = \bar{\mu}(C \setminus B) + \bar{\mu}(B \setminus A) = 0.$$

We now show that $\mathcal{A} \subseteq \mathcal{M}^*$. Suppose that $E \subseteq X$ and $\mu^*(E) < \infty$. Choose $B_n \in \mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\mu^*(E) \le \sum_{n=1}^{\infty} \mu(B_n) - \epsilon.$$

Since $\mu(B_n) = \mu(B_n \cap A) + \mu(B_n \cap A^c)$,

$$\begin{aligned}\mu^*(E) + \epsilon \ge \sum \mu(B_n) &= \sum \mu(B_n \cap A) + 2\mu(B_n \cap A^c) \\ &\ge \mu^*(E \cap A) + \mu^*(E \cap A^c)\end{aligned}$$

which implies that

$$\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

■

Theorem 6.10. *Suppose that μ_0 is a $\sigma-$ finite premeasure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$,*

$$\begin{aligned}&\mu^* \text{ is the associated outer measure,} \\ &\mathcal{M} = \mathcal{M}(\mu^*), \\ &\bar{\mu} = \mu^*|\mathcal{M} \text{ and} \\ &\mu = \bar{\mu}|_{\sigma(\mathcal{A})}.\end{aligned}$$

Then $\mathcal{M} = \overline{\sigma(\mathcal{A})} = \overline{\sigma(\mathcal{A})}^{\mu}$ – the completion of $\sigma(\mathcal{A})$ with respect to μ.

Proof. For $B \in \mathcal{M}$ there exists $A \in \mathcal{A}_{\delta\sigma}$ and $C \in \mathcal{A}_{\sigma\delta}$ such that $A \subseteq B \subseteq C$ and

$$\mu(C \setminus A) = \bar{\mu}(C \setminus A) = 0.$$

This implies $B \in \overline{\sigma(\mathcal{A})}$ and hence that $\mathcal{M} \subseteq \overline{\sigma(\mathcal{A})}$.

For the reverse inclusion, suppose $N \in \overline{\sigma(\mathcal{A})}$, i.e. $N \subseteq F \in \sigma(\mathcal{A})$ such that $0 = \mu(F) = \mu^*(F)$. Then $\mu^*(N) = 0$ and

$$\mu^*(E) \le \mu^*(E \cap N) + \mu^*(E \cap N^c) = 0 + \mu^*(E \cap N^c) \le \mu^*(E)$$

which shows that $N \in \mathcal{M}$. ■

6.4. **Construction of measures on a simple product space.**

Exercise 6.11. Let $Y \equiv \{0,1\}^{\mathbb{N}}$ (the set of sequences $y = (y_1, y_2, \ldots)$ with $y_i \in X \equiv \{0,1\}$, $Y_n \equiv \{0,1\}^n$ for all $n \in \mathbb{N}$, and $\pi_n : Y \to Y_n$ be defined by $\pi_n(y) = (y_1, y_2, \ldots, y_n)$. $\mathcal{A}$ denote the collection of "cylinder sets" in Y, i.e. sets of the form

$$A = \pi_n^{-1}(C) \text{ where } n \in \mathbb{N} \text{ and } C \subset Y_n. \tag{6.9}$$

In words a cylinder set is a subset of Y which is determined by restricting the values of only a finite number of coordinates of $y \in Y$. For example $A \equiv \{y \in Y : y_{2i} = 0 \text{ for } i \in N\}$ is **not** a cylinder set.

a) Show that $\mathcal{A}$ is an algebra.
b) Show that if $A_n \in \mathcal{A}$ and $A_n \downarrow \emptyset$ then $A_n = \emptyset$ for all n sufficiently large.
c) Conclude that any finitely additive measure μ_0 on $\mathcal{A}$ is a premeasure.

(1) **Solution** a) Let $\mathcal{A}_n \equiv \pi_n^{-1}(\mathcal{P}(Y_n))$ then $\mathcal{A}_n$ is the pull-back of an algebra and hence an algebra. Next notice that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for each n. To see this, let $C \subset Y_n$ and set $A = \pi_n^{-1}(C)$. Then $A = \pi_n^{-1}(C \times X)$ where $X \equiv \{0, 1\}$. This shows that $A \in \mathcal{A}_{n+1}$ and hence that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. Now it is not hard to check an increasing union of algebras is still an algebra and hence $\mathcal{A} = \cup_n \mathcal{A}_n$ is an algebra.

b) We will prove the contrapositive of part b). Namely, suppose that $A_k \in \mathcal{A}$ with A_k decreasing as k increases and $A_N = \cap_{k=1}^N A_k \neq \emptyset$ for each $N \in \mathbb{N}$, then $\cap_{k=1}^\infty A_k \neq \emptyset$. Using the ideas in the proof of part a) above, we may choose an increasing sequence $\{n_k\}$ and sets $D_k \subset Y_{n_k}$ such that $A_k = \pi_{n_k}^{-1}(D_k)$ for all $k \in \mathbb{N}$. To simplify notation replace the sequence $\{A_1, A_2, A_3, \dots\}$ by the sequence of sets

$$\{B_1, B_2, B_3, \dots\} \equiv \{\overbrace{Y, \dots, Y}^{n_1 - 1}, \overbrace{A_1, \dots A_1}^{n_2 - n_1}, \overbrace{A_2, \dots, A_2}^{n_3 - n_2}, \dots\}.$$

Then B_i is also decreasing, $\cap_{i=1}^N B_i \neq \emptyset$ for all $N \in \mathbb{N}$, and $\cap_{i=1}^\infty B_i = \cap_{n=1}^\infty A_n$. Moreover, the B_i may be written in the form $B_i = \pi_i^{-1}(C_i)$ where $C_i \subset Y_i$ for all $i \in \mathbb{N}$.

We will now finish the proof by of part b) by showing that $\cap_{i=1}^\infty B_i \neq \emptyset$. To do this, notice that $B_i = \pi_i^{-1}(C_i) \supset B_{i+1} = \pi_{i+1}^{-1}(C_{i+1})$ implies $C_{i+1} \subset C_i \times X$ for each $i \in \mathbb{N}$. So by induction,

$$C_j \subset C_i \times X^{(j-i)} \text{ for all } j > i. \tag{6.10}$$

Since no B_i is empty by assumption, no C_i is empty either. In particular this implies that $\{x_1 | x \in C_i\}$ is not empty for each i. Choose ϵ_1 so the that for infinitely many i's, $\epsilon_1 \in \{x_1 | x \in C_i\}$. Then by (6.10) it must happen that $\epsilon_1 \in \{x_1 | x \in C_i\}$ for all i. By similar logic $\{x_2 | x \in C_i \text{ with } x_1 = \epsilon_1\}$ is non empty for all $i \geq 2$. Hence we may choose $\epsilon_2 \in \{0, 1\}$ such that $(\epsilon_1, \epsilon_2) \in \{(x_1, x_2) | x \in C_i\}$ for all $i \geq 2$. By induction, one may show there exists $\epsilon_1, \epsilon_2, \epsilon_3, \dots \in \{0, 1\}$ such that $(\epsilon_1, \dots, \epsilon_j) \in \{(x_1, x_2, \dots, x_j) | x \in C_i\}$ for all $i \geq j$ and in particular $(\epsilon_1, \dots, \epsilon_j) \in C_j$ for all $j \in \mathbb{N}$. Set $\epsilon \equiv (\epsilon_1, \epsilon_2, \dots) \in Y$, then we have shown that

$$\pi_j(\epsilon) = (\epsilon_1, \dots, \epsilon_j) \in C_j \text{ for all } j \in \mathbb{N}.$$

This shows that $\epsilon \in B_j$ for all j, i.e. $\epsilon \in \cap B_j = \cap A_n$ and hence $\cap_n A_n \neq \emptyset$.

c) It follows, by Homework 3.1.1 and what we have just proved, that any finitely additive measure μ_0 on $\mathcal{A}$ is actually countably additive on $\mathcal{A}$, i.e. μ_0 is a premeasure on $\mathcal{A}$.

Exercise 6.12. Show there is a unique finitely additive measure μ_0 on $\mathcal{A}$ such that $\mu_0(A) = 2^{-n}$ if A is a set of the form

$$A = \{y \in Y | y_i = \epsilon_i \text{ for } i = 1, 2, \dots, n\}, \tag{6.11}$$

where each $\epsilon_i \in \{0, 1\}$. Use the above problems to conclude there exists unique measure μ on $\mathcal{M} \equiv \sigma(\mathcal{A})$ such that $\mu(A) = 2^{-n}$ if A is as in (6.11).

(1) **Solution**: For $A \in \mathcal{P}(Y_n)$ let $\mu_n(A) = 2^{-n}\#(A)$, where $\#(A)$ denotes the number of elements in A. Then μ_n is a measure on $\mathcal{P}(Y_n)$. If $A \in \mathcal{A}_n \subset \mathcal{A}$ is of the form $A = \pi_n^{-1}(C)$ with $C \subset Y_n$, set $\mu_0(A) = \mu_n(C)$. We must show that μ_0 is well defined. For this suppose that $A = \pi_m^{-1}(D)$ for some $D \subset Y_m$. Without loss of generality assume that $m > n$, then D must be given by $D = C \times X^{(m-n)}$. Therefore

$$\mu_m(D) = 2^{-m}\#(C \times X^{(m-n)}) = 2^{-m}2^{(m-n)}\#(C) = 2^{-n}\#(C) = \mu_n(C),$$

which shows that μ_0 is well defined. Now it is easily checked that μ_0 is a measure on $\mathcal{A}$ since μ_m is a measure on $\mathcal{A}_m$ for each m. Therefore by the last problem, μ_0 is in fact a premeasure on $\mathcal{A}$. By Theorem 1.14 of Folland, it follows that μ extends to a measure on $\mathcal{M} = \sigma(\mathcal{A})$.

Remark 6.13 (A Cryptic Remark). The measure μ is essentially Lebesgue measure on the unit interval $[0, 1]$.

7. Continuous and measurable functions

We are mostly interested in measurable functions, nevertheless it is instructive to first reformulate the notion of a continuous function between two metric spaces.

Lemma 7.1. *Suppose that (X, ρ) and (Y, d) are two metric spaces and $f : X \to Y$ is a function. Then following are equivalent:*

(1) *f is continuous.*
(2) *$f^{-1}(V) \in \tau_\rho$ for all $V \in \tau_d$, i.e. $f^{-1}(V)$ is open in X if V is open in Y.*
(3) *$f^{-1}(C)$ is closed in X if C is closed in Y.*
(4) *For all convergent sequences $\{x_n\} \subset X$, $\{f(x_n)\}$ is convergent in Y and*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

Proof. 1. $\Rightarrow$ 2. For all $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ if $\rho(x, x') < \delta$. i.e.

$$B_x(\delta) \subset f^{-1}(B_{f(x)}(\epsilon))$$

So if $V \subset_0 Y$ and $x \in f^{-1}(V)$ we may choose $\epsilon > 0$ such that $B_{f(x)}(\epsilon) \subseteq V$ then

$$B_x(\delta) \subseteq f^{-1}(B_{f(x)}(\epsilon)) \subseteq f^{-1}(V)$$

showing that $f^{-1}(V)$ is open.

2. $\iff$ 3. If C is closed in Y, then $C^c \subset_o Y$ and hence $f^{-1}(C^c) \subset_o X$. Since $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$, this shows that $f^{-1}(C)$ is the compliment of an open set and hence closed. Similarly one shows that 3. $\Rightarrow$ 2.

2. $\Rightarrow$ 1. Let $\epsilon > 0$ and $x \in X$, then, since $f^{-1}(B_{f(x)}(\epsilon)) \subset_o X$,there exists $\delta > 0$ such that $B_x(\delta) \subseteq f^{-1}(B_{f(x)}(\epsilon))$ i.e. if $\rho(x, x') < \delta$ then $d(f(x'), f(x)) < \epsilon$.

1. $\Rightarrow$ 4. If f is continuous and $x_n \to x$ in X, let $\epsilon > 0$ and choose $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ when $\rho(x, x') < \delta$. There exists an $N > 0$ such that $\rho(x, x_n) < \delta$ for all $n \geq N$ and therefore $d(f(x), f(x_n)) < \epsilon$ for all $n \geq N$. That is to say $\lim_{n\to\infty} f(x_n) = f(x)$ as $n \to \infty$.

4. $\Rightarrow$ 1.We will show that not 1. $\Rightarrow$ not 4. not 1 implies there exists $\epsilon > 0$, a point $x \in X$ and a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $d(f(x), f(x_n)) \geq \epsilon$ while $\rho(x, x_n) < \frac{1}{n}$. Clearly this sequence $\{x_n\}$ violates 4. ∎

Notation 7.2. For a general topological space (X,τ), the **Borel** σ **– algebra** is the σ – algebra, $\mathcal{B}_X = \sigma(\tau)$. We will use $\mathcal{B}_{\mathbb{R}}$ to denote the Borel σ - algebra on $\mathbb{R}$ and recall that $\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,\infty) : a \in \mathbb{R}\})$.

Our notion of a "measurable" function will be similar to conditions 2. and 3. above for continuity. For motivational purposes, suppose $(X,\mathcal{M},\mu)$ is a measure space $f : X \to \mathbb{R}_+$. Roughly speaking, in the next section we are going to define $\int_X f d\mu$ by

$$\int_X f d\mu = \lim_{\text{mesh}\to 0} \sum_{0<a_1<a_2<a_3<\dots}^{\infty} a_i \mu(f^{-1}(a_i, a_{i+1}]).$$

For this to make sense we will need to require $f^{-1}((a,b]) \in \mathcal{M}$ for all $a < b$. Because of Lemma 7.7 below, this last condition is equivalent to the condition

$$f^{-1}(\mathcal{B}_{\mathbb{R}}) \subseteq \mathcal{M},$$

where we are using the following notation.

Notation 7.3. If $f : X \to Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$ let

$$f^{-1}\mathcal{E} \equiv f^{-1}(\mathcal{E}) \equiv \{f^{-1}(E) | E \in \mathcal{E}\}.$$

If $\mathcal{G} \subset \mathcal{P}(X)$, let

$$f_*\mathcal{G} \equiv \{A \in \mathcal{P}(Y) | f^{-1}(A) \in \mathcal{G}\}.$$

It is easily checked that $f^{-1}\mathcal{E}$ and $f_*\mathcal{G}$ are σ – algebras (topologies) provided $\mathcal{E}$ and $\mathcal{G}$ are σ – algebras (topologies). (You should check these statements.)

Definition 7.4. Let $(X,\mathcal{M})$ and $(Y,\mathcal{F})$ be measurable (topological) spaces. A function $f : X \to Y$ is **measurable (continuous)** if $f^{-1}(\mathcal{F}) \subseteq \mathcal{M}$. We will also say that f is $\mathcal{M}/\mathcal{F}$ – measurable (continuous) or $(\mathcal{M},\mathcal{F})$ – measurable (continuous).

Remark 7.5. Let $f : X \to Y$ be a function. Given a σ – algebra (topology) $\mathcal{F} \subset \mathcal{P}(Y)$, the σ – algebra (topology) $\mathcal{M} := f^{-1}(\mathcal{F})$ is the smallest σ – algebra (topology) on X such that f is $(\mathcal{M},\mathcal{F})$ - measurable (continuous). Similarly, if $\mathcal{M}$ is a σ - algebra (topology) on X then $\mathcal{F} = f_*\mathcal{M}$ is the largest σ – algebra (topology) on Y such that f is $(\mathcal{M},\mathcal{F})$ - measurable (continuous).

Lemma 7.6. *Suppose that $(X,\mathcal{M})$, $(Y,\mathcal{F})$ and $(Z,\mathcal{G})$ are measurable (topological) spaces. If $f : (X,\mathcal{M}) \to (Y,\mathcal{F})$ and $g : (Y,\mathcal{F}) \to (Z,\mathcal{G})$ are measurable (continuous) functions then $g \circ f : (X,\mathcal{M}) \to (Z,\mathcal{G})$ is measurable (continuous) as well.*

Proof. This is easy since by assumption $g^{-1}(\mathcal{G}) \subset \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subset \mathcal{M}$ so that

$$(g \circ f)^{-1}(\mathcal{G}) = f^{-1}\left(g^{-1}(\mathcal{G})\right) \subset f^{-1}(\mathcal{F}) \subset \mathcal{M}.$$

■

Lemma 7.7. *Suppose that $f : X \to Y$ is a function and $\mathcal{E} \subset \mathcal{P}(Y)$, then*

$$\sigma\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\sigma(\mathcal{E})) \text{ and} \tag{7.1}$$

$$\tau\left(f^{-1}(\mathcal{E})\right) = f^{-1}(\tau(\mathcal{E})). \tag{7.2}$$

Moreover, if $\mathcal{F} = \sigma(\mathcal{E})$ (or $\mathcal{F} = \tau(\mathcal{E})$) and $\mathcal{M}$ is a σ – algebra (topology) on X, then f is $(\mathcal{M},\mathcal{F})$ – measurable (continuous) iff $f^{-1}(\mathcal{E}) \subseteq \mathcal{M}$.

Proof. We will prove Eq. (7.1), the proof of Eq. (7.2) being analogous. If $\mathcal{E} \subset \mathcal{F}$, then $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ and therefore, (because $f^{-1}(\sigma(\mathcal{E}))$ is a σ – algebra)

$$\mathcal{G} := \sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$$

which proves half of Eq. (7.1). For the reverse inclusion notice that

$$f_*\mathcal{G} = \left\{B \subset Y : f^{-1}(B) \in \mathcal{G}\right\}.$$

is a σ – algebra which contains $\mathcal{E}$ and thus $\sigma(\mathcal{E}) \subset f_*\mathcal{G}$. Hence if $B \in \sigma(\mathcal{E})$ we know that $f^{-1}(B) \in \mathcal{G}$, i.e.

$$f^{-1}(\sigma(\mathcal{E})) \subset \mathcal{G}.$$

The last assertion of the Lemma is an easy consequence of Eqs. (7.1) and (7.2). ■

Corollary 7.8. *Suppose that $(X, \mathcal{M})$ is a measurable space. Then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ - measurable iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{Q}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$, etc. Similarly, if $(X, \mathcal{M})$ is a topological space, then $f : X \to \mathbb{R}$ is $(\mathcal{M}, \tau_{\mathbb{R}})$ - continuous iff $f^{-1}((a, b)) \in \mathcal{M}$ for all $-\infty < a < b < \infty$ iff $f^{-1}((a, \infty)) \in \mathcal{M}$ and $f^{-1}((-\infty, b)) \in \mathcal{M}$ for all $a, b \in \mathbb{Q}$.*

Proof. An exercise in using Lemma 7.7. ■

We will often deal with function $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$\mathcal{B}_{\bar{\mathbb{R}}} := \sigma\left(\left\{[a, \infty] : a \in \bar{\mathbb{R}}\right\}\right). \tag{7.3}$$

The following Corollary of Lemma 7.7 is a direct analogue of Corollary 7.8.

Corollary 7.9. *$f : X \to \bar{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ - measurable iff $f^{-1}((a, \infty]) \in \mathcal{M}$ for all $a \in \bar{\mathbb{R}}$ iff $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \bar{\mathbb{R}}$, etc.*

Proposition 7.10. *A subset $A \subset \bar{\mathbb{R}}$ is in $\mathcal{B}_{\bar{\mathbb{R}}}$ iff $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$. In particular $\{\pm\infty\}$, $\{\infty\}$ and $\{-\infty\}$ are in $\mathcal{B}_{\bar{\mathbb{R}}}$.*

Proof. Let $i : \mathbb{R} \to \bar{\mathbb{R}}$ be the inclusion map. Since $i^{-1}(([a, \infty])) = [a, \infty) \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$ for all $a \in \bar{\mathbb{R}}$, i is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\bar{\mathbb{R}}})$– measurable. In particular if $A \in \mathcal{B}_{\bar{\mathbb{R}}}$, then $i^{-1}(A) = A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$.

For the converse, we begin with the observations:

$$\begin{aligned}
\{-\infty\} &= \cap_{n=1}^{\infty} [-\infty, -n) = \cap_{n=1}^{\infty} [-n, \infty]^c \in \mathcal{B}_{\bar{\mathbb{R}}}, \\
\{\infty\} &= [\infty, \infty] \in \mathcal{B}_{\bar{\mathbb{R}}} \text{ and} \\
\mathbb{R} &= \bar{\mathbb{R}} \setminus \{\pm\infty\} \in \mathcal{B}_{\bar{\mathbb{R}}}.
\end{aligned}$$

Using these facts you may easily shows that

$$\mathcal{M} = \{A \subset \mathbb{R} : A \in \mathcal{B}_{\bar{\mathbb{R}}}\}$$

is a σ – algebra on $\mathbb{R}$ which contains (a, ∞) for all $a \in \mathbb{R}$. Hence $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$, i.e. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\bar{\mathbb{R}}}$. Using these observations, if $A \subset \bar{\mathbb{R}}$ and $A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}$, then

$$A = (A \cap \mathbb{R}) \cup (A \cap \{\pm\infty\}) \in \mathcal{B}_{\bar{\mathbb{R}}}.$$

■

Proposition 7.11 (Closure under sups, infs and limits)**.** *Suppose that $(X, \mathcal{M})$ is a measurable space and $f_j : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ is a sequence of $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. Then*

$$\sup_j f_j, \quad \inf_j f_j, \quad \lim_{j\to\infty} \sup\ f_j \text{ and } \lim_{j\to\infty} \inf\ f_j$$

are all $\mathcal{M}/\mathcal{B}_{\overline{\mathbb{R}}}$ – measurable functions. (Note that this result is in generally false when $(X, \mathcal{M})$ is a topological space and measurable is replaced by continuous in the statement.)

Proof. Define $g_+(x) := \sup_j f_j(x)$, then

$$\begin{aligned}\{x : g_+(x) \le a\} &= \{x : f_j(x) \le a \; \forall j\} \\ &= \cap_j \{x : f_j(x) \le a\} \in \mathcal{M}\end{aligned}$$

so that g_+ is measurable. Similarly if $g_-(x) = \inf_j f_j(x)$ then

$$\{x : g_-(x) \ge a\} = \cap_j \{x ; f_j(x) \ge a\} \in \mathcal{M}.$$

Since

$$\begin{aligned}\limsup \; f_j &= \inf_n \sup \{f_j : j \ge n\} \text{ and} \\ \underline{\lim} \; f_j &= \sup_n \inf \{f_j : j \ge n\}\end{aligned}$$

we are done by what we have already proved. ■

Lemma 7.12. *Suppose that $(X, \mathcal{M})$ is a measurable space, (Y, τ) be a topological space and $f_j : X \to Y$ is $(\mathcal{M}, \mathcal{B}_Y)$ – measurable for all j. Also assume that for each $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Then $f : X \to Y$ is also $(\mathcal{M}, \mathcal{B}_Y)$ – measurable.*

Proof. Suppose that $V \subset Y$ is an open set, then

$$\begin{aligned}f^{-1}(V) &= \{x : f(x) \in V\} = \{x : f_n(x) \in V \text{ for almost all } n\} \\ &= \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} f_n^{-1}(V) \in \mathcal{M}\end{aligned}$$

since $f_n^{-1}(V) \in \mathcal{M}$ because each f_n is measurable. Therefore $f^{-1}(\tau) \subset \mathcal{M}$ and thus

$$f^{-1}(\mathcal{B}_Y) = f^{-1}(\sigma(\tau)) = \sigma\left(f^{-1}(\tau)\right) \subset \mathcal{M}.$$

■

Definition 7.13. A function $f : X \to Y$ between to topological spaces is **Borel measurable** if $f^{-1}(\mathcal{B}_Y) \subseteq \mathcal{B}_X$.

Proposition 7.14. *Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function. Then f is Borel measurable.*

Proof. Using Lemma 7.7 it suffices to recall $\mathcal{B}_Y = \sigma(\tau_Y)$ and to notice that $f^{-1}(V) \in \tau_X \subset \mathcal{B}_X$ for all V open in Y. ■

Definition 7.15. A function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue measurable** if $f^{-1}(\mathcal{B}_{\mathbb{R}}) \subseteq \mathcal{L} := \overline{\mathcal{B}_{\mathbb{R}}}^m$ – the completion of $\mathcal{B}_{\mathbb{R}}$ relative to Lebesgue measure m.

7.1. Relative Topologies and $\sigma -$ Algebras.

Definition 7.16. Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets, $A \subset X$, $i_A : A \to X$ be the inclusion map $(i_A(x) = x)$ for all $x \in A$, and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

Proposition 7.17. *Suppose that $A \subset X$, $\mathcal{M} \subset \mathcal{P}(X)$ is a σ – algebra and $\tau \subset \mathcal{P}(X)$ is a topology, then $\mathcal{M}_A \subset \mathcal{P}(A)$ is a σ – algebra and $\tau_A \subset \mathcal{P}(A)$ is a topology. (The topology τ_A is called the relative topology on A.) Moreover if $\mathcal{E} \subset \mathcal{P}(X)$ is such that $\mathcal{M} = \sigma(\mathcal{E})$ $(\tau = \tau(\mathcal{E}))$ then $\mathcal{M}_A = \sigma(\mathcal{E}_A)$ $(\tau_A = \tau(\mathcal{E}_A))$.*

Proof. The first assertion os easy to check as remarked after Notation 7.3. The second assertion is a consequence of Lemma 7.7. Indeed,

$$\mathcal{M}_A = i_A^{-1}(\mathcal{M}) = i_A^{-1}(\sigma(\mathcal{E})) = \sigma(i_A^{-1}(\mathcal{E})) = \sigma(\mathcal{E}_A)$$

and similarly

$$\tau_A = i_A^{-1}(\tau) = i_A^{-1}(\tau(\mathcal{E})) = \tau(i_A^{-1}(\mathcal{E})) = \tau(\mathcal{E}_A).$$

■

Definition 7.18. Let $A \subset X$, $f : A \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology, then we say that $f|_A$ is measurable (continuous) if $f|_A$ is $\mathcal{M}_A$ – measurable (τ_A continuous).

Proposition 7.19. *Let $A \subset X$, $f : X \to \mathbb{C}$ be a function, $\mathcal{M} \subset \mathcal{P}(X)$ be a σ – algebra and $\tau \subset \mathcal{P}(X)$ be a topology. If f is $\mathcal{M}$ – measurable (τ continuous) then $f|_A$ is $\mathcal{M}_A$ measurable (τ_A continuous). Moreover if $A_n \in \mathcal{M}$ ($A_n \in \tau$) such that $X = \cup_{n=1}^{\infty} A_n$ and $f|A_n$ is $\mathcal{M}_{A_n}$ measurable (τ_{A_n} continuous) for all n, then f is $\mathcal{M}$ – measurable (τ continuous).*

Proof. Notice that i_A is $(\mathcal{M}_A, \mathcal{M})$ – measurable (τ_A, τ) – continuous) hence $f|_A = f \circ i_A$ is $\mathcal{M}_A$ measurable (τ_A – continuous). Let $B \subset \mathbb{C}$ be a Borel set and consider

$$f^{-1}(B) = \cup_{n=1}^{\infty} \left(f^{-1}(B) \cap A_n\right) = \cup_{n=1}^{\infty} f|_{A_n}^{-1}(B).$$

If $A \in \mathcal{M}$ ($A \in \tau$), then it is easy to check that

$$\begin{aligned} \mathcal{M}_A &= \{B \in \mathcal{M} : B \subset A\} \subset \mathcal{M} \text{ and} \\ \tau_A &= \{B \in \tau : B \subset A\} \subset \tau. \end{aligned}$$

The second assertion is now an easy consequence of the previous three equations. ■

7.2. **Product Spaces.** The reader who finds this section a little too heavy may wish to first read Appendix 7.4 below where the most important special cases are covered. The material in this subsection before Corollary 7.28 may then be skipped on the first reading.

Definition 7.20. Let X and A be sets, and suppose for $\alpha \in A$ we are give a measurable (topological) space $(Y_\alpha, \mathcal{F}_\alpha)$ and a function $f_\alpha : X \to Y_\alpha$. We will write $\sigma(f_\alpha : \alpha \in A)$ ($\tau(f_\alpha : \alpha \in A)$) for the smallest σ-algebra (topology) on X such that each f_α is measurable (continuous), i.e.

$$\begin{aligned} &\sigma(f_\alpha : \alpha \in A) = \sigma(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)) \text{ and} \\ &\tau(f_\alpha : \alpha \in A) = \tau(\cup_\alpha f_\alpha^{-1}(\mathcal{F}_\alpha)). \end{aligned}$$

Proposition 7.21. *Assuming the notation in Definition 7.20 and additionally let $(Z, \mathcal{M})$ be a measurable (topological) space and $g : Z \to X$ be a function. Then g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable ($(\mathcal{M}, \tau(f_\alpha : \alpha \in A))$ – continuous) iff $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$–measurable (continuous) for all $\alpha \in A$.*

Proof. ($\Rightarrow$) If g is $(\mathcal{M}, \sigma(f_\alpha : \alpha \in A))$ – measurable, then the composition $f_\alpha \circ g$ is $(\mathcal{M}, \mathcal{F}_\alpha)$ – measurable by Lemma 7.6.

($\Leftarrow$) Let

$$\mathcal{G} = \sigma(f_\alpha : \alpha \in A) = \sigma\left(\cup_{\alpha \in A} f_\alpha^{-1}(\mathcal{F}_\alpha)\right).$$

If $f_\alpha \circ g$ is $(\mathcal{M},\mathcal{F}_\alpha)$ – measurable for all α, then

$$g^{-1}f_\alpha^{-1}(\mathcal{F}_\alpha) \subseteq \mathcal{M}\,\forall\,\alpha \in A$$

and therefore

$$g^{-1}\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right) = \cup_{\alpha\in A}g^{-1}f_\alpha^{-1}(\mathcal{F}_\alpha) \subseteq \mathcal{M}.$$

Hence

$$g^{-1}\left(\mathcal{G}\right) = g^{-1}\left(\sigma\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right)\right) = \sigma(g^{-1}\left(\cup_{\alpha\in A}f_\alpha^{-1}(\mathcal{F}_\alpha)\right) \subseteq \mathcal{M}$$

which shows that g is $(\mathcal{M},\mathcal{G})$ – measurable.

The topological case is proved in the same way. ■

Definition 7.22. Suppose$(X_\alpha\mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of measurable spaces and let X be the product space

$$X = \prod_{\alpha\in A} X_\alpha$$

and $\pi_\alpha : X \to X_\alpha$ be the canonical projection maps. Then the product σ – algebra, $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha \equiv \sigma(\pi_\alpha : \alpha\in A) = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Similarly if $(X_\alpha\mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of topological, the product topology $\bigotimes_\alpha \mathcal{M}_\alpha$, is defined by

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha \equiv \tau(\pi_\alpha : \alpha\in A) = \tau\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right).$$

Remark 7.23. Let $(Z,\mathcal{M})$ be a measurable (topological) space and

$$\left(X = \prod_{\alpha\in A} X_\alpha, \bigotimes_{\alpha\in A}\mathcal{M}_\alpha\right)$$

be as in Definition 7.22. By Proposition 7.21, a function $f : Z \to X$ is measurable (continuous) iff $\pi_\alpha \circ f$ is $(\mathcal{M},\mathcal{M}_\alpha)$ – measurable (continuous) for all $\alpha \in A$.

Proposition 7.24. *Suppose that $(X_\alpha\mathcal{M}_\alpha)_{\alpha\in A}$ is a collection of measurable (topological) spaces and $\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha$ generates $\mathcal{M}_\alpha$ for each $\alpha\in A$, then*

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \quad \left(\tau\left(\cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)\right)\right) \tag{7.4}$$

Moreover, suppose that A is either finite or countably infinite, $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha\in A$, and $\mathcal{M}_\alpha = \sigma(\mathcal{E}_\alpha)$ is a σ – algebra for all $\alpha\in A$. Then the product σ – algebra satisfies

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha\in A\right\}\right). \tag{7.5}$$

Similarly if A is finite and $\mathcal{M}_\alpha = \tau(\mathcal{E}_\alpha)$, then the product topology satisfies

$$\bigotimes_{\alpha\in A}\mathcal{M}_\alpha = \tau\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha\in A\right\}\right). \tag{7.6}$$

Proof. We will prove Eq. (7.4) in the measure theoretic case since a similar proof works in the topological category. Since $\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \cup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)$, it follows that

$$\mathcal{F} := \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha)\right) = \bigotimes_\alpha \mathcal{M}_\alpha.$$

Conversely,

$$\mathcal{F} \supset \sigma(\pi_\alpha^{-1}(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\sigma(\mathcal{E}_\alpha)) = \pi_\alpha^{-1}(\mathcal{M}_\alpha)$$

holds for all α implies that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{M}_\alpha) \subset \mathcal{F}$$

and hence that $\bigotimes_\alpha \mathcal{M}_\alpha \subseteq \mathcal{F}$.

We now prove Eq. (7.5). Since we are assuming that $X_\alpha \in \mathcal{E}_\alpha$ for each $\alpha \in A$, we see that

$$\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}$$

and therefore by Eq. (7.4)

$$\bigotimes_{\alpha\in A} \mathcal{M}_\alpha = \sigma\left(\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha)\right) \subset \sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right).$$

This last statement is true independent as to whether A is countable or not. For the reverse inclusion it suffices to notice that since A is countable,

$$\prod_{\alpha\in A} E_\alpha = \cap_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha) \in \bigotimes_{\alpha\in A} \mathcal{M}_\alpha$$

and hence

$$\sigma\left(\left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \text{ for all } \alpha \in A\right\}\right) \subset \bigotimes_{\alpha\in A} \mathcal{M}_\alpha.$$

Let us record the following special case of Proposition 7.24. ■

Corollary 7.25. *Suppose $(X_i, \mathcal{M}_i)$ are measurable (topological) spaces, $\mathcal{E}_i \subset \mathcal{P}(X_i)$ such that $X_i \in \mathcal{E}_i$ for $i = 1, 2$ and $\mathcal{M}_i = \sigma(\mathcal{E}_i)$ $(\mathcal{M}_i = \tau(\mathcal{E}_i))$. Then*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = \sigma(\mathcal{E}_1 \times \mathcal{E}_2) \quad (\tau(\mathcal{E}_1 \times \mathcal{E}_2)).$$

Proposition 7.26. *Let $\{X_\alpha\}_{\alpha\in A}$ be a sequence of sets where A is at most countable. Suppose for each $\alpha \in A$ we are given a countable set $\mathcal{E}_\alpha \subset \mathcal{P}(X_\alpha)$. Let $\tau_\alpha = \tau(\mathcal{E}_\alpha)$ be the topology on X_α generated by $\mathcal{E}_\alpha$ and X be the product space $\prod_{\alpha\in A} X_\alpha$ with equipped with the product topology $\tau := \otimes_{\alpha\in A}\tau(\mathcal{E}_\alpha)$. Then the Borel σ – algebra $\mathcal{B}_X = \sigma(\tau)$ is the same as the product σ – algebra:*

$$\mathcal{B}_X = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha},$$

where $\mathcal{B}_{X_\alpha} = \sigma(\tau(\mathcal{E}_\alpha)) = \sigma(\mathcal{E}_\alpha)$ for all $\alpha \in A$.

Proof. By Proposition 7.24, the topology τ may be described as the smallest topology containing $\mathcal{E} = \cup_{\alpha\in A}\pi_\alpha^{-1}(\mathcal{E}_\alpha)$. Now $\mathcal{E}$ is the countable union of countable sets so is still countable. Therefore by Proposition 3.14 and Proposition 7.24 we have

$$\mathcal{B}_X = \sigma(\tau) = \sigma(\tau(\mathcal{E})) = \sigma(\mathcal{E}) = \otimes_{\alpha\in A}\sigma(\mathcal{E}_\alpha) = \otimes_{\alpha\in A}\sigma(\tau_\alpha) = \otimes_{\alpha\in A}\mathcal{B}_{X_\alpha}.$$

■

Proposition 7.27. *If (X_i, ρ_i) for $i = 1, \ldots, n$ be metric spaces such that for each i there a countable dense subset $D_i \subseteq X_i$, then*

$$\bigotimes_i \mathcal{B}_{X_i} = \mathcal{B}_{(X_1 \times \cdots \times X_n)}$$

where $\mathcal{B}_{X_i}$ is the Borel σ – algebra on X_i and $\mathcal{B}_{(X_1 \times \cdots \times X_n)}$ is the Borel σ – algebra on $X_1 \times \cdots \times X_n$ equipped with the topology coming from the product metric

$$\rho((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum_{i=1}^n \rho_i(x_i, y_i).$$

Proof. Let $(x_1, x_2, \ldots, x_n) \in X_1 \times \cdots \times X_n$ and $\epsilon > 0$ be given. It is easily seen that

$$\prod_i B(x_i, \epsilon/n) \subset B((x_1, x_2, \ldots, x_n), \epsilon) \subset \prod_i B(x_i, \epsilon).$$

Because of this equation, it is easily seen that $\tau_\rho = \tau_{\rho_1} \otimes \tau_{\rho_2} \cdots \otimes \tau_{\rho_n}$ is just the product topology on $X_1 \times \cdots \times X_n$. Proof is now complete by applying Proposition 7.26 with

$$\mathcal{E}_i := \{B(x, \epsilon) \subset X_i : x \in D_i \text{ and } \epsilon \in \mathbb{Q} \cap (0, \infty)\}.$$

■

Corollary 7.28. *Let $\mathcal{B}_{\mathbb{R}^n}$ denote the Borel σ –algebra on $\mathbb{R}^n$, then $\mathcal{B}_{\mathbb{R}^n} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \cdots \otimes \mathcal{B}_{\mathbb{R}}$. Moreover if $(X, \mathcal{M})$ is a measurable space, then*

$$f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$$

is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}^n})$ – measurable iff $f_i : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable for each i. In particular, a function $f : X \to \mathbb{C}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ – measurable.

Corollary 7.29. *Let $(X, \mathcal{M})$ be a measurable space and $f, g : X \to \mathbb{C}$ be $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable functions. Then $f \pm g$ and $f \cdot g$ are also $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable.*

Proof. Define $F : X \to \mathbb{C} \times \mathbb{C}$, $A_\pm : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ by $F(x) = (f(x), g(x))$, $A_\pm(w, z) = w \pm z$ and $M(w, z) = wz$. Then $A_\pm$ and M are continuous and hence $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$ – measurable. Also F is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}}) = (\mathcal{M}, \mathcal{B}_{\mathbb{C}^2})$ – measurable since $\pi_1 \circ F = f$ and $\pi_2 \circ F = g$ are $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable measurable. Therefore $A_\pm \circ F = f \pm g$ and $M \circ F = f \cdot g$ being the composition of measurable functions are also measurable. ■

Lemma 7.30. *Let $\alpha \in \mathbb{C}$, $(X, \mathcal{M})$ be a measurable space and $f : X \to \mathbb{C}$ be a $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$ – measurable function. Then*

$$F(x) := \begin{cases} \frac{1}{f(x)} & \text{if} \quad f(x) \neq 0 \\ \alpha & \text{if} \quad f(x) = 0 \end{cases}$$

is measurable.

Proof. Define $i : \mathbb{C} \to \mathbb{C}$ by

$$i(z) = \begin{cases} \frac{1}{z} & \text{if} \quad z \neq 0 \\ \alpha & \text{if} \quad z = 0. \end{cases}$$

For any open set $V \subset \mathbb{C}$ we have

$$i^{-1}(V) = i^{-1}(V \setminus \{0\}) \cup i^{-1}(V \cap \{0\})$$

Because i is continuous except at $z = 0$, $i^{-1}(V \setminus \{0\})$ is an open set and hence in $\mathcal{B}_{\mathbb{C}}$. Moreover, $i^{-1}(V \cap \{0\}) \in \mathcal{B}_{\mathbb{C}}$ since $i^{-1}(V \cap \{0\})$ is either the empty set or the one point set $\{\alpha\}$. Therefore $i^{-1}(\tau_{\mathbb{C}}) \subseteq \mathcal{B}_{\mathbb{C}}$ and hence $i^{-1}(\mathcal{B}_{\mathbb{C}}) = i^{-1}(\sigma(\tau_{\mathbb{C}})) = \sigma(i^{-1}(\tau_{\mathbb{C}})) \subseteq \mathcal{B}_{\mathbb{C}}$ which shows that i is Borel measurable. Since $F = i \circ f$ is the composition of measurable functions, F is also measurable. ■

7.3. **Measurability on Complete Measure Spaces.** In this subsection we will discuss a couple of measurability results concerning completions of measure spaces. We will first need to introduce the notion of simple functions.

Definition 7.31. A function $f : (X, \mathcal{M}) \to \mathbb{C}$ is a **simple function** if f is measurable and the range of f is finite, i.e. $\#(f(X)) < \infty$.

A function $f : (X, \mathcal{M}) \to \mathbb{C}$ is a **characteristic function** if f is measurable and $f(X) = \{0, 1\}$. If we let $A := f^{-1}(\{1\})$, we will write

$$f(x) = 1_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

If f is simple function then f may be written as a finite linear combination of characteristic functions:

$$f = \sum_{z \in \mathbb{C}} z 1_{f^{-1}(\{z\})}$$

where the above sum is really a finite since the range of f is a finite set.

Theorem 7.32 (Approximation Theorem)**.** *Let $f : X \to [0, \infty]$ be measurable and define*

$$\begin{aligned} \phi_n(x) &\equiv \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)}(x) + 2^n 1_{f^{-1}((2^n, \infty])}(x) \\ &= \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{\left\{\frac{k}{2^n} < f \le \frac{k+1}{2^n}\right\}}(x) + 2^n 1_{\{f > 2^n\}}(x) \end{aligned}$$

then $\phi_n \le f$ for all n, $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on the sets $X_M := \{x \in X : f(x) \le M\}$ for all $M \in (0, \infty)$. Moreover, if $f : X \to \mathbb{C}$ is a measurable function, then there exists simple functions ϕ_n such that $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all x and $|\phi_n| \uparrow |f|$ as $n \to \infty$.

Proof. It is clear by construction that $\phi_n(x) \le f(x)$ for all x and that $0 \le f(x) - \phi_n(x) \le 2^{-n}$ if $x \in X_{2^n}$. From this it follows that $\phi_n(x) \uparrow f(x)$ for all $x \in X$ and $\phi_n \uparrow f$ uniformly on bounded sets.

Also notice that

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right] = \left(\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right] = \left(\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right] \cup \left(\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]$$

and for $x \in f^{-1}\left(\left(\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right]\right)$, $\phi_n(x) = \phi_{n+1}(x) = \frac{2k}{2^{n+1}}$ and for $x \in f^{-1}\left(\left(\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]\right)$, $\phi_n(x) = \frac{2k}{2^{n+1}} \le \frac{2k+1}{2^{n+1}} = \phi_{n+1}(x)$. Similarly since

$$(2^n, \infty] = (2^n, 2^{n+1}] \cup (2^{n+1}, \infty],$$

for $x \in f^{-1}((2^{n+1}, \infty])$ $\phi_n(x) = 2^n < 2^{n+1} = \phi_{n+1}(x)$ and for $x \in f^{-1}((2^n, 2^{n+1}])$, $\phi_{n+1}(x) \ge 2^n = \phi_n(x)$. Therefore $\phi_n \le \phi_{n+1}$ for all n and we have completed the proof of the first assertion.

For the second assertion, first assume that $f : X \to \mathbb{R}$ is a measurable function and choose $\phi_n^{\pm}$ to be simple functions such that $\phi_n^{\pm} \uparrow f_{\pm}$ as $n \to \infty$ and define $\phi_n = \phi_n^+ - \phi_n^-$. Then

$$|\phi_n| = \phi_n^+ + \phi_n^- \le \phi_{n+1}^+ + \phi_{n+1}^- = |\phi_{n+1}|$$

and clearly $|\phi_n| = \phi_n^+ + \phi_n^- \uparrow f_+ + f_- = |f|$ and $\phi_n = \phi_n^+ - \phi_n^- \to f_+ - f_- = f$ as $n \to \infty$.

Now suppose that $f : X \to \mathbb{C}$ is measurable. We may now choose simple function u_n and v_n such that $|u_n| \uparrow |\mathrm{Re}\, f|$, $|v_n| \uparrow |\mathrm{Im}\, f|$, $u_n \to \mathrm{Re}\, f$ and $v_n \to \mathrm{Im}\, f$ as $n \to \infty$. Let $\phi_n = u_n + iv_n$, then

$$|\phi_n|^2 = u_n^2 + v_n^2 \uparrow |\mathrm{Re}\, f|^2 + |\mathrm{Im}\, f|^2 = |f|^2$$

and $\phi_n = u_n + iv_n \to \mathrm{Re}\, f + i\,\mathrm{Im}\, f = f$ as $n \to \infty$. ■

Proposition 7.33. *Suppose that $(X, \mathcal{M}, \mu)$ is a complete measure space and $f : X \to \mathbb{R}$ is measurable.*

(1) *If $g : X \to \mathbb{R}$ is a function such that $f(x) = g(x)$ for μ – a.e. x, then g is measurable.*
(2) *If $f_n : X \to \mathbb{R}$ are measurable and $f : X \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} f_n = f$, μ - a.e., then f is measurable as well.*

Proof. 1. Let $E = \{x : f(x) \ne g(x)\}$ which is assumed to be in $\mathcal{M}$ and $\mu(E) = 0$. Then $g = 1_{E^c} f + 1_E g$ since $f = g$ on E^c. Now $1_{E^c} f$ is measurable so g will be measurable if we show $1_E g$ is measurable. For this consider,

$$(1_E g)^{-1}(A) = \begin{cases} E^c \cup (1_E g)^{-1}(A \setminus \{0\}) & \text{if } 0 \in A \\ (1_E g)^{-1}(A) & \text{if } 0 \notin A \end{cases} \tag{7.7}$$

Since $(1_E g)^{-1}(B) \subseteq E$ if $0 \notin B$ and $\mu(E) = 0$, it follow by completeness of $\mathcal{M}$ that $(1_E g)^{-1}(B) \in \mathcal{M}$ if $0 \notin B$ Therefore Eq. (7.7) shows that $1_E g$ is measurable.

2. Let $E = \{x : \lim\limits_{n\to\infty} f_n(x) \ne f(x)\}$ by assumption $E \in \mathcal{M}$ and $\mu(E) = 0$. Since $g \equiv 1_E f = \lim_{n\to\infty} 1_{E^c} f_n$, g is measurable. Because $f = g$ on E^c and $\mu(E) = 0$, $f = g$ a.e. so by part 1. f is also measurable. ■

The above results are in general false if $(X, \mathcal{M}, \mu)$ is not complete. For example, let $X = \{0, 1, 2\}$ $\mathcal{M} = \{\{0\}, \{1, 2\}, X, \phi\}$ and $\mu = \delta_0$ Take $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, then $g = 0$ a.e. yet g is not measurable.

Lemma 7.34. *Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $\bar{\mathcal{M}}$ is the completion of $\mathcal{M}$ relative to μ and $\bar{\mu}$ is the extension of μ to $\bar{\mathcal{M}}$. Then a function $f : X \to \mathbb{R}$ is $(\bar{\mathcal{M}}, \mathcal{B} = \mathcal{B}_{\mathbb{R}})$ – measurable iff there exists a function $g : X \to \mathbb{R}$ that is $(\mathcal{M}, \mathcal{B})$ – measurable such $E = \{x : f(x) \ne g(x)\} \in \bar{\mathcal{M}}$ and $\bar{\mu}(E) = 0$, i.e. $f(x) = g(x)$ for $\bar{\mu}$ – a.e. x.*

Proof. Suppose first that such a function g exists so that $\bar{\mu}(E) = 0$. Since g is also $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, we see from Proposition 7.33 that f is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable.

Conversely if f is $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable, by considering $f_{\pm}$ we may assume that $f \ge 0$. Choose $(\bar{\mathcal{M}}, \mathcal{B})$ – measurable simple function $\phi_n \ge 0$ such that $\phi_n \uparrow f$ as $n \to \infty$. Writing

$$\phi_n = \sum a_k 1_{A_k}$$

with $A_k \in \bar{\mathcal{M}}$, we may choose $B_k \in \mathcal{M}$ such that $B_k \subset A_k$ and $\bar{\mu}(A_k \setminus B_k) = 0$. Letting

$$\tilde{\phi}_n := \sum a_k 1_{B_k}$$

we have produced a $(\mathcal{M}, \mathcal{B})$ – measurable simple function $\tilde{\phi}_n \geq 0$ such that $E_n := \{\phi_n \neq \tilde{\phi}_n\}$ has zero $\bar{\mu}$ – measure. Since $\bar{\mu}(\cup_n E_n) \leq \sum_n \bar{\mu}(E_n)$, there exists $F \in \mathcal{M}$ such that $\cup_n E_n \subset F$ and $\mu(F) = 0$. It now follows that

$$1_F \tilde{\phi}_n = 1_F \phi_n \uparrow g := 1_F f \text{ as } n \to \infty.$$

This shows that $g = 1_F f$ is $(\mathcal{M}, \mathcal{B})$ – measurable and that $\{f \neq g\} \subset F$ has $\bar{\mu}$ – measure zero. ■

7.4. **Appendix: Special cases of the Product Theorems.**

Theorem 7.35. *Suppose that X_1 and X_2 are sets, $\tau_1 \subset \mathcal{P}(X_1)$ and $\tau_2 \subset \mathcal{P}(X_2)$ are topologies, $\mathcal{M}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{M}_2 \subset \mathcal{P}(X_2)$ are σ – algebras and $\mathcal{E}_1 \subset \mathcal{P}(X_1)$ and $\mathcal{E}_2 \subset \mathcal{P}(X_2)$ are such that $X_1 \in \mathcal{E}_1$, $X_2 \in \mathcal{E}_2$, $\tau_1 = \tau(\mathcal{E}_1)$, $\tau_2 = \tau(\mathcal{E}_2)$, $\mathcal{M}_1 = \sigma(\mathcal{E}_1)$, and $\mathcal{M}_2 = \sigma(\mathcal{E}_2)$. Then*

1. $\mathcal{M}_1 \otimes \mathcal{M}_2 = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$, *i.e.* $\sigma(\sigma(\mathcal{E}_1) \times \sigma(\mathcal{E}_2)) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$.
2. $\tau_1 \otimes \tau_2 = \tau(\mathcal{E}_1 \times \mathcal{E}_2)$, *i.e.* $\tau((\tau(\mathcal{E}_1) \times \tau(\mathcal{E}_2)) = \tau(\mathcal{E}_1 \times \mathcal{E}_2)$.
3. *If we further assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are countable and let $\mathcal{B}_1 = \sigma(\tau_1)$ and $\mathcal{B}_2 = \sigma(\tau_2)$ then*
   (a) $\mathcal{B}_1 = \sigma(\mathcal{E}_1) = \mathcal{M}_1$ *and* $\mathcal{B}_2 = \sigma(\mathcal{E}_2) = \mathcal{M}_2$.
   (b) $\sigma(\tau_1 \otimes \tau_2) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2) = \mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{B}_1 \otimes \mathcal{B}_2$. *That is the Borel σ – algebra on $X_1 \times X_2$ with the product topology is the product of the Borel σ – algebras on X_1 and X_2.*

Proof. Since $\mathcal{M}_1 \otimes \mathcal{M}_2 = \sigma(\mathcal{M}_1 \times \mathcal{M}_2) \supset \mathcal{M}_1 \times \mathcal{M}_2 \supset \mathcal{E}_1 \times \mathcal{E}_2$ it follows that $\mathcal{M}_1 \otimes \mathcal{M}_2 \supset \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$. For the reverse inclusion, let

$$\mathcal{M} := \{A \in \mathcal{M}_1 : A \times X_2 \in \sigma(\mathcal{E}_1 \times \mathcal{E}_2)\}.$$

It is routine to check that $\mathcal{M} \subset \mathcal{P}(X_1)$ is σ – algebra on X_1 and that $\mathcal{E}_1 \subset \mathcal{M}$. Therefore $\mathcal{M}_1 = \sigma(\mathcal{E}_1) \subset \mathcal{M}$, i.e. we have shown that $A \times X_2 \in \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$ for all $A \in \mathcal{M}_1$. By symmetry, we may show that $X_1 \times B \in \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$ for all $B \in \mathcal{M}_2$ and therefore

$$A \times B = (A \times X_2) \cap (X_1 \times B) \in \sigma(\mathcal{E}_1 \times \mathcal{E}_2) \; \forall A \in \mathcal{M}_1 \text{ and } B \in \mathcal{M}_2.$$

Therefore $\mathcal{M}_1 \otimes \mathcal{M}_2 \subset \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$ and we have proved item 1. Item 2. is proved in exactly the same way.

Item 3a. was already proved in Proposition 3.14. Similarly, by Item 2. $\tau_1 \otimes \tau_2 = \tau(\mathcal{E}_1 \times \mathcal{E}_2)$ by item 1. and because $\mathcal{E}_1 \times \mathcal{E}_2$ is still countable, we may apply Proposition 3.14 to shows that $\sigma(\tau_1 \otimes \tau_2) = \sigma(\mathcal{E}_1 \times \mathcal{E}_2)$. The other assertions in Item 3b. now follow from the previous assertions in the Theorem. ■

Corollary 7.36. $\mathcal{B}_{\mathbb{R}^{m+n}} = \mathcal{B}_{\mathbb{R}^n} \otimes \mathcal{B}_{\mathbb{R}^m}$.

Proof. Let $\mathcal{E}_1$ denote the collection of open balls in $\mathbb{R}^n$ with centers in $\mathbb{Q}^n$ and rational or infinite radii and let $\mathcal{E}_2$ denote the collection of open balls in $\mathbb{R}^m$ with centers in $\mathbb{Q}^m$ and rational or infinite radii. It is easy to check that $\tau(\mathcal{E}_1)$ and $\tau(\mathcal{E}_2)$ are the standard topologies on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. We may now finish the proof using Theorem 7.35 provided that we show $\tau(\mathcal{E}_1 \times \mathcal{E}_2)$ is the standard topology on $\mathbb{R}^{n+m}$. This is a consequence of the following assertions:

(1) Every set $E_1 \times E_2 \in \mathcal{E}_1 \times \mathcal{E}_2$ may be written as a union of open balls in $\mathbb{R}^{n+m}$, i.e. every set Every set $E_1 \times E_2 \in \mathcal{E}_1 \times \mathcal{E}_2$ is open in $\mathbb{R}^{n+m}$ with the standard topology, $\tau_{\mathbb{R}^{n+m}}$. (From this we conclude that $\tau(\mathcal{E}_1 \times \mathcal{E}_2) \subset \tau_{\mathbb{R}^{n+m}}$.
(2) Every open ball $B((x,y),\delta) \subset \mathbb{R}^{n+m}$ may be written as a union of elements in $\mathcal{E}_1 \times \mathcal{E}_2$. This shows that $B((x,y),\delta) \in \tau(\mathcal{E}_1 \times \mathcal{E}_2)$ and hence that $\tau_{\mathbb{R}^{n+m}} \subset \tau(\mathcal{E}_1 \times \mathcal{E}_2)$.

The proof of these assertions are left to the reader who may find the following observations useful. If $B(x,\delta) \in \mathcal{E}_1$ and $B(y,\epsilon) \in \mathcal{E}_2$ and $(a,b) \in B(x,\delta) \times B(y,\epsilon)$ then

$$|x-a|^2 + |y-b|^2 < \delta^2 + \epsilon^2$$

and hence

$$B(x,\delta) \times B(y,\epsilon) \subset B((x,y), \sqrt{\delta^2+\epsilon^2}). \tag{7.8}$$

Moreover we have

$$B((x,y), \min(\delta,\epsilon)) \subset B(x,\delta) \times B(y,\epsilon). \tag{7.9}$$

■

Lemma 7.37. *Suppose $(X_1,\mathcal{M}_1)$, $(X_2,\mathcal{M}_2)$ and $(X,\mathcal{M})$ are measurable spaces. A function $F(x) = (F_1(x), F_2(x))$ from $X \to X_1 \times X_2$ is $(\mathcal{M}, \mathcal{M}_1 \otimes \mathcal{M}_2)$ measurable iff $F_i : X \to X_i$ is $(\mathcal{M},\mathcal{M}_i)$ – measurable for $i = 1,2$.*

Proof. Let $\pi_i : X_1 \times X_2 \to X_i$ be the projection maps, i.e. $\pi_1(x_1,x_2) = x_1$ and $\pi_2(x_1,x_2) = x_2$. Since $\pi_1^{-1}(A) = A \times X_2 \in \mathcal{M}_1 \otimes \mathcal{M}_2$ for all $A \in \mathcal{M}_1$, π_1 is measurable and a similar argument shows that π_2 is measurable. Therefore if $F : X \to X_1 \times X_2$ is $(\mathcal{M}, \mathcal{M}_1 \otimes \mathcal{M}_2)$ measurable then $F_i = \pi_i \circ F$ is $(\mathcal{M},\mathcal{M}_i)$ – measurable for $i = 1,2$. Conversely if $F_i = \pi_i \circ F$ is $(\mathcal{M},\mathcal{M}_i)$ – measurable for $i = 1,2$, then

$$\begin{aligned} F^{-1}(A \times X_2) &= F_1^{-1}(A) \in \mathcal{M} \text{ and} \\ F^{-1}(X_1 \times B) &= F_2^{-1}(B) \in \mathcal{M} \end{aligned}$$

for all $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$. Since

$$\{A \times X_2, X_1 \times B : A \in \mathcal{M}_1 \text{ and } B \in \mathcal{M}_2\}$$

generate $\mathcal{M}_1 \otimes \mathcal{M}_2$, it follows that F is $(\mathcal{M}, \mathcal{M}_1 \otimes \mathcal{M}_2)$ – measurable. ■

8. Integration theory

8.1. **Integral of Simple functions.**

Let $(X,\mathcal{M},\mu)$ be a fixed measure space in this section.

Definition 8.1. Let $\mathbb{F} = \mathbb{C}$ or $[0,\infty]$ and suppose that $\phi : X \to \mathbb{F}$ is a simple function. If $\mathbb{F} = \mathbb{C}$ assume further that $\mu(\phi^{-1}(\{y\})) < \infty$ for all $y \neq 0$ in $\mathbb{C}$. For such functions ϕ we define $\int \phi = \int_X \phi \, d\mu$ by

$$\int_X \phi \, d\mu = \sum_{y \in \mathbb{F}} y \mu(\phi^{-1}(\{y\})).$$

Proposition 8.2. *The integral has the following properties.*

(1) *Suppose that* $\lambda \in \mathbb{C}$ *then*

$$\int_X \lambda f d\mu = \lambda \int_X f d\mu. \tag{8.1}$$

(2) *Suppose that* ϕ *and* ψ *are two simple functions, then*

$$\int (\phi + \psi) d\mu = \int \psi d\mu + \int \phi d\mu.$$

(3) *If* ϕ *and* ψ *are non-negative simple functions such that* $\phi \le \psi$ *then*

$$\int \phi d\mu \le \int \psi d\mu.$$

(4) *If* ϕ *is a non-negative simple function then* $A \to \nu(A) := \int_A \phi \, d\mu \equiv \int_X 1_A \phi \, d\mu$ *is a measure.*

Proof. Let us write $\{\phi = y\}$ for the set $\phi^{-1}(\{y\}) \subset X$ and $\mu(\phi = y)$ for $\mu(\{\phi = y\}) = \mu(\phi^{-1}(\{y\}))$ so that

$$\int \phi = \sum_{y \in \mathbb{C}} y \mu(\phi = y).$$

We will also write $\{\phi = a, \psi = b\}$ for $\phi^{-1}(\{a\}) \cap \psi^{-1}(\{b\})$. This notation is more intuitive for the purposes of this proof. Suppose that $\lambda \in \mathbb{F}$ then

$$\begin{aligned}
\int_X \lambda\phi \; d\mu &= \sum_{y \in \mathbb{F}} y \; \mu(\lambda\phi = y) \\
&= \sum_{y \in \mathbb{F}} y \; \mu(\phi = y/\lambda) \\
&= \sum_{z \in \mathbb{F}} \lambda z \; \mu(\phi = z) = \lambda \int_X \phi \; d\mu
\end{aligned}$$

provided that $\lambda \neq 0$. The case $\lambda = 0$ is clear, so we have proved 1.

Suppose that ϕ and ψ are two simple functions, then

$$\begin{aligned}
\int (\phi + \psi) \; d\mu &= \sum_{z \in \mathbb{F}} z \; \mu(\phi + \psi = z) \\
&= \sum_{z \in \mathbb{F}} z \; \mu\left(\cup_{\omega \in \mathbb{F}} \{\phi = \omega, \; \psi = z - \omega\}\right) \\
&= \sum_{z \in \mathbb{F}} z \sum_{\omega \in \mathbb{F}} \mu(\phi = \omega, \; \psi = z - \omega) \\
&= \sum_{z, \omega \in \mathbb{F}} (z + \omega) \mu(\phi = \omega, \; \psi = z) \\
&= \sum_{z \in \mathbb{F}} z \; \mu(\psi = z) + \sum_{\omega \in \mathbb{F}} \omega \; \mu(\phi = \omega) \\
&= \int \psi \; d\mu + \int \phi \; d\mu.
\end{aligned}$$

which proves 2.

For 3. if ϕ and ψ are non-negative simple functions such that $\phi \le \psi$

$$\begin{aligned}\int \phi &= \sum_{a\ge 0} a\mu(\phi = a) \\ &= \sum_{a,b\ge 0} a\mu(\phi = a, \psi = b) \\ &\le \sum_{a,b\ge 0} b\mu(\phi = a, \psi = b) \\ &= \sum_{b\ge 0} b\mu(\psi = b) = \int \psi,\end{aligned}$$

where in the third inequality we have used $\{\phi = a, \psi = b\} = \emptyset$ if $a > b$.

Finally for 4., write $\phi = \sum \lambda_i 1_{B_i}$ with $\lambda_i > 0$ and $B_i \in \mathcal{M}$, then

$$\nu(A) = \int 1_A \phi \, d\mu = \sum_{i=1}^N \lambda_i \; \mu(A \cap B_i).$$

The latter expression for ν is easily checked to be a measure. ■

8.2. **Integrals of positive functions.**

Definition 8.3. Let $L^+ = \{f : X \to [0, \infty] : f \text{ is measurable}\}$. Define

$$\int_X f d\mu = \sup \left\{ \int_X \phi \, d\mu : \phi \text{ is simple and } \phi \le f \right\}.$$

Because of item 3. of Proposition 8.2, this definition is consistent with our previous definition of the integral on non-negative simple functions. We say the $f \in L^+$ is **integrable** if

$$\int_X f d\mu < \infty.$$

Remark 8.4. Notice that we still have the monotonicity property: $0 \le f \le g$ then

$$\int_X f \le \int_X g$$

and for $c > 0$

$$\int_X cf = c \int_X f.$$

Also notice that if f is integrable, then $\mu(\{f = \infty\}) = 0$.

Lemma 8.5. *Let X be a set and $\rho : X \to [0, \infty]$ be a function, let $\mu = \sum_{x\in X} \rho(x)\delta_x$ on $\mathcal{M} = \mathcal{P}(X)$, i.e.*

$$\mu(A) = \sum_{x\in A} \rho(x).$$

If $f : X \to [0, \infty]$ is a function (which is necessarily measurable), then

$$\int_X f d\mu = \sum_X \rho f.$$

Proof. Suppose that $\phi : X \to [0,\infty]$ is a simple function, then $\phi = \sum_{z\in[0,\infty]} z1_{\phi^{-1}(\{z\})}$ and

$$\begin{aligned}\sum_X \rho\phi &= \sum_{x\in X}\rho(x)\sum_{z\in[0,\infty]} z1_{\phi^{-1}(\{z\})}(x)\\ &= \sum_{z\in[0,\infty]} z\sum_{x\in X}\rho(x)1_{\phi^{-1}(\{z\})}(x)\\ &= \sum_{z\in[0,\infty]} z\mu(\phi^{-1}(\{z\})) = \int_X \phi d\mu.\end{aligned}$$

So on simple function $\phi : X \to [0,\infty]$,

$$\sum_X \rho\phi = \int_X \phi d\mu.$$

Suppose that $\phi : X \to [0,\infty)$ is a simple function such that $\phi \le f$, then

$$\int_X \phi d\mu = \sum_X \rho\phi \le \sum_X \rho f.$$

Taking the sup over ϕ in this last equation then shows that

$$\int_X f d\mu \le \sum_X \rho f.$$

For the reverse inequality, let $\Lambda \subset\subset X$ be a finite set and $N \in (0,\infty)$. Set $f^N(x) = \min\{N, f(x)\}$ and let $\phi_{N,\Lambda}$ be the simple function given by $\phi_{N,\Lambda}(x) := 1_\Lambda(x)f^N(x)$. Because $\phi_{N,\Lambda}(x) \le f(x)$,

$$\sum_\Lambda \rho f^N = \sum_X \rho\phi_{N,\Lambda} = \int_X \phi_{N,\Lambda}d\mu \le \int_X f d\mu.$$

Since $f^N \uparrow f$ as $N \to \infty$, we may let $N \to \infty$ in this last equation to concluded that

$$\sum_\Lambda \rho f \le \int_X f d\mu$$

and since Λ is arbitrary we learn that

$$\sum_X \rho f \le \int_X f d\mu.$$

■

Theorem 8.6 (Monotone Convergence Theorem)**.** *Suppose $f_n \in L^+$ is a sequence of functions such that $f_n \uparrow f$ (necessarily in L^+) then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

Proof. Since $f_n \le f_m \le f$, for all $n \le m < \infty$,

$$\int f_n \le \int f_m \le \int f$$

from which if follows $\int f_n$ is increasing in n and

$$\lim_{n\to\infty}\int f_n \le \int f.$$

For the opposite inequality, let ϕ be a simple function such that $0 \le \phi \le f$ and let $\alpha \in (0,1)$. Notice that

$$E_n \equiv \{f_n \ge \alpha\phi\} \uparrow X \text{ as } n \to \infty$$

and that, by Proposition 8.2,

$$\int f_n \ge \int 1_{E_n} f_n \ge \int_{E_n} \alpha\phi = \alpha \int_{E_n} \phi. \tag{8.2}$$

Because $E \to \alpha \int_E \phi$ is a measure and $E_n \uparrow X$,

$$\lim_{n\to\infty} \int_{E_n} \phi = \int_X \phi d\mu.$$

Hence we may pass to the limit in Eq. (8.2) to get

$$\lim_{n\to\infty} \int f_n \ge \alpha \int \phi.$$

Because this equation is valid for all simple functions $0 \le \phi \le f$, by the definition of $\int f$ we have

$$\lim_{n\to\infty} \int f_n \ge \alpha \int f.$$

Since $\alpha \in (0,1)$ is arbitrary we conclude that

$$\lim_{n\to\infty} \int f_n \ge \int f.$$

■

Corollary 8.7. *If $f_n \in L^+$ is a sequence of functions then*

$$\int \sum_n f_n = \sum_n \int f_n.$$

Proof. First off we show that

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$

by choosing non-negative simple function ϕ_n and ψ_n such that $\phi_n \uparrow f_1$ and $\psi_n \uparrow f_2$. Then $(\phi_n + \psi_n)$ is simple as well and $(\phi_n + \psi_n) \uparrow (f_1 + f_2)$ so that by the monotone convergence theorem,

$$\begin{aligned}\int (f_1 + f_2) &= \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \left(\int \phi_n + \int \psi_n \right) \\ &= \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2.\end{aligned}$$

Now to the general case. Let $g_N \equiv \sum_{n=1}^{N} f_n$ and $g = \sum_1^\infty f_n$, then $g_N \uparrow g$ and so by monotone convergence theorem and the additivity just proved,

$$\begin{aligned}\sum_{n=1}^\infty \int f_n &:= \lim_{N\to\infty} \sum_{n=1}^N \int f_n = \lim_{N\to\infty} \int \sum_{n=1}^N f_n \\ &= \lim_{N\to\infty} \int g_N = \int g = \sum_{n=1}^\infty \int f_n.\end{aligned}$$

■

The following Lemma is a simple application of this Corollary.

Lemma 8.8 (First Borell-Carnteli- Lemma.)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, $A_n \in \mathcal{M}$, and set*

$$\begin{aligned} \{A_n \text{ i.o.}\} &= \{x \in X : x \in A_n \text{ for infinitely many } n\text{'s}\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} A_n. \end{aligned}$$

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(\{A_n \text{ i.o.}\}) = 0$.

Proof. (First Proof.) Let us first observe that

$$\{A_n \text{ i.o.}\} = \left\{ x \in X : \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty \right\}.$$

Hence if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\infty > \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \int_X 1_{A_n} \, d\mu = \int_X \sum_{n=1}^{\infty} 1_{A_n} \, d\mu$$

implies that $\sum\limits_{n=1}^{\infty} 1_{A_n}(x) < \infty$ for μ - a.e. x. That is to say $\mu(\{A_n \text{ i.o.}\}) = 0$.

(Second Proof.) Of course we may give a strictly measure theoretic proof of this fact:

$$\begin{aligned} \mu(A_n \text{ i.o.}) &= \lim_{N \to \infty} \mu \left(\bigcup_{n \geq N} A_n \right) \\ &\leq \lim_{N \to \infty} \sum_{n \geq N} \mu(A_n) \end{aligned}$$

and the last limit is zero since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. ■

Example 8.9. Suppose that $f \in C([0,1])$ and $f \geq 0$. Let $\pi_k = \{0 = a_0 < a_1 < \cdots < a_{n_k} = 1\}$ be a sequence of refining partitions such that $\text{mesh}(\pi_k) \to 0$ as $k \to \infty$. Let

$$f_k(x) = f(0) 1_{\{0\}} + \sum_{\pi_k} \min \{f(x) : a_k \leq x \leq a_{k+1}\} 1_{(a_k, a_{k+1}]}(x)$$

then $f_k \uparrow f$ as $k \to \infty$ so that by the monotone convergence theorem,

$$\begin{aligned} \int_0^1 f dm &= \lim_{k \to \infty} \int_0^1 f_k \, dm \\ &= \lim_{k \to \infty} \sum_{\pi_k} \min \{f(x) : a_k \leq x \leq a_{k+1}\} \, m\left((a_{k+1} a_k]\right) \\ &= \int_0^1 f(x) dx \end{aligned}$$

where the latter integral is the Riemann integral.

Example 8.10. Let m be Lebesgue measure on $\mathbb{R}$, then

$$\begin{aligned}\int_{(0,1]} \frac{1}{x^p}\, dm(x) &= \lim_{n\to\infty} \int_0^1 1_{(\frac{1}{n},1]}(x)\frac{1}{x^p} dm(x) \\ &= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{1}{x^p} dx = \lim_{n\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_{1/n}^1 \\ &= \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \infty & \text{if } p>1 \end{cases}\end{aligned}$$

If $p = 1$ we find

$$\int_{(0,1]} \frac{1}{x^p}\, dm(x) = \lim_{n\to\infty} \int_{\frac{1}{n}}^1 \frac{1}{x} dx = \lim_{n\to\infty} \ln(x)|_{1/n}^1 = \infty.$$

Example 8.11. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of the points in $\mathbb{Q} \cap [0,1]$ and define

$$\frac{1}{\sqrt{|x-r_n|}} = 5 \text{ if } x = r_n$$

and

$$f(x) = \sum_{n=1}^\infty 2^{-n} \frac{1}{\sqrt{|x-r_n|}}.$$

Then

$$\int_0^1 \frac{1}{\sqrt{|x-r_n|}}\, dx \le 4 \int_0^1 f(x)dx \le 4$$

and hence

$$\int_{[0,1]} f(x)dm(x) \le 4 < \infty$$

which shows that $m(f = \infty) = 0$, i.e. that $f < \infty$ for almost every $x \in [0,1]$ and this implies that

$$\sum_{n=1}^\infty 2^{-n} \frac{1}{\sqrt{|x-r_n|}} < \infty \text{ for a.e. } x.$$

The following simple lemma will often be useful.

Lemma 8.12 (Chevbyshev's Iinequality)**.** *Suppose that $f \ge 0$ is a measurable function, then for any $\epsilon > 0$,*

$$\mu(\{f \ge \epsilon\}) \le \frac{1}{\epsilon} \int_X f d\mu. \tag{8.3}$$

Proof. Since $1_{\{f\ge\epsilon\}} \le 1_{\{f\ge\epsilon\}} \frac{1}{\epsilon} f \le \frac{1}{\epsilon} f$,

$$\mu(\{f \ge \epsilon\}) = \int_X 1_{\{f\ge\epsilon\}} d\mu \le \int_X 1_{\{f\ge\epsilon\}} \frac{1}{\epsilon} f d\mu \le \frac{1}{\epsilon} \int_X f d\mu.$$

■

Proposition 8.13. *Suppose that $f \ge 0$ is a measurable function. Then $\int_X f d\mu = 0$ iff $f = 0$ a.e. Also if $f, g \ge 0$ are measurable functions such that $f \le g$ a.e. then $\int f d\mu \le \int g d\mu$. In particular if $f = g$ a.e. then $\int f d\mu = \int g d\mu$.*

Proof. If $f = 0$ a.e. and $\phi \leq f$ is a simple function then $\phi = 0$ a.e. This implies that $\mu(\phi^{-1}(\{y\})) = 0$ for all $y > 0$ and hence $\int_X \phi d\mu = 0$ and therefore $\int_X f d\mu = 0$.

Conversely, if $\int f d\mu = 0$, let $E_n = \{f \geq \frac{1}{n}\}$. Then

$$0 = \int_{E_n} f \geq \int_{E_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(E_n)$$

which shows that $\mu(E_n) = 0$ for all n. Since $\{f > 0\} = \cup E_n$, we have

$$\mu(\{f > 0\}) \leq \sum_n \mu(E_n) = 0,$$

i.e. $f = 0$ a.e.

For the second assertion let $E \in \mathcal{M}$ be a set such that $\mu(E^c) = 0$ and $1_E f \leq 1_E g$ everywhere. Because $g = 1_E g + 1_{E^c} g$ and $1_{E^c} g = 0$ a.e.,

$$\int g d\mu = \int 1_E g d\mu + \int 1_{E^c} g d\mu = \int 1_E g d\mu$$

and similarly $\int f d\mu = \int 1_E f d\mu$. Since $1_E f \leq 1_E g$ everywhere,

$$\int f d\mu = \int 1_E f d\mu \leq \int 1_E g d\mu = \int g d\mu.$$

■

Corollary 8.14. *Suppose that $\{f_n\}$ is a sequence of non-negative functions and f is a measurable function such that off a set of measure zero, $f_n \uparrow f$, then*

$$\int f_n \uparrow \int f \text{ as } n \to \infty.$$

Proof. Let $E \subseteq X$ such that $\mu(X \setminus E) = 0$ and $f_n 1_E \uparrow f 1_E$. Then by the monotone convergence theorem,

$$\int f_n = \int f_n 1_E \uparrow \int f 1_E = \int f \text{ as } n \to \infty.$$

■

Lemma 8.15 (Fatou's Lemma)**.** *If $f_n : X \to [0, \infty]$ is a sequence of measurable functions then*

$$\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$$

Proof. Define $g_k \equiv \inf_{n \geq k} f_n$ so that $g_k \uparrow \liminf_{n\to\infty} f_n$ as $k \to \infty$. Since $g_k \leq f_n$ for all $k \leq n$ we have

$$\int g_k \leq \int f_n \text{ for all } n \geq k$$

and therefore

$$\int g_k \leq \lim \inf_{n\to\infty} \int f_n \text{ for all } k.$$

We may now use the monotone convergence theorem to let $k \to \infty$ to find

$$\int \lim \inf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k \overset{\text{MCT}}{=} \lim_{k\to\infty} \int g_k \leq \lim \inf_{n\to\infty} \int f_n.$$

■

8.2.1. *Integrals of Complex Valued Functions.*

Definition 8.16. A measurable function $f : \mathbb{R} \rightarrow [-\infty, \infty]$ is **integrable** if $f_+ \equiv f 1_{\{f\geq 0\}}$ and $f_- = -f\ 1_{\{f\leq 0\}}$ are **integrable**. We write L^1 for the space of integrable functions. For $f \in L^1$, let

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

Remark 8.17. Notice that if f is integrable, then

$$f_\pm \leq |f| \leq f_+ + f_-$$

so that f is integrable iff

$$\int |f|\ d\mu < \infty.$$

Proposition 8.18. *The map*

$$f \in L^1 \rightarrow \int_X f d\mu \in \mathbb{R}$$

is linear. Also if $f, g \in L^1$ are real valued functions such that $f \leq g$, the $\int f d\mu \leq \int g d\mu$.

Proof. If $f, g \in L^1$ and $a, b \in \mathbb{R}$, then

$$|af + bg| \leq |a||f| + |b|\,|g| \in L^1.$$

For $a \in \mathbb{R}$, say $a < 0$,

$$(af)_+ = -af_- \text{ and } (af)_- = -af_+$$

so that

$$\int af = -a \int f_- + a \int f_+ = a(\int f_+ - \int f_-) = a \int f.$$

A similar calculation works for $a > 0$ and the case $a = 0$ is trivial so we have shown that

$$\int af = a \int f.$$

Now set $h = f + g$. Since $h = h_+ - h_-$,

$$h_+ - h_- = f_+ - f_- + g_+ - g_-$$

or

$$h_+ + f_- + g_- = h_- + f_+ + g_+.$$

Therefore,

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

and hence

$$\int h = \int h_+ - \int h_- = \int f_+ + \int g_+ - \int f_- - \int g_- = \int f + \int g.$$

Finally if $f_+ - f_- = f \leq g = g_+ - g_-$ then $f_+ + g_- \leq g_+ + f_-$ which implies that

$$\int f_+ + \int g_- \leq \int g_+ + \int f_-$$

or equivalently that

$$\int f = \int f_+ - \int f_- \leq \int g_+ - \int g_- = \int g.$$

■

Definition 8.19. A measurable function $f : X \to \mathbb{C}$ is integrable if $\int_X |f| \, d\mu < \infty$, again we write $f \in L^1$. One shows that $\int |f| \, d\mu < \infty$ iff

$$\int |\mathrm{Re}\, f| \, d\mu + \int |\mathrm{Im}\, f| \, d\mu < \infty.$$

For $f \in L^1$ define

$$\int f \, d\mu = \int \mathrm{Re}\, f \, d\mu + i \int \mathrm{Im}\, f \, d\mu.$$

It is routine to show that the integral is still linear on the complex L^1 (prove!).

Proposition 8.20. *Suppose that $f \in L^1$, then*

$$\left| \int_X f d\mu \right| \le \int_X |f| d\mu.$$

Proof. Start by writing $\int_X f \, d\mu = Re^{i\theta}$. Then

$$\begin{aligned} \left| \int_X f d\mu \right| &= R = e^{-i\theta} \int_X f d\mu = \int_X e^{-i\theta} f d\mu \\ &= \int_X \mathrm{Re}\left(e^{-i\theta} f\right) d\mu. \end{aligned}$$

Let $g := \mathrm{Re}\left(e^{-i\theta} f\right) = g_+ - g_-$ then combining the previous equation with the following estimate proves the theorem.

$$\begin{aligned} \int_X g &= \int_X g_+ - \int_X g \le \int_X g_+ + \int_X g_- \\ &= \int_X g_+ + g_- = \int_X |g| d\mu \\ &= \int_X |\mathrm{Re}(e^{-i\theta} f)| d\mu \le \int_X |f| d\mu. \end{aligned}$$

■

Proposition 8.21. *Let $f, g \in L^1$, then*

(1) *The set $\{f \neq 0\}$ is σ-finite, i.e. there exists $E_n \in \mathcal{M}$ such that $\mu(E_n) < \infty$ and $E_n \uparrow \{f \neq 0\}$.*
(2) *The following are equivalent*
   (a) $\int_E f = \int_E g$ *for all $E \in \mathcal{M}$*
   (b) $\int_X |f - g| = 0$
   (c) $f = g$ *a.e.*

Proof. 1. The sets $E_n := \{|f| \ge \frac{1}{n}\}$ satisfy the conditions in item 1. since clearly $E_n \uparrow \{f \neq 0\}$ and by Chevbyshev's inequality (8.3),

$$\mu(E_n) \le \frac{1}{\epsilon} \int_X |f| d\mu < \infty.$$

2. (a) $\Longrightarrow$ (c)Notice that

$$\int_E f = \int_E g \Leftrightarrow \int_E (f - g) = 0$$

for all $E \in \mathcal{M}$. Taking $E = \{\operatorname{Re}(f-g) > 0\}$ and using $1_E \operatorname{Re}(f-g) \geq 0$, we learn that

$$0 = \operatorname{Re} \int_E (f-g) d\mu = \int 1_E \operatorname{Re}(f-g) \Longrightarrow 1_E \operatorname{Re}(f-g) = 0 \text{ a.e.}$$

This implies that $1_E = 0$ a.e. which happens iff

$$\mu\left(\{\operatorname{Re}(f-g) > 0\}\right) = \mu(E) = 0.$$

Similar $\mu(\operatorname{Re}(f-g) < 0) = 0$ so that $\operatorname{Re}(f-g) = 0$ a.e. Similarly, $\operatorname{Im}(f-g) = 0$ a.e and hence $f - g = 0$ a.e., i.e. $f = g$ a.e.

(c) $\Longrightarrow$ (b) is clear and so is (b) $\Longrightarrow$ (a) since

$$\left|\int_E f - \int_E g\right| \leq \int |f-g| = 0.$$

■

Corollary 8.22. *Suppose that $(X, \mathcal{M}, \mu)$ be a measure space and $\{A_n\}_{n=1}^\infty \subset \mathcal{M}$ is a collection of sets such that $\mu(A_i \cap A_j) = 0$ for all $i \neq j$, then*

$$\mu\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n).$$

Proof. Since

$$\begin{aligned} \mu\left(\cup_{n=1}^\infty A_n\right) &= \int_X 1_{\cup_{n=1}^\infty A_n} d\mu \text{ and} \\ \sum_{n=1}^\infty \mu(A_n) &= \int_X \sum_{n=1}^\infty 1_{A_n} d\mu \end{aligned}$$

it suffices to show that

$$\sum_{n=1}^\infty 1_{A_n} = 1_{\cup_{n=1}^\infty A_n} \ \mu - \text{a.e.} \tag{8.4}$$

Now $\sum_{n=1}^\infty 1_{A_n} \geq 1_{\cup_{n=1}^\infty A_n}$ and $\sum_{n=1}^\infty 1_{A_n}(x) \neq 1_{\cup_{n=1}^\infty A_n}(x)$ iff $x \in A_i \cap A_j$ for some $i \neq j$, that is

$$\left\{x : \sum_{n=1}^\infty 1_{A_n}(x) \neq 1_{\cup_{n=1}^\infty A_n}(x)\right\} = \cup_{i<j} A_i \cap A_j$$

and the later set has measure 0 being the countable union of sets of measure zero. This proves Eq. (8.4) and hence the corollary. ■

Definition 8.23. Let $(X, \mathcal{M}, \mu)$ be a measure space and $L^1(\mu) = L^1(X, \mathcal{M}, \mu)$ denote the set of L^1 functions modulo the equivalence relation $f \sim g$ iff $f = g$ a.e. We make this into a normed space using the norm

$$\|f - g\|_{L^1} = \int |f-g| \, d\mu$$

and into a metric space using $\rho_1(f,g) = \|f-g\|_{L^1}$.

Remark 8.24. More generally we may define $L^p(\mu) = L^p(X, \mathcal{M}, \mu)$ for $p \in [1, \infty)$ as the set of measurable functions f such that

$$\int_X |f|^p \, d\mu < \infty$$

modulo the equivalence relation $f \sim g$ iff $f = g$ a.e.

We make $L^p(\mu) = L^p(X, \mathcal{M}, \mu)$ into a normed space using the norm

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu\right)^{1/p}.$$

We will see next quarter that $\|\cdot\|_{L^p}$ has the following properties:

$$\begin{aligned} \|\lambda f\|_{L^p} &= |\lambda| \, \|f\|_{L^p} \text{ and} \\ \|f+g\|_{L^p} &\leq \|f\|_{L^p} + \|g\|_{L^p} \end{aligned} \tag{8.5}$$

for all $f, g \in L^p$. In particular $\rho_p(f,g) = \|f-g\|_{L^p}$ is a metric on L^p.

Theorem 8.25 (Dominated Convergence Theorem). *Suppose $f_n \to f$ a.e. $|f_n| \leq g \in L^1$. Then $f \in L^1$ and*

$$\int_X f d\mu = \lim_{h\to\infty} \int_X f_n d\mu.$$

Proof. Notice that $|f| = \lim |f_n| \leq g$ a.e. so that $f \in L^1$. By considering the real and imaginary parts of f separately, it suffices to prove the theorem in the case where f is real. By Fatou's Lemma,

$$\begin{aligned} \int_X (g \pm f) d\mu &= \int_X \liminf \, (g \pm f_n) \, d\mu \leq \liminf \int_X (g \pm f_n) \, d\mu \\ &= \int_X g d\mu + \liminf \left(\pm \int_X f_n d\mu\right). \end{aligned}$$

Since $\liminf(-a_n) = -\limsup a_n$, we have shown,

$$\int_X g d\mu \pm \int_X f d\mu \leq \int_X g d\mu + \begin{cases} \liminf \int_X f_n d\mu \\ -\limsup \int_X f_n d\mu \end{cases}$$

and therefore

$$\limsup \int_X f_n d\mu \leq \int_X f d\mu \leq \liminf \int_X f_n d\mu.$$

This shows that $\lim_{n\to\infty} \int_X f_n d\mu$ exists and is equal to $\int_X f d\mu$. ∎

Corollary 8.26 (Differentiation Under the Integral). *Suppose that $J \subset \mathbb{R}$ is an open interval and $f : J \times X \to \mathbb{C}$ is a function such that*

(1) *$f(t, \cdot) \in L^1$ for all $t \in J$,*
(2) *$\frac{\partial f}{\partial t}(t,x)$ exists for all (t,x)*
(3) *There is a function $g \in L^1$ such that $\left|\frac{\partial f}{\partial t}(t,x)\right| \leq g(x) \in L^1$.*
Then

$$\frac{\partial}{\partial t} \int f(t,x) d\mu(x) = \int \frac{\partial f}{\partial t}(t,x) d\mu(x).$$

Proof. The proof is the same as that case for sums that you did in one of your homework problems. ∎

Exercise 8.27. Show

$$\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n dm(x) = 1.$$

Solution. Let $f_n(x) = (1 - \frac{x}{n})^n 1_{[0,n]}(x)$ and notice that $\lim_{n\to\infty} f_n(x) = e^{-x}$. We will now show

$$0 \le f_n(x) \le e^{-x} \text{ for all } x \ge 0.$$

It suffices to consider $x \in [0, n]$. Let $g(x) = e^x f_n(x)$, then for $x \in (0, n)$,

$$\frac{d}{dx} \ln g(x) = 1 + n \frac{1}{(1 - \frac{x}{n})}(-\frac{1}{n}) = 1 - \frac{1}{(1 - \frac{x}{n})} \le 0$$

which shows that $\ln g(x)$ and hence $g(x)$ is decreasing on $[0, n]$. Therefore $g(x) \le g(0) = 1$, i.e

$$0 \le f_n(x) \le e^{-x}.$$

Now by the Monotone convergence theorem,

$$\int_0^\infty e^{-x} dm(x) = \lim_{M\to\infty} \int_0^M e^{-x} dm(x) = \lim_{M\to\infty} (1 - e^{-M}) = 1 < \infty,$$

so that e^{-x} is an integrable function on $[0, \infty)$. Therefore we may now apply the dominated convergence theorem to learn

$$\begin{aligned} \lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n dm(x) &= \lim_{n\to\infty} \int_0^\infty f_n(x) dm(x) \\ &= \int_0^\infty \lim_{n\to\infty} f_n(x) dm(x) = \int_0^\infty e^{-x} dm(x) = 1. \end{aligned}$$

■

8.3. **Comparison to the Riemann Integral.** In this section, suppose $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$ be a bounded function. To each partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\} \tag{8.6}$$

of $[a, b]$ let

$$\begin{aligned} S_P f &= \sum M_j (t_j - t_{j-1}) \\ s_P f &= \sum m_j (t_j - t_{j-1}) \end{aligned}$$

where

$$\begin{aligned} M_j &= \sup\{f(x) : t_j < x \le t_{j-1}\} \\ m_j &= \inf\{f(x) : t_j < x \le t_{j-1}\} \end{aligned}$$

and define the upper and lower Riemann integrals by

$$\begin{aligned} \overline{\int_a^b} f(x) dx &= \inf_P S_P f \text{ and} \\ \underline{\int_b^a} f(x) dx &= \sup_P \; s_P f \end{aligned}$$

respectively.

Fact 8.28. Recall the following fact from the theory of Riemann integrals. There exists a refining sequence of partitions P_k (i.e. the P_k 's are increasing) such that

$$S_{P_k} f \searrow \overline{\int_a^b} f \text{ as } k \to \infty \text{ and}$$
$$s_{P_k} f \uparrow \underline{\int_a^b} f \text{ as } k \to \infty.$$

Definition 8.29. The function f is **Riemann integrable** iff $\overline{\int_a^b} f = \underline{\int_a^b} f$ and which case the Riemann integral $\int_a^b f$ is defined to be the common value:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

For a partition P as in Eq. (8.6) let

$$G_P = \sum_1^n M_j 1_{(t_{j-1},t_j]} \text{ and } g_P = \sum_1^n m_j 1_{(t_{j-1},t_j]}.$$

If P_k is a sequence of refining partitions as in Fact 8.28, then G_{P_k} is a decreasing sequence, g_{P_k} is an increasing sequence and $g_{P_k} \le f \le G_{P_k}$ for all k. Define

$$G \equiv \lim_{k\to\infty} G_{P_k} \text{ and } g \equiv \lim_{k\to\infty} g_{P_k}. \tag{8.7}$$

and notice that $g \le f \le G$. By the dominated convergence theorem,

$$\int_{[a,b]} gdm = \lim_{k\to\infty} \int_{[a,b]} g_{P_k} = \lim_{k\to\infty} s_{P_k} f = \underline{\int_a^b} f(x)dx$$
and
$$\int_{[a,b]} Gdm = \lim_{k\to\infty} \int_{[a,b]} G_{P_k} = \lim_{k\to\infty} S_{P_k} f = \overline{\int_a^b} f(x)dx.$$

Therefore f is Riemann integrable iff $\int_{[a,b]} G = \int_{[a,b]} g$ i.e. iff $\int_{[a,b]} G - g = 0$. Since $G \ge f \ge g$ this happens iff $G = g$ a.e. Hence we have proved the following theorem.

Theorem 8.30. *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff the Borel measurable functions $g, G : [a,b] \to \mathbb{R}$ defined in Eq. (8.7) satisfy $g(x) = G(x)$ for m – a.e. $x[a,b]$. Moreover if f is Riemann integrable, then*

$$\int_a^b f(x)dx = \int_{[a,b]} gdm = \int_{[a,b]} Gdm.$$

The function f need not be Borel measurable but it is necessarily Lebesgue measurable, i.e. f is $\mathcal{L}/\mathcal{B}$ – measurable where $\mathcal{L}$ is the Lebesgue σ – algebra and $\mathcal{B}$ is the Borel σ – algebra on $[a,b]$. If we let $\bar{m}$ denote the completion of m, then we may also write

$$\int_a^b f(x)dx = \int_{[a,b]} fd\bar{m}.$$

9. Modes of Convergence

As usual let $(X, \mathcal{M}, \mu)$ be a fixed measure space and let $\{f_n\}$ be a sequence of measurable functions on X. Also let $f : X \to \mathbb{C}$ be a measurable function.

Definition 9.1. We have the following notions of convergence.

1. $f_n \to f$ a.e. if there is a set $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $\lim_{n\to\infty} 1_E f_n = 1_E f$.
2. $f_n \to f$ in μ – measure if $\lim_{n\to\infty} \mu(|f_n - f| > \epsilon) = 0$ for all $\epsilon > 0$. We will abbreviate this by saying $f_n \to f$ in L^0 or by $f_n \xrightarrow{\mu} f$
3. $f_n \to f$ in L^p iff $f \in L^p$ and $f_n \in L^p$ for all n, and $\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0$.

Definition 9.2. We have the following notions of Cauchy sequences.

1. $\{f_n\}$ is a.e. Cauchy if there is a set $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and$\{1_E\, f_n\}$ is a pointwise Cauchy sequences
2. $\{f_n\}$ is Cauchy in μ – measure if $\lim_{m,n\to\infty} \mu(|f_n - f_m| > \epsilon) = 0$ for all $\epsilon > 0$.
3. $\{f_n\}$ is Cauchy in L^p if $\lim_{m,n\to\infty} \int |f_n - f_m|^p d\mu = 0$.

Lemma 9.3. *Suppose $a_n \in \mathbb{R}$ or $\mathbb{C}$ and $|a_{n+1} - a_n| \le \epsilon_n$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Then $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ or $\mathbb{C}$ exists and $|a - a_n| \le \delta_n \equiv \sum_{k=n}^{\infty} \epsilon_k$.*

Proof. Let $m > n$ then

$$|a_m - a_n| = \left|\sum_{k=n}^{m-1} (a_{k+1} - a_k)\right| \le \sum_{k=nj}^{m-1} |a_{k+1} - a_k| \le \sum_{k=n}^{\infty} \epsilon_k \equiv \delta_n \quad . \tag{9.1}$$

So $|a_m - a_n| \le \delta_{\min(m,n)} \to 0$ as $,m, n \to \infty$, i.e. $\{a_n\}$ is Cauchy. Let $m \to \infty$ in (9.1) to find $|a - a_n| \le \delta_n$. ■

Theorem 9.4. *Suppose $\{f_n\}$ is L^0-Cauchy. Then there exists a subsequence $g_j = f_{n_j}$ of $\{f_n\}$ such that $\lim g_j \equiv f$ exists a.e. and $f_n \xrightarrow{\mu} f$ as $n \to \infty$. Moreover if g is a measurable function such that $f_n \xrightarrow{\mu} g$ as $n \to \infty$, then $f = g$ a.e.*

Proof. Let $\epsilon_n > 0$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$ ($\epsilon_n = 2^{-n}$ would do) and set $\delta_n = \sum_{k=n}^{\infty} \epsilon_k$. Choose $g_j = f_{n_j}$ such that ($n_j \uparrow$) and

$$\mu(\{|g_{j+1} - g_j| > \epsilon_j\}) \le \epsilon_j.$$

Let $E_j = \{|g_{j+1} - g_j| > \epsilon_j\}$, $F_N = \bigcup_{j=N}^{\infty} E_j$ and

$$E \equiv \bigcap_{N=1}^{\infty} F_N = \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} E_j = \{E_j \text{ i.o.}\}.$$

Then $\mu(E) = 0$ since

$$\mu(E) \le \sum_{j=N}^{\infty} \mu(E_j) \le \sum_{j=N}^{\infty} \epsilon_j = \delta_N \to 0 \text{ as } N \to \infty.$$

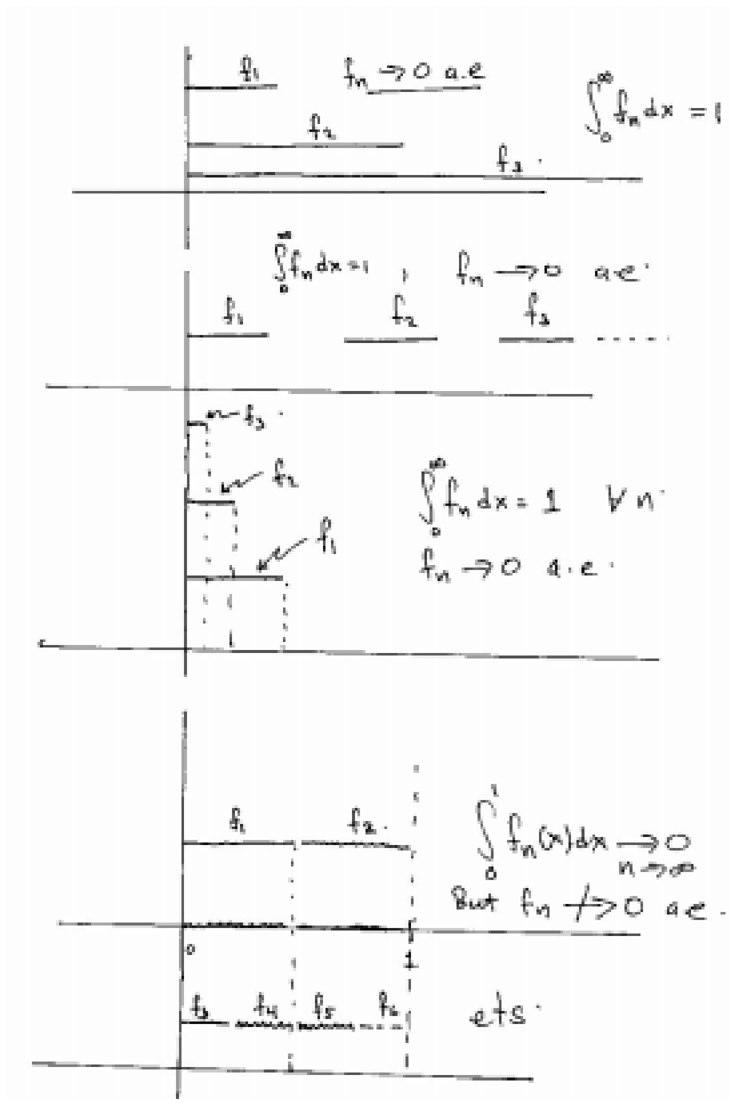

Figure 7. Modes of convergence examples.

For $x \notin F_N$, $|g_{j+1}(x) - g_j(x)| \le \epsilon_j$ for all $j \ge N$ and by Lemma 9.3, $f(x) = \lim_{j\to\infty} g_j(x)$ exists and $|f(x)-g_j(x)| \le \delta_j$ for all $j \ge N$. Therefore, $\lim_{j=\infty} g_j(x) = f(x)$ exists for all $x \notin E$. Moreover, $\{x : |f(x) - f_j(x)| > \delta_j\} \subseteq F_j$ for all $j \ge N$ and hence

$$\mu(|f - g_j| > \delta_j) \le \mu(F_j) \le \delta_j \to 0 \text{ as } j \to \infty.$$

Therefore $g_j \xrightarrow{\mu} f$ as $j \to \infty$.

Since

$$\begin{aligned}\{|f_n - f| > \epsilon\} &= \{|f - g_j + g_j - f_n| > \epsilon\}\\ &\subseteq \{|f - g_j| > \epsilon/2\} \cup \{|g_j - f_n| > \epsilon/2\},\end{aligned}$$

$$\mu(\{|f_n - f| > \epsilon\}) \le \mu(\{|f - g_j| > \epsilon/2\}) + \mu(|g_j - f_n| > \epsilon/2)$$

and

$$\mu(\{|f_n - f| > \epsilon\}) \le \lim_{j\to\infty}\sup \mu(|g_j - f_n| > \epsilon/2) \to 0 \text{ as } n \to \infty.$$

Finally we show that the limiting function is unique up to measure zero. If also $f_n \xrightarrow{\mu} g$ as $n \to \infty$, then arguing as above

$$\mu(|f - g| > \epsilon) \le \mu(\{|f - f_n| > \epsilon/2\}) + \mu(|g - f_n| > \epsilon/2) \to 0 \text{ as } n \to \infty.$$

That is to say $\mu(|f - g| > \epsilon) = 0$ for all $\epsilon > 0$ and hence

$$\mu(|f - g| > 0) = \mu(\cup_{n=1}^\infty \left\{|f - g| > \frac{1}{n}\right\}) \le \sum_{n=1}^\infty \mu(|f - g| > \frac{1}{n}) = 0,$$

i.e. $f = g$ a.e. ■

Theorem 9.5 (Completeness of $L^p(\mu)$)**.** *Suppose that $\{f_n\} \subseteq L^p(\mu)$ is Cauchy, then there exists $f \in L^p(\mu)$ such that $f_n \xrightarrow{L^p} f$. Moreover f is unique modulo the equivalence relation of being equal off sets of measure zero.*

Proof. Write

$$\|f\| = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

By Chebyschev's inequality (8.3),

$$\begin{aligned} \mu\left(|f_n - f_m| \ge \epsilon\right) &= \mu\left(|f_n - f_m|^p \ge \epsilon^p\right) \\ &\le \frac{1}{\epsilon^p}\int_X |f_n - f_m|^p \, d\mu = \frac{1}{\epsilon^p}\|f_n - f_m\|^p \to 0 \text{ as } m, n \to \infty \end{aligned}$$

for all $\epsilon > 0$. This shows that $\{f_n\}$ is L^0-Cauchy (i.e. Cauchy in measure) so there exists $\{g_j\} \subseteq \{f_n\}$ such that $g_j \to f$ a.e. Now by Fatou's Lemma,

$$\begin{aligned} \|g_j - f\|^p &= \int \lim_{k\to\infty} \inf |g_j - g_k|^p \le \lim_{k\to\infty} \inf \int |g_j - g_k|^p d\mu \\ &= \lim_{k\to\infty} \inf \|g_j - g_k\|^p \to 0 \text{ as } j \to \infty. \end{aligned}$$

In particular, $\|f\| \le \|g_j - f\| + \|g_j\| < \infty$ so the $f \in L^p$ and $g_j \xrightarrow{L^p} f$. The proof is finished because,

$$\|f_n - f\| \le \|f_n - g_j\| + \|g_j - f\| \to 0 \text{ as } j, n \to \infty.$$

■

Theorem 9.6 (Egoroff's Theorem)**.** *Suppose $\mu(X) < \infty$ and $f_n \to f$ a.e. Then for all $\epsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. In particular $f_n \xrightarrow{\mu} f$ as $n \to \infty$.*

Proof. Let $f_n \to f$ a.e. Then $\mu(\{|f_n - f| > \frac{1}{k} \text{ i.o. } n\}) = 0$ for all $k > 0$, i.e.

$$\mu\left(\bigcap_{N=1}^\infty \bigcup_{n\ge N} \{|f_n - f| > \frac{1}{k}\}\right) = 0.$$

So

$$\lim_{N\to\infty} \mu\left(\bigcup_{n\ge N} \{|f_n - f| > \frac{1}{k}\}\right) = 0.$$

Choose and increasing sequence $\{N_k\}_{k=1}^{\infty}$ such that $\mu(E_k) < \epsilon 2^{-k}$ where

$$E_k := \bigcup_{n \geq N_k} \{|f_n - f| > \frac{1}{k}\}.$$

Thus $E := \cup E_k$ satisfies $\mu(E) < \epsilon$ and if $x \notin E$, $|f_n - f| \leq \frac{1}{k}$ for all $n \geq N_k$ and all k,. i.e. $f_n \to f$ uniformly on E^c. ■

10. Density Theorems

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{S}_f$ denote the collection of simple functions ϕ with additional property that $\mu(\phi = z) < \infty$ for all $z \neq 0$. Notice that for $\phi \in \mathcal{S}_f$ and $p \in [1, \infty)$, that $|\phi|^p = \sum_{z \neq 0} |z|^p 1_{\{\phi = z\}}$ and hence

$$\int |\phi|^p \, d\mu = \sum_{z \neq 0} |z|^p \mu(\phi = z) < \infty.$$

That is $\mathcal{S}_f \subset L^p(\mu)$.

Lemma 10.1 (Simple Functions are Dense)**.** *Let simple functions $\mathcal{S}_f$ form a dense subspace of $L^p(\mu)$ for all $1 \leq p < \infty$.*

Proof. Since $f = u + iv$ with $u, v \in L^p(\mu)$ and for f real valued $f = f_+ - f_-$ with $f_\pm \in L^p$ we may assume without loss of generality that $f \in L^p \cap L^+$. Choose simple functions $\phi_n \uparrow f$ as in Theorem 7.32. Since

$$|f - \phi_n|^p \leq (|f| + |\phi_n|)^p \leq 2^p |f|^p \in L^1$$

we may apply the dominated convergence theorem to find

$$\lim_{n \to \infty} \int |f - \phi_n|^p d\mu = \int \lim_{n \to \infty} |f - \phi_n|^p d\mu = 0.$$

■

Definition 10.2. Let (X, d) be a metric space. We say that a set $A \subset X$ is **bounded** provided that $A \subset B(x, R)$ for some $x \in X$ and $R \in (0, \infty)$. (By the triangle inequality A is bounded iff for all $x \in X$, there exists $R < \infty$ such that $A \subset B(x, R)$.) Also let $BC_b(X)$ denote the set of bounded continuous functions $f : X \to \mathbb{C}$ such that f is identically zero off a bounded subset of X.

Theorem 10.3 (Continuous Functions are Dense)**.** *Let (X, d) be a metric space, τ_d be the topology on X generated by d and $\mathcal{B}_X = \sigma(\tau_d)$ be the Borel σ – algebra.*

1. *Suppose $\mu : \mathcal{B}_X \to [0, \infty]$ is a measure such that μ is finite on bounded sets, then $BC_b(X) \subset L^p(\mu)$ is a dense subspace.*
2. *Suppose that there is a sequence of compact set $K_n \subset X$ such that $K_n^o \subset K_n \subset K_{n+1}^o$ for all n, $X = \cup_{n=1}^{\infty} K_n$ and $\mu(K_n) < \infty$ for all n, where for $E \subset X$, $E^o = \cup_{V \in \tau, V \subset E} V$ is the interior of E. Then $C_c(X)$ (the collection of continuous functions with compact support) is dense in $L^p(\mu)$.*

Proof. Part 1. Since $\mathcal{S}_f$ is dense in $L^p(\mu)$ it suffices to shows that any $\phi \in \mathcal{S}_f$ may be well approximated by $f \in BC_b(X)$. Moreover, to prove this it suffices to show that for $A \in \mathcal{M}$ with $\mu(A) < \infty$ that 1_A may be well approximated by an $f \in BC_b(X)$. Let $x_0 \in X$ and $n \in \mathbb{N}$ and set $A_n := A \cap B(x_0, n)$ so that $A_n \uparrow A$ as $n \to \infty$. Therefore

$$\int |1_A - 1_{A_n}|^p \, d\mu = \mu(A \setminus A_n) \to 0 \text{ as } n \to \infty.$$

This shows that we may further assume that A is bounded, i.e. $A \subset B(x_0, N)$ for some large N.

By Exercises 4.21 and 4.22, for any $\epsilon > 0$ there exists $F \subset A \subset V$ where F is closed, V is open and $\mu(V \setminus F) < \epsilon$. Since $F \subset A$, F is automatically bounded. By replacing V by the bounded open set $V \cap B(x_0, N)$ if necessary, we may also assume that V is bounded.

Define

$$f(x) = \frac{d_{V^c}(x)}{d_F(x) + d_{V^c}(x)} \tag{10.1}$$

where $d_F(x) = \inf\{d(x, y) : y \in F\}$. By Lemma 2.37, d_F and d_{V^c} are continuous functions on X. Since F and V^c are closed, $d_F(x) > 0$ if $x \notin F$ and $d_{V^c}(x) > 0$ if $x \in V$. Since $F \cap V^c = \emptyset$, $d_F(x) + d_{V^c}(x) > 0$ for all x and $(d_F + d_{V^c})^{-1}$ is continuous as well. Therefore, $f : X \to [0, 1]$ is continuous and $f(x) = 1$ for $x \in F$ and $f(x) = 0$ if $x \notin V$ and since V is bounded we have shown that $f \in BC_b(X)$.

Since $|1_A - f| \le 1_{V \setminus F}$,

$$\int |1_A - f|^p \, d\mu \le \int 1_{V \setminus F} d\mu = \mu(V \setminus F) \le \epsilon \tag{10.2}$$

or equivalently

$$\|1_A - f\| \le \epsilon^{1/p}.$$

Since $\epsilon > 0$ is arbitrary, we have shown that 1_A can be approximated in $L^p(\mu)$ arbitrarily well by a continuous function $f \in BC_b(X)$.

Part 2. The proof of the second assertion is similar. Again it suffices to show that for $A \in \mathcal{M}$ with $\mu(A) < \infty$ that 1_A may be well approximated by an $f \in C_c(X)$. Now let $A_n := A \cap K_n^o$ so that $A_n \uparrow A$ as $n \to \infty$. Therefore

$$\int |1_A - 1_{A_n}|^p \, d\mu = \mu(A \setminus A_n) \to 0 \text{ as } n \to \infty.$$

This shows that we may further assume that $A \subset K_N^o$ for some N.

By Exercises 4.21 and 4.22, for any $\epsilon > 0$ there exists $F \subset A \subset V$ where F is closed, V is open and $\mu(V \setminus F) < \epsilon$. Since F is a closed subset of K_N a compact set, F is also compact. By replacing V by the open set $V \cap K_N^o$ if necessary, we may also assume that $V \subset K_N$ as well. Again define f to be the continuous function in Eq. (10.1) and as before Eq. (10.2) still holds. Moreover f has support in the compact set K_N which shows that $f \in C_c(X)$. ∎

Corollary 10.4. *Suppose $V \subset \mathbb{R}^n$ is an open set, $\mathcal{B}_V$ is the Borel σ – algebra on V and μ is a measure on $(V, \mathcal{B}_V)$ which is finite on compact sets. Then $C_c(V)$ is dense in $L^p(\mu)$ for all $p \in [1, \infty)$.*

Proof. It suffices to shows that there exists compact subset $K_n \subset V$ such $K_n^o \subset K_n \subset K_{n+1}^o \uparrow V$ as $n \to \infty$. This is essentially done in the proof of Theorem 12.2 below, also see Figure 8 below. ∎

Later on we will show that the space $C_c^\infty(\mathbb{R})$ of infinitely differentiable functions with compact support is also dense in $L^p(m)$ for all $p \in [1, \infty)$. A key point in proving this fact is the following lemma.

Lemma 10.5. *Suppose $f \in L^1(dm)$ and $\phi \in C_c^\infty(\mathbb{R})$ then*

$$h(x) = f * \phi(x) = \int f(y)\phi(x - y) dy$$

is a smooth function, i.e. $h(x)$ is infinitely differentiable.

Proof. Let $F(x,y) := f(y)\phi(x-y)$, then if $C<\infty$ is a bound on the derivative of ϕ,

$$\left|\frac{\partial F}{\partial x}(x,y)\right| = \left|f(y)\phi'(x-y)\right| \le C|f(y)| \in L^1(dy).$$

Therefore by Corollary 8.26,

$$\frac{d}{dx}(f*\phi)(x) = \int F(y)\phi'(x-y)dy = (f*\phi')(x).$$

Working inductively one may now shows that $(f*\phi)^{(n)}(x) = f*\phi^{(n)}(x)$. ■

Theorem 10.6. *Suppose that $\mathcal{M}$ is a σ – algebra on X and $\mathcal{A}\subset\mathcal{M}$ is an algebra such that*

(1) $\sigma(\mathcal{A}) = \mathcal{M}$
(2) *$\mathcal{A}$ is countable*
(3) *μ is σ-finite on $\mathcal{A}$.*

Then $L^p(X,\mathcal{M},\mu)$ is separable for all $1\le p<\infty$. Moreover

$$\mathbb{D} = \{\sum a_j x_{A_j} : a_j \in \mathbb{Q}+i\mathbb{Q},\ A_j\in\mathcal{A} \text{ and } \mu(A_j)<\infty\}$$

is a countable dense subspace.

Proof. Given $\epsilon>0$, by Corollary 4.15, for all $E\in\mathcal{M}$ such that $\mu(E)<\infty$, there exists $A\in\mathcal{A}$ such that$\mu(E\triangle A)<\epsilon$ and therefore

$$\int |1_E - 1_A|^p d\mu = \mu(E\triangle A) < \epsilon. \tag{10.3}$$

This equation shows that any simple function in $\mathcal{S}_f$ may be approximated arbitrary well by an element from $\mathbb{D}$ and hence $\mathbb{D}$ is also dense in $L^p(\mu)$. ■

Corollary 10.7 (Riemann Lebesgue Lemma)**.** *Suppose that $f\in L^1(\mathbb{R},m)$, then*

$$\lim_{\lambda\to\pm\infty}\int_{\mathbb{R}} f(x)e^{i\lambda x}dm(x) = 0.$$

Proof. Let $\mathcal{A}$ denote the algebra on $\mathbb{R}$ generated by the half open intervals, i.e. $\mathcal{A}$ consists of sets of the form

$$\coprod_{k=1}^{n}(a_k,b_k]\cap\mathbb{R}$$

where $a_k,b_k\in\bar{\mathbb{R}}$. By Theorem 10.6, given $\epsilon>0$ there exists $\phi=\sum_{k=1}^n c_k 1_{(a_k,b_k]}$ with $a_k,b_k\in\mathbb{R}$ such that

$$\int_{\mathbb{R}}|f-\phi|dm<\epsilon.$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}}\phi(x)e^{i\lambda x}dm(x) &= \int_{\mathbb{R}}\sum_{k=1}^n c_k 1_{(a_k,b_k]}(x)e^{i\lambda x}dm(x)\\ &= \sum_{k=1}^n c_k\int_{a_k}^{b_k} e^{i\lambda x}dm(x) = \sum_{k=1}^n c_k\lambda^{-1}e^{i\lambda x}|_{a_k}^{b_k}\\ &= \lambda^{-1}\sum_{k=1}^n c_k\left(e^{i\lambda b_k}-e^{i\lambda a_k}\right)\to 0 \text{ as } |\lambda|\to\infty.\end{aligned}$$

Combining these two equations with

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x)e^{i\lambda x}dm(x)\right| &\le \left|\int_{\mathbb{R}} (f(x)-\phi(x))\, e^{i\lambda x}dm(x)\right| + \left|\int_{\mathbb{R}} \phi(x)e^{i\lambda x}dm(x)\right| \\ &\le \int_{\mathbb{R}} |f-\phi| dm + \left|\int_{\mathbb{R}} \phi(x)e^{i\lambda x}dm(x)\right| \\ &\le \epsilon + \left|\int_{\mathbb{R}} \phi(x)e^{i\lambda x}dm(x)\right|\end{aligned}$$

we learn that

$$\lim\sup_{|\lambda|\to\infty}\left|\int_{\mathbb{R}} f(x)e^{i\lambda x}dm(x)\right| \le \epsilon + \lim\sup_{|\lambda|\to\infty}\left|\int_{\mathbb{R}} \phi(x)e^{i\lambda x}dm(x)\right| = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have proven the lemma. ■

11. Fubini's Theorem

This next example gives a "real world" example of the fact that it is not always possible to interchange order of integration.

Example 11.1. Consider

$$\begin{aligned}\int_0^1 dy \int_1^\infty dx(e^{-xy} - 2e^{-2xy}) &= \int_0^1 dy \left\{\frac{e^{-y}}{-y} - 2\frac{e^{-2xy}}{-2y}\right\}\Bigg|_{x=1}^{\infty} \\ &= \int_0^1 dy \left[\frac{e^{-y}-e^{-2y}}{y}\right] \\ &= \int_0^1 dy\ e^{-y}\left(\frac{1-e^{-y}}{y}\right) \in (0,\infty).\end{aligned}$$

Note well that $\left(\frac{1-e^{-y}}{y}\right)$ has not singularity at 0. On the other hand

$$\begin{aligned}\int_1^\infty dx \int_0^1 dy(e^{-xy} - 2e^{-2xy}) &= \int_1^\infty dx \left\{\frac{e^{-xy}}{-x} - 2\frac{e^{-2xy}}{-2x}\right\}\Bigg|_{y=0}^{1} \\ &= \int_1^\infty dx \left\{\frac{e^{-2x}-e^{-x}}{x}\right\} \\ &= -\int_1^\infty e^{-x}\left[\frac{1-e^{-x}}{x}\right] dx \in (-\infty,0).\end{aligned}$$

Moral $\int dx \int dy\ f(x,y) \neq \int dy \int dx\ f(x,y)$ is **not always true**.

11.1. **Product Measure.** Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces.

Notation 11.2. Suppose that $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ are functions, let $f \otimes g$ denote the function on $X \times Y$ given by

$$f \otimes g(x,y) = f(x)g(y).$$

Notice that if f, g are measurable, then $f \otimes g$ is $(\mathcal{M}\otimes\mathcal{N}, \mathcal{B}_{\mathbb{C}})$ – measurable. To prove this, first suppose that $f = 1_A$ and $g = 1_B$ with $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Then $f \otimes g = 1_A \otimes 1_B = 1_{A\times B}$ which is measurable since $A \times B \in \mathcal{M}\otimes\mathcal{N}$. Now if f and g are simple functions then

$$f \otimes g = \sum_{w,z\in\mathbb{C}} wz1_{\{f=w\}\times\{g=z\}}$$

which is again measurable. For general f and g choose simple function ϕ_n and ψ_n converging to f and g respectively, then since $f \otimes g = \lim_{n\to\infty} \phi_n \otimes \psi_n$ we learn that $f \otimes g$ is measurable as well.

Let $\mathcal{E} \subset \mathcal{P}(X \times Y)$ be given by

$$\mathcal{E} = \mathcal{M} \times \mathcal{N} = \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}$$

and recall from Exercise 3.11 that $\mathcal{E}$ is an elementary family. Hence the algebra $\mathcal{A} = \mathcal{A}(\mathcal{E})$ generated by $\mathcal{E}$ consists of sets which may be written as disjoint unions of sets from $\mathcal{E}$.

Definition 11.3. Define $\pi^0 : \mathcal{E} \to [0, \infty]$ by

$$\pi^0 (A \times B) = \mu(A)\nu(B)$$

for $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Notice that

$$\begin{aligned} \int_Y 1_{A\times B}(x, y)d\nu(y) &= \int_Y 1_A(x)1_B(y)d\nu(y) \\ &= 1_A(x)\nu(B) \end{aligned}$$

and hence

$$\int_X \left(\int_Y 1_{A\times B}(x, y)d\nu(y) \right) d\mu(x) = \mu(A)\nu(B) = \pi^0(A \times B) \tag{11.1}$$

and similarly,

$$\int_Y \left(\int_X 1_{A\times B}(x, y)d\mu(x) \right) d\nu(y) = \pi^0(A \times B). \tag{11.2}$$

Theorem 11.4. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure space, then there exists a unique measure π on $\mathcal{M} \otimes \mathcal{N}$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$. We will denote this "product" measure by $\mu \times \nu$.*

Proof. 1. π^0 is σ – finite on $\mathcal{E}$. Indeed, if $X_n \in \mathcal{M}$ and $Y_n \in \mathcal{N}$ are such that $X_n \uparrow X$ and $Y_n \uparrow Y$ and $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$ for all n, then $X_n \times Y_n \in \mathcal{E}$, $X_n \times Y_n \uparrow X \times Y$ and $\pi^0(X_n \times Y_n) = \mu(X_n)\nu(Y_n) < \infty$ for all n.

2. The function π^0 is σ – additive on $\mathcal{E} = \mathcal{M} \times \mathcal{N}$. To prove this suppose that $E \in \mathcal{M} \otimes \mathcal{N}$ and $E_k = A_k \times B_k \in \mathcal{E}$ are such that $E = \coprod_{k=1}^{\infty}(A_k \times B_k)$, then

$$1_E = \sum_{k=1}^{\infty} 1_{A_k} \otimes 1_{B_k}$$

so that

$$y \to 1_E(x, y) = \sum_{k=1}^{\infty} 1_{A_k}(x)1_{B_k}(y)$$

is measurable for all $x \in X$ and

$$\begin{aligned} \int_Y 1_E(x, y)d\nu(y) &= \int_Y \sum_{k=1}^{\infty} 1_{A_k}(x)1_{B_k}(y)d\nu(y) = \sum_{k=1}^{\infty} 1_{A_k}(x) \int_Y 1_{B_k} d\nu \\ &= \sum_{k=1}^{\infty} 1_{A_k}(x)\nu(B_k). \end{aligned}$$

The latter expression is measurable and

$$\begin{aligned}\int_X \left(\int_Y 1_E(x,y)d\nu(y)\right) d\mu(x) &= \int_X \sum_{k=1}^{\infty} 1_{A_k}(x)\nu(B_k)d\mu(x) \\ &= \sum_{k=1}^{\infty}\mu(A_k)\nu(B_k) = \sum_{k=1}^{\infty}\pi^0(A_k\times B_k). \qquad (11.3)\end{aligned}$$

Similarly one shows that

$$\int_Y \left(\int_X 1_E(x,y)d\mu(x)\right) d\nu(y) = \sum_{k=1}^{\infty}\pi^0(A_k\times B_k). \qquad (11.4)$$

Taking $E = A\times B\in\mathcal{E}$, Eqs. (11.4) and (11.2) shows that

$$\pi^0(A\times B) = \sum_{k=1}^{\infty}\pi^0(A_k\times B_k)$$

that is to say π^0 is σ − additive on $\mathcal{E}$.

By items 1. and 2. just proved and Theorem 5.7, there exists a unique measure π on $\mathcal{M}\otimes\mathcal{N} = \sigma(\mathcal{A}) = \sigma(\mathcal{E})$ such $\pi(A\times B) = \mu(A)\nu(B)$ for all $A\in\mathcal{M}$ and $B\in\mathcal{N}$. ■

The proof gives a bit more than stated. Namely

$$\begin{aligned}\pi(E) &= \int_X \left[\int_Y 1_E(x,y)d\nu(y)\right] d\mu(x) \\ &= \int_Y \left(\int_X 1_E(x,y)d\mu(x)\right) d\nu(y). \qquad (11.5)\end{aligned}$$

for all $E\in\mathcal{A}_\sigma$. This follows by writing $E = \coprod_{k=1}^{\infty}(A_k\times B_k)$ with $A_k\times B_k\in\mathcal{E}$ and using the identity,

$$\pi(E) = \sum_{k=1}^{\infty}\pi(A_k\times B_k) = \sum_{k=1}^{\infty}\pi^0(A_k\times B_k)$$

along with Eqs. (11.3) and (11.4).

Theorem 11.5 (Tonelli's Theorem)**.** *Suppose $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ-finite measure spaces and $\pi = \mu\times\nu$ be the product measure on $\mathcal{M}\otimes\mathcal{N}$. If $f\in L^+(X\times Y,\mathcal{M}\otimes\mathcal{N})$, then $f(\cdot,y)\in L^+(X,\mathcal{M})$ for all $y\in Y$, $f(x,\cdot)\in L^+(Y,\mathcal{N})$ for all $x\in X$,*

$$x\to g(x) = \int_Y f(x,y)d\nu(y)\in L^+(X,\mathcal{M}),$$

$$y\to h(y) = \int_X f(x,y)d\mu(x)\in L^+(Y,\mathcal{N})$$

and

$$\int_{X\times Y} f\, d\pi = \int_X d\mu(x)\int_Y d\nu(y)f(x,y) \qquad (11.6)$$

$$= \int_Y d\nu(y)\int_X d\mu(x)f(x,y) \qquad (11.7)$$

Proof. By linearity and the monotone convergence limiting arguments it suffices to prove the assertion in the theorem when $f = 1_E$ with $E \in \mathcal{M} \otimes \mathcal{N}$. First **assume** $\mu(X) < \infty$ and $\nu(Y) < \infty$. Let $\mathcal{C}$ be the collection of all $E \in \mathcal{M} \otimes \mathcal{N}$ such that theorem holds when $f = 1_E$. The collection $\mathcal{C}$ satisfies:

(1) By the proof of Theorem 11.4 and Eq. (11.5) $\mathcal{A}(\mathcal{E}) \subseteq \mathcal{C}$.
(2) $\mathcal{C}$ is a monotone class. (We will check this shortly.)

By the monotone class lemma we may conclude that $\mathcal{C} = \sigma(\mathcal{A}(\mathcal{E})) = \mathcal{M} \otimes \mathcal{N}$ which proves the theorem in the finite measure case.

To check item 2., suppose that $E_n \in \mathcal{C}$ and $E_n \uparrow E$. Then then $1_{E_n} \uparrow 1_E$ as $n \to \infty$ so that $1_{E_n}(x, \cdot) \uparrow 1_E(x, \cdot)$ so that $1_E(x, \cdot) \in L^+(Y, \mathcal{N})$ for all $x \in X$ and by the monotone convergence theorem

$$\int_Y 1_{E_n}(x,y) d\nu(y) \uparrow \int_Y 1_E(x,y) d\nu(y)$$

so that $x \to \int_Y 1_E(x,y) d\nu(y) \in L^+(X, \mathcal{M})$. Again by the monotone convergence theorem we may take limits of the identity

$$\int 1_{E_n} \, d\pi = \int d\mu(x) \int d\nu(y) 1_{E_n}(x,y)$$

to prove Eq. (11.6). One shows in the same way that the analogous statements holds for the opposite order of integration, i.e. $E \in \mathcal{C}$.

If $E_n \in \mathcal{C}$ and $E_n \downarrow E$, we again shows that $E \in \mathcal{C}$ by the same techniques using the dominated convergence theorem instead of the monotone convergence theorem. Therefore $\mathcal{C}$ is a monotone class and hence by the monotone class theorem $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$ and Tonelli's theorem is proved when μ and ν are finite measures.

For the σ – finite case, choose $X_n \in \mathcal{M}$, $Y_n \in \mathcal{N}$ such that $X_n \uparrow X$, $Y_n \uparrow Y$, $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$ for all n. Then define $\mu_n(A) = \mu(X_n \cap A)$ and $\nu_n(B) = \nu(Y_n \cap B)$ for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$ and let $\pi_n = \mu_n \times \nu_n$. Since

$$\pi_n(A \times B) = \mu_n(A)\nu_n(B) = \mu(X_n \cap A)\nu(Y_n \cap B) = \pi\left((X_n \times Y_n) \cap (A \times B)\right)$$

for all $A, B \in \mathcal{M} \times \mathcal{N}$. Since σ – finite measure on $\mathcal{M} \otimes \mathcal{N}$ are determined by their values on $\mathcal{M} \times \mathcal{N}$, we see that

$$\pi_n(E) = \pi\left((X_n \times Y_n) \cap E\right) \text{ for all } E \in \mathcal{M} \otimes \mathcal{N}.$$

Let us also observe that

$$\int_Y g d\nu_n = \int_{Y_n} g d\nu$$

for all measurable functions $g \in L^+(Y, \mathcal{N})$. This is proved first for simple functions and then for general functions by passing to the limit.

With this notation, we may apply the finite version of the theorem to conclude

$$x \to g_n(x) = \int_Y f(x,y) d\nu_n(y) = \int_Y 1_{Y_n}(y) f(x,y) d\nu(y) \in L^+(X, \mathcal{M}),$$

$$y \to h_n(y) = \int_X f(x,y) d\mu_n(x) = \int_X 1_{X_n}(x) f(x,y) d\mu(x) \in L^+(Y, \mathcal{N})$$

and

$$\int_{X\times Y} 1_{X_n\times Y_n} f\, d\pi = \int_{X\times Y} f\, d\pi_n$$
$$= \int_X d\mu_n(x)\int_Y d\nu_n(y) f(x,y) = \int_X d\mu(x)\int_Y d\nu(y)\left(1_{X_n\times Y_n} f\right)(x,y)$$
$$= \int_Y d\nu_n(y)\int_X d\mu_n(x) f(x,y) = \int_Y d\nu(y)\int_X d\mu(x)\left(1_{X_n\times Y_n} f\right)(x,y).$$

Passing to the limit in these identities with the aid of the monotone convergence theorem concludes the proof of the theorem. ■

Theorem 11.6 (Fubini's Theorem). *Suppose $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ are σ-finite measure spaces and $\pi=\mu\times\nu$ be the product measure on $\mathcal{M}\otimes\mathcal{N}$. If $f\in L^1(\pi)$ then for μ a.e. x, $f(x,\cdot)\in L^1(\nu)$ and for ν a.e. y, $f(\cdot,y)\in L^1(\mu)$. The functions*

$$g(x) = \int_Y f(x,y)dv(y) \text{ and } h(y) = \int_X f(x,y)d\mu(x)$$

are in $L^1(\mu)$ and $L^1(\nu)$ respectively and Eq. (11.7) holds.

Proof. If $f\in L^1(X\times Y)\cap L^+$ then by Eq. (11.6),

$$\int_X\left(\int_Y f(x,y)d\nu(y)\right)d\mu(x) < \infty$$

so $\int_Y f(x,y)d\nu(y)<\infty$ for μ a.e. x, i.e. for μ a.e. x, $f(x,\cdot)\in L^1(\nu)$. Similarly for ν a.e. y $f(\cdot,y)\in L^1(\mu)$. Let f be a real valued function in $f\in L^1(X\times Y)$ and let $f=f_+-f_-$. Apply the results just proved to $f_\pm$ to conclude, $f_\pm(x,\cdot)\in L^1(\nu)$ for μ a.e. x and that

$$\int_Y f_\pm(\cdot,y)d\nu(y)\in L^1(\mu).$$

Therefore for μ a.e. $.x$,

$$f(x,\cdot) = f_+(x,\cdot) - f_-(x,\cdot)\in L^1(\nu)$$

and

$$x\to\int f(x,y)d\nu(y) = \int f_+(x,\cdot)d\nu(y) - \int f_-(x,\cdot)d\nu(y)$$

is a μ − almost everywhere defined function such that

$$\int f(\cdot,y)d\nu(y)\in L^1(\mu).$$

Because

$$\int f_\pm(x,y)d\left(\mu\times\nu\right) = \int(\int f_\pm(x,y)d\nu(y))d\mu(x),$$

$$\int f\, d(\mu\times\nu) = \int f_+ d(\mu\times\nu) - \int f_- d(\mu\times\nu)$$
$$= \int\left(\int f_+ d\nu\right)d\mu - \int\left(\int f_- d\nu\right)d\mu$$
$$= \int d\mu\left(\int f_+ d\nu - \int f_- d\nu\right)$$
$$= \int d\mu\int d\nu(f_+ - f_-) = \int d\mu\int d\nu f.$$

The proof that

$$\int f\,d(\mu\times\nu) = \int d\nu(y)\int d\mu(x) f(x,y)$$

is analogous. ■

Notation 11.7. Given $E \subset X\times Y$ and $x\in X$, let

$${}_xE := \{y\in Y : (x,y)\in E\}.$$

Similarly if $y\in Y$ is given let

$$E_y := \{x\in X : (x,y)\in E\}.$$

If $f : X\times Y\to\mathbb{C}$ is a function let $f_x = f(x,\cdot)$ and $f^y := f(\cdot,y)$ so that $f_x : Y\to\mathbb{C}$ and $f^y : X\to\mathbb{C}$.

Theorem 11.8. *Suppose $(X,\mathcal{M},\mu),(Y,\mathcal{N},\nu)$ are complete σ-finite measure space. Let $(X\times Y,\mathcal{L},\lambda)$ be the completion of $(X\times Y,\mathcal{M}\otimes\mathcal{N},\mu\times\nu)$. If f is $\mathcal{L}$-measurable and (a) $f\geq 0$ or (b) $f\in L^1(\lambda)$ then f_x is $\mathcal{N}$-measurable for μ a.e. x and f^y is $\mathcal{M}$-measurable for ν a.e. y and in case (b) $f_x\in L^1(\nu)$ and $f^y\in L^1(\mu)$ for μ a.e. x and ν a.e. y respectively. Moreover,*

$$x\to\int f_x d\nu \text{ and } y\to\int f^y d\mu$$

are measurable and

$$\int f d\lambda = \int\left(\int f d\mu\right)\,d\nu = \int\left(\int f d\nu\right)\,d\mu.$$

Proof. If $E\in\mathcal{M}\otimes\mathcal{N}$ is a $\mu\times\nu$ null set ($(\mu\times\nu)(E)=0$), then

$$0 = (\mu\times\nu)(E) = \int_X \nu({}_xE)d\mu(x) = \int_X \mu(E_y)d\nu(y).$$

This shows that

$$\mu(\{x : \nu({}_xE)\neq 0\}) = 0 \text{ and } \nu(\{y : \mu(E_y)\neq 0\}) = 0,$$

i.e. $\nu({}_xE)=0$ for μ a.e. x and $\mu(E_y)=0$ for ν a.e. y.

If h is $\mathcal{L}$ measurable and $h=0$ for λ- a.e., then there exists $E\in\mathcal{M}\otimes\mathcal{N}\ni \{(x,y) : h(x,y)\neq 0\}\subseteq E$ and $(\mu\times\nu)(E)=0$. Therefore $|h(x,y)|\leq 1_E(x,y)$ and $(\mu\times\nu)(E)=0$. Since

$$\begin{aligned}\{h_x\neq 0\} &= \{y\in Y : h(x,y)\neq 0\}\subset {}_xE \text{ and}\\ \{h_y\neq 0\} &= \{x\in X : h(x,y)\neq 0\}\subset E_y\end{aligned}$$

we learn that for μ a.e. x and ν a.e. y that $\{h_x\neq 0\}\in\mathcal{M}$, $\{h_y\neq 0\}\in\mathcal{N}$, $\nu(\{h_x\neq 0\})=0$ and a.e. and $\mu(\{h_y\neq 0\})=0$. This implies

$$\begin{aligned}&\text{for } \nu \text{ a.e. } y, \int h(x,y)d\nu(y) \text{ exists and equals } 0\\ &\text{and}\\ &\text{for } \mu \text{ a.e. } x, \int h(x,y)d\mu(y) \text{ exists and equals } 0.\end{aligned}$$

Therefore

$$0 = \int h d\lambda = \int\left(\int h d\mu\right)\,d\nu = \int\left(\int h d\nu\right)\,d\mu.$$

For general $f \in L^1(\lambda)$, we may choose $g \in L^1(\mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ such that $f(x,y) = g(x,y)$ for $\lambda-$ a.e. (x,y). Define $h \equiv f - g$. Then $h = 0,\ \lambda-$ a.e. Hence by what we have just proved and Theorem 11.5 $f = g + h$ has the following properties:

(1) For μ a.e. x, $y \to f(x,y) = g(x,y) + h(x,y)$ is in $L^1(\nu)$ and
$$\int f(x,y) d\nu(y) = \int g(x,y) d\nu(y).$$
(2) For ν a.e. y, $x \to f(x,y) = g(x,y) + h(x,y)$ is in $L^1(\mu)$ and
$$\int f(x,y) d\mu(x) = \int g(x,y) d\mu(x).$$

From these assertions and Theorem 11.5, it follows that
$$\begin{aligned} \int d\mu(x) \int d\nu(y) f(x,y) &= \int d\mu(x) \int d\nu(y) g(x,y) \\ &= \int d\nu(y) \int d\nu(x) g(x,y) \\ &= \int g(x,y) d(\mu \times \nu)(x,y) \\ &= \int f(x,y) d\lambda(x,y) \end{aligned}$$
and similarly we shows
$$\int d\nu(y) \int d\mu(x) f(x,y) = \int f(x,y) d\lambda(x,y).$$
■

Exercise 11.9. For $M, \Lambda \in (0,\infty)$, show

$$\left| \int_0^M \frac{\sin x}{x} e^{-\Lambda x} dx - \pi/2 \right| \le \arctan(\Lambda) + \delta(M) \tag{11.8}$$

where $\lim_{M\to\infty} \delta(M) = 0$. As special cases of this expression we learn that

$$\lim_{M\to\infty} \int_0^M \frac{\sin x}{x} dx = \pi/2 \tag{11.9}$$

and

$$\lim_{\Lambda\to 0} \int_0^\infty \frac{\sin x}{x} e^{-\Lambda x} dx = \pi/2. \tag{11.10}$$

Solution. Define
$$f_M(\lambda) := \int_0^M e^{-\lambda x} \frac{\sin x}{x} dx$$
and notice that
$$\begin{aligned} f_M'(\lambda) &= -\int_0^M e^{-\lambda x} \sin x dx = \frac{e^{-\lambda x}}{\lambda^2 + 1} (\cos x + \lambda \sin x) \left|_{x=0}^{x=M} \right. \\ &= \frac{e^{-M\lambda} \cos M + \lambda e^{-M\lambda} \sin M - 1}{\lambda^2 + 1}. \end{aligned}$$

By the fundamental theorem of calculus,

$$\begin{aligned} f_M(\Lambda)-f_M(0) &= \int_0^\Lambda f_M'(\lambda)d\lambda \\ &= \int_0^\Lambda \frac{e^{-M\lambda}\cos M+\lambda e^{-M\lambda}\sin M-1}{\lambda^2+1}d\lambda \\ &= \epsilon(M,\Lambda)-\arctan(\Lambda) \end{aligned} \tag{11.11}$$

where

$$\epsilon(M,\Lambda)=\int_0^\Lambda \frac{e^{-M\lambda}\cos M+\lambda e^{-M\lambda}\sin M}{\lambda^2+1}d\lambda.$$

Using $|\sin x|\le|x|$ [4], we have the estimates:

$$\begin{aligned} |\epsilon(M,\Lambda)| &\le \int_0^\Lambda \left|\frac{e^{-M\lambda}\cos M+\lambda e^{-M\lambda}\sin M}{\lambda^2+1}\right|d\lambda \\ &\le \int_0^\Lambda e^{-M\lambda}\frac{1+\lambda}{\lambda^2+1}d\lambda \\ &\le \int_0^\infty e^{-M\lambda}\frac{\lambda+1}{\lambda^2+1}d\lambda \to 0 \text{ as } M\to\infty \text{ (DCT)} \end{aligned}$$

and

$$|f_M(\Lambda)|\le\int_0^M\left|\frac{\sin x}{x}\right|e^{-\lambda x}dx\le\int_0^M e^{-\lambda x}dx=\frac{1-e^{-M\Lambda}}{\Lambda}.$$

Using these estimates, we may let $M\to\infty$ in Eq. (11.11) to find

$$\limsup_{M\to\infty}|f_M(0)-\arctan(\Lambda)|\le 1/\Lambda$$

and then letting $\Lambda\to\infty$ we find

$$\lim_{M\to\infty}f_M(0)=\pi/2.$$

Hence $\delta_1(M):=|\pi/2-f_M(0)|\to 0$ as $M\to\infty$ and $\delta_2(M):=\sup_{\Lambda\ge 0}|\epsilon(M,\Lambda)|\to 0$ as $M\to\infty$. Therefore by Eq. (11.11),

$$\begin{aligned} |f_M(\Lambda)-\pi/2| &= |\epsilon(M,\Lambda)+f_M(0)-\pi/2-\arctan(\Lambda)| \\ &\le \delta_2(M)+\delta_1(M)+\arctan(\Lambda) \\ &= \delta(M)+\arctan(\Lambda) \end{aligned}$$

where $\delta(M)=\delta_1(M)+\delta_2(M)\to 0$ as $M\to\infty$, which proves Eq. (11.8). ■

12. Lebesgue measure on $\mathbb{R}^d$

In this section let

$$m^d:=\overbrace{m\times\cdots\times m}^{d\text{ times}}$$

and $(\mathcal{L}_d,m):=\left(\overline{\mathcal{B}_{\mathbb{R}^d}}^{m^d},\bar{m}^d\right)$ denote the completion of $\left(\mathcal{B}_{\mathbb{R}^d},m^d\right)$. The measure λ is called Lebesgue measure on the Lebesgue measurable set $\mathcal{L}_d$.

[4]By the fundamental theorem of calculus,

$$|\sin x|=\left|\int_0^x\cos y dy\right|\le\left|\int_0^x|\cos y|\,dy\right|\le\left|\int_0^x 1dy\right|=|x|.|$$

Theorem 12.1. *Lebesgue measure λ is translation invariant.*

Proof. Let $A = J_1 \times \cdots \times J_d$ with $J_i \in \mathcal{B}_{\mathbb{R}}$ and $x \in \mathbb{R}^d$. Then

$$x + A = (x_1 + J_1) \times (x_2 + J_2) \times \cdots \times (x_d + J_d)$$

and therefore by translation invariance of m on $\mathcal{B}_{\mathbb{R}}$ we find that

$$m^d(x + A) = m(x_1 + J_1) \ldots m(x_d + J_d) = m(J_1) \ldots m(J_d) = m^d(A)$$

and hence $m^d(x + A) = m^d(A)$ for all $A \in \mathcal{B}_{\mathbb{R}^d}$ by Theorem 4.7. From this fact we see that the measure $m^d(x + \cdot)$ and $m^d(\cdot)$ have the same null sets. Using this it is easily seen that $m(x + A) = m(A)$ for all $A \in \mathcal{L}_d$. ■

In the remainder of this section, let $T : \Omega \to T(\Omega) \subset_o \mathbb{R}^d$ be a C^1 – diffeomorphism.

Theorem 12.2 (Change of Variables Theorem)**.** *Let $\Omega \subset_o \mathbb{R}^d$ be an open set and $T : \Omega \to T(\Omega) \subset_o \mathbb{R}^d$ be a C^1 – diffeomorphism. Then for any Borel measurable $f : T(\Omega) \to [0, \infty]$ we have*

$$\int_\Omega f \circ T |\det T'| dm = \int_{T(\Omega)} f \; dm. \tag{12.1}$$

Proof. We will carry out the proof in a number of steps.

Step 1. Suppose that

$$\Omega \xrightarrow{T} T(\Omega) \xrightarrow{S} S(T(\Omega))$$

are two C^1 – diffeomorphisms and Theorem 12.2 holds for T and separately, then it holds for $S \circ T$. Indeed

$$\begin{aligned} \int_\Omega f \circ S \circ T |\det{(S \circ T)'}| dm &= \int_\Omega f \circ S \circ T |\det{(S' \circ T)} T'| dm \\ &= \int_\Omega (|\det S'| \, f \circ S) \circ T |T'| dm \\ &= \int_{T(\Omega)} |\det S'| \, f \circ S dm = \int_{S(T(\Omega))} f dm. \end{aligned}$$

Step 2. Eq. (12.1) holds when $\Omega = \mathbb{R}^n$ and T is linear and invertible. This will be proved in Theorem 12.3 below. The proof is a simple application of Fubini's theorem, the scaling and translation invariance properties of one dimensional Lebesgue measure and the fact that by row reduction arguments T may be written as a product of "elementary" transformations. It is here that we use the result in Step 1.

Step 3. For all $A \in \mathcal{B}_\Omega$,

$$m(T(A)) \leq \int_A |\det T'| \, dm. \tag{12.2}$$

This will be proved in Theorem 12.6 below.

Step 4. Step 3. implies the general case. To see this, let $B \in \mathcal{B}_{T(\Omega)}$ and $A = T^{-1}(B)$ in Eq. (12.2) to learn that

$$\int_\Omega 1_A dm = m(A) \leq \int_{T^{-1}(A)} |\det T'| \, dm = \int_\Omega 1_A \circ T \, |\det T'| \, dm.$$

Using linearity we may conclude from this equation that

$$\int_{T(\Omega)} f dm \leq \int_{\Omega} f \circ T \left|\det T'\right| dm \tag{12.3}$$

for all non-negative simple functions f on $T(\Omega)$. Using Theorem 7.32 and the monotone convergence theorem one easily extends this equation to hold for all nonnegative measurable functions f on $T(\Omega)$.

Applying Eq. (12.3) with Ω replaced by $T(\Omega)$, T replaced by T^{-1} and f by $g : \Omega \to [0,\infty]$, we see that

$$\int_{\Omega} g dm = \int_{T^{-1}(T(\Omega))} g dm \leq \int_{T(\Omega)} g \circ T^{-1} \left|\det \left(T^{-1}\right)'\right| dm \tag{12.4}$$

for all Borel measurable g. Taking $g = (f \circ T) \left|\det T'\right|$ in this equation shows,

$$\begin{aligned} \int_{\Omega} f \circ T \left|\det T'\right| dm &\leq \int_{T(\Omega)} f \left|\det T' \circ T^{-1}\right| \left|\det \left(T^{-1}\right)'\right| dm \\ &= \int_{T(\Omega)} f dm \end{aligned} \tag{12.5}$$

wherein the last equality we used the fact that $T \circ T^{-1} = id$ so that $\left(T' \circ T^{-1}\right)\left(T^{-1}\right)' = id$ and hence $\det T' \circ T^{-1} \det \left(T^{-1}\right)' = 1$.

Combining Eqs. (12.3) and (12.5) proves Eq. (12.1). ■

We now fill in the missing details in the proof.

Theorem 12.3. *Suppose $T \in GL(d,\mathbb{R})$.*

1. *$f : \mathbb{R}^d \to \mathbb{R}$ is $(\mathcal{L}_d, \mathcal{B})$ measurable then so is $f \circ T$ if $f \geq 0$ on $f \in L^1$ then*

$$\int f(y)dy = |\det T| \int f \circ T(x) dx \tag{12.6}$$

2. *If $E \in \mathcal{L}_d$ and $E \subset \Omega$ then $T(E) \in \mathcal{L}_d$ and $m(T(E)) = |\det T|\, m(E)$.*

Proof. Since f is **Borel** measurable and $T : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and hence Borel measurable, $f \circ T$ is also Borel measurable. We now break the proof of Eq. (12.6) into a number of cases. In each case we make use Tonelli's theorem and the basic properties of one dimensional Lebesgue measure.

1. Suppose that $i < k$ and

$$T(x_1, x_2 \dots, x_d) = (x_1, \dots, x_{i-1}, x_k, x_{i+1} \dots, x_{k-1}, x_i, x_{k+1}, \dots x_d)$$

then by Tonelli's theorem,

$$\begin{aligned} \int f \circ T(x_1, \dots, x_d) &= \int f(x_1, \dots, x_k, \dots x_i, \dots x_d) dx_1 \dots dx_d \\ &= \int f(x_1, \dots, x_d) dx_1 \dots dx_d \end{aligned}$$

which prove Eq. (12.6) in this case since $|\det T| = 1$.

(2) Suppose that $c \in \mathbb{R}$ and $T(x_1, \dots x_k, \dots, x_d) = (x_1, \dots, cx_k, \dots x_d)$, then

$$\int f \circ T(x_1, \dots, x_d) dm = \int f(x_1, \dots, cx_k, \dots x_i, \dots x_d) dx_1 \dots dx_k \dots dx_d$$
$$= |c|^{-1} \int f(x_1, \dots, x_d) dx_1 \dots dx_d$$
$$= |\det T|^{-1} \int f \, dm$$

which again proves Eq. (12.6) in this case.

(3) Suppose that

$$T(x_1, x_2 \dots, x_d) = (x_1, \dots, x_i + cx_k, \dots x_k, \dots x_d).$$

Then

$$\int f \circ T(x_1, \dots, x_d) dm = \int f(x_1, \dots, x_i + cx_k, \dots x_k, \dots x_d) dx_1 \dots dx_i \dots dx_k \dots dx_d$$
$$= \int f(x_1, \dots, x_i, \dots x_k, \dots x_d) dx_1 \dots dx_i \dots dx_k \dots dx_d$$
$$= \int f(x_1, \dots, x_d) dx_1 \dots dx_d$$

where in the second inequality we did the x_i integral first and used translation invariance of Lebesgue measure. Again this proves Eq. (12.6) in this case since $\det(T) = 1$.

Since every invertible matrix is a product of matrices of the type occurring in steps 1. – 3. above, it follows that Eq. (12.6) holds in general. For the second assertion, let $E \in \mathcal{B}_{\mathbb{R}^d}$ and take $f = 1_E$ in Eq. (12.6) to learn that

$$|\det T| m(T^{-1}(E)) = |\det T| \int 1_{T^{-1}(E)} dm = |\det T| \int 1_E \circ T dm = \int 1_E dm = m(E).$$

Replacing T by T^{-1} in this equation shows that

$$m(T(E)) = |\det T| \, m(E)$$

for all $E \in \mathcal{B}_{\mathbb{R}^d}$. In particular this shows that $m \circ T$ and m have the same null sets and therefore the completion of $\mathcal{B}_{\mathbb{R}^d}$ is $\mathcal{L}_d$ for both measures. It is also clear that

$$m(T(E)) = |\det T| \, m(E)$$

for all $E \in \mathcal{L}_d$. ■

Notation 12.4. For $a, b \in \mathbb{R}^d$ we will write $a \le b$ is $a_i \le b_i$ for all i and $a < b$ if $a_i < b_i$ for all i. Given $a < b$ let $[a, b] = \prod_{i=1}^d [a_i, b_i]$ and $(a, b] = \prod_{i=1}^d (a_i, b_i]$. (Notice that the closure of $(a, b]$ is $[a, b]$.) We will say that $Q = (a, b]$ is a cube provided that $b_i - a_i = 2\delta > 0$ is a constant independent of i. When Q is a cube, let

$$x_Q := a + (\delta, \delta, \dots, \delta)$$

be the center of the cube.

Notice that with this notation, if Q is a cube of side length 2δ,

$$\bar{Q} = \{x \in \mathbb{R}^d : |x - x_Q| \le \delta\} \tag{12.7}$$

and the interior (Q^0) of Q may be written as

$$Q^0 = \{x \in \mathbb{R}^d : |x - x_Q| < \delta\}.$$

Notation 12.5. For $a \in \mathbb{R}^d$, let $|a| = \max_i |a_i|$ and if T is a $d \times d$ matrix let $\|T\| = \max_i \sum_j |T_{ij}|$.

A key point of this notation is that

$$\begin{aligned} |Ta| &= \max_i \left| \sum_j T_{ij} a_j \right| \le \max_i \sum_j |T_{ij}| \, |a_j| \\ &\le \|T\| \, |a| . \end{aligned} \tag{12.8}$$

Theorem 12.6. *Let $\Omega \subset_o \mathbb{R}^d$ be an open set and $T : \Omega \to T(\Omega) \subset_o \mathbb{R}^d$ be a C^1 – diffeomorphism. Then for any $A \in \mathcal{B}_\Omega$,*

$$m(T(A)) \le \int_A |\det T'(x)| dx. \tag{12.9}$$

Proof. Step 1. We will first assume that $A = Q = (a, b]$ is a cube such that $\bar{Q} = [a, b] \subset \Omega$. Let $\delta = (b_i - a_i)/2$ be half the side length of Q. By the fundamental theorem of calculus (for Riemann integrals) for $x \in Q$,

$$\begin{aligned} T(x) &= T(x_Q) + \int_0^1 T'(x_Q + t(x - x_Q))(x - x_Q) dt \\ &= T(x_Q) + T'(x_Q) S(x) \end{aligned}$$

where

$$S(x) = \left[\int_0^1 T'(x_Q)^{-1} T'(x_Q + t(x - x_Q)) dt \right] (x - x_Q).$$

Therefore $T(Q) = T(x_Q) + T'(x_Q) S(Q)$ and hence

$$\begin{aligned} m(T(Q)) &= m \left(T(x_Q) + T'(x_Q) S(Q) \right) = m \left(T'(x_Q) S(Q) \right) \\ &= |\det T'(x_Q)| \, m \left(S(Q) \right) . \end{aligned} \tag{12.10}$$

Now for $x \in \bar{Q}$, i.e. $|x - x_Q| \le \delta$,

$$\begin{aligned} |S(x)| &\le \left\| \int_0^1 T'(x_Q)^{-1} T'(x_Q + t(x - x_Q)) dt \right\| |x - x_Q| \\ &\le h(x_Q, x) \delta \end{aligned}$$

where

$$h(x_Q, x) := \int_0^1 \left\| T'(x_Q)^{-1} T'(x_Q + t(x - x_Q)) \right\| dt. \tag{12.11}$$

Hence

$$S(Q) \subset \max_{x \in Q} h(x_Q, x) \{ x \in \mathbb{R}^d : |x| \le \delta \max_{x \in Q} h^d(x_Q, x) \}$$

and

$$m(S(Q)) \le \max_{x \in Q} h(x_Q, x)^d \, (2\delta)^d = \max_{x \in Q} h^d(x_Q, x) m(Q). \tag{12.12}$$

Combining Eqs. (12.10) and (12.12) shows that

$$m(T(Q)) \le |\det T'(x_Q)| \, m(Q) \cdot \max_{x \in Q} h^d(x_Q, x). \tag{12.13}$$

To refine this estimate, we will subdivide Q into smaller cubes, i.e. for $n \in \mathbb{N}$ let

$$\mathcal{Q}_n = \left\{ (a, a + \frac{2}{n}(\delta, \delta, \dots, \delta)] + \frac{2\delta}{n} \xi : \xi \in \{0, 1, 2, \dots, n\}^d \right\}.$$

Notice that $Q = \coprod_{A\in\mathcal{Q}_n} A$. By Eq. (12.13),

$$m(T(A)) \le |\det T'(x_A)|\, m(A) \cdot \max_{x\in A} h^d(x_A, x)$$

and summing the equation on A gives

$$m(T(Q)) = \sum_{A\in\mathcal{Q}_n} m(T(A)) \le \sum_{A\in\mathcal{Q}_n} |\det T'(x_A)|\, m(A) \cdot \max_{x\in A} h^d(x_A, x).$$

Since $h^d(x,x) = 1$ for all $x \in \bar{Q}$ and $h^d : \bar{Q}\times\bar{Q} \to [0,\infty)$ is continuous function on a compact set, for any $\epsilon > 0$ there exists n such that if $x, y \in \bar{Q}$ and $|x-y| \le \delta/n$ then $h^d(x,y) \le 1+\epsilon$. Using this in the previously displayed equation, we find that

$$\begin{aligned} m(T(Q) &\le (1+\epsilon)\sum_{A\in\mathcal{Q}_n} |\det T'(x_A)|\, m(A) \\ &= (1+\epsilon)\int_Q \sum_{A\in\mathcal{Q}_n} |\det T'(x_A)|\, 1_A(x) dm(x). \end{aligned} \tag{12.14}$$

Since $|\det T'(x)|$ is continuous on the compact set $\bar{Q}$, it easily follows by uniform continuity that

$$\sum_{A\in\mathcal{Q}_n} |\det T'(x_A)|\, 1_A(x) \to |\det T'(x)| \text{ as } n\to\infty$$

and the convergence in uniform on $\bar{Q}$. Therefore the dominated convergence theorem enables us to pass to the limit, $n\to\infty$, in Eq. (12.14) to find

$$m(T(Q)) \le (1+\epsilon)\int_Q |\det T'(x)|\, dm(x).$$

Since $\epsilon > 0$ is arbitrary we are done we have shown that

$$m(T(Q)) \le \int_Q |\det T'(x)|\, dm(x).$$

Step 2. We will now show that Eq. (12.9) is valid when $A = U$ is an open subset of Ω. For $n\in\mathbb{N}$, let

$$\mathcal{Q}_n = \left\{(0, (\delta,\delta,\dots,\delta)] + 2^{-n}\xi : \xi\in\mathbb{Z}^d\right\}$$

so that $\mathcal{Q}_n$ is a partition of $\mathbb{R}^d$. Let $\mathcal{F}_1 := \left\{A\in\mathcal{Q}_1 : \bar{A}\subset U\right\}$ and define $\mathcal{F}_n \subset \cup_{k=1}^n \mathcal{Q}_k$ inductively as follows. Assuming $\mathcal{F}_{n-1}$ has been defined, let

$$\begin{aligned} \mathcal{F}_n &= \mathcal{F}_{n-1}\cup\left\{A\in\mathcal{Q}_n : \bar{A}\subset U \text{ and } A\cap B = \emptyset \text{ for all } B\in\mathcal{F}_{n-1}\right\} \\ &= \mathcal{F}_{n-1}\cup\left\{A\in\mathcal{Q}_n : \bar{A}\subset U \text{ and } A\nsubseteq B \text{ for any } B\in\mathcal{F}_{n-1}\right\} \end{aligned}$$

Now set $\mathcal{F} = \cup\mathcal{F}_n$ (see Figure 8) and notice that $U = \coprod_{A\in\mathcal{F}} A$. Indeed by construction, the sets in $\mathcal{F}$ are pairwise disjoint subset of U so that $\coprod_{A\in\mathcal{F}} A \subset U$. If $x\in U$, there exists an n and $A\in\mathcal{Q}_n$ such that $x\in A$ and $\bar{A}\subset U$. Then by construction of $\mathcal{F}$, either $A\in\mathcal{F}$ or there is a set $B\in\mathcal{F}$ such that $A\subset B$. In either case $x\in\coprod_{A\in\mathcal{F}} A$ which shows that $U = \coprod_{A\in\mathcal{F}} A$. Therefore by step 1.,

$$\begin{aligned} m(T(U)) &= m(T(\cup_{A\in\mathcal{F}} A)) = m((\cup_{A\in\mathcal{F}} T(A))) \\ &= \sum_{A\in\mathcal{F}} m(T(A)) \le \sum_{A\in\mathcal{F}} \int_A |\det T'(x)|\, dm(x) \\ &= \int_U |\det T'(x)|\, dm(x) \end{aligned}$$

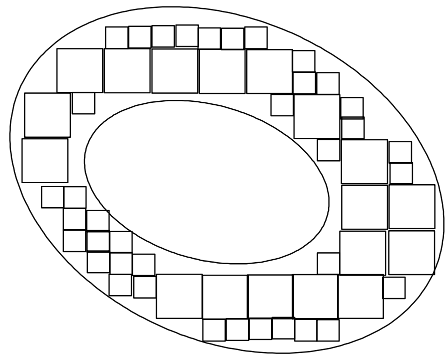

FIGURE 8. Filling out an open set with almost disjoint cubes. We have drawn $\mathcal{F}_2$.

which proves step 2.

Step 3. For general $A \in \mathcal{B}_\Omega$ let μ be the measure,

$$\mu(A) := \int_A |\det T'(x)|\, dm(x).$$

Then $m \circ T$ and μ are (σ – finite measures as you should check) on $\mathcal{B}_\Omega$ such that $m \circ T \leq \mu$ on open sets. By regularity of these measures, we may conclude that $m \circ T \leq \mu$. Indeed, if $A \in \mathcal{B}_\Omega$,

$$m\left(T(A)\right) = \inf_{U \subset_o \Omega} m\left(T(U)\right) \leq \inf_{U \subset_o \Omega} \mu(U) = \mu(A) = \int_A |\det T'(x)|\, dm(x).$$

■

12.1. **Polar Coordinates and Surface Measure.** Let

$$S^{d-1} = \{x \in \mathbb{R}^d : |x|^2 := \sum_{i=1}^d x_i^2 = 1\}$$

be the unit sphere in $\mathbb{R}^d$. Let $\Phi : \mathbb{R}^d \setminus (0) \to (0,\infty) \times S^{d-1}$ and Φ^{-1} be the inverse map given by

$$\Phi(x) := (|x|, \frac{x}{|x|}) \text{ and } \Phi^{-1}(r,\omega) = r\omega \tag{12.15}$$

respectively. Since Φ and Φ^{-1} are continuous, they are Borel measurable.

Consider the measure $\Phi_* m$ on $\mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{S^{d-1}}$ given by

$$\Phi_* m(A) := m\left(\Phi^{-1}(A)\right)$$

for all $A \in \mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{S^{d-1}}$. For $E \in \mathcal{B}_{S^{d-1}}$ and $a > 0$, let

$$E_a := \{r\omega : r \in (0,a] \text{ and } \omega \in E\} = \Phi^{-1}((0,a] \times E) \in \mathcal{B}_{\mathbb{R}^d}.$$

Noting that $E_a = aE_1$, we have for $0 < a < b$, $E \in \mathcal{B}_{S^{d-1}}, E$ and $A = (a,b] \times E$ that

$$\Phi^{-1}(A) = \{r\omega : r \in (a,b] \text{ and } \omega \in E\} \tag{12.16}$$

$$= bE_1 \setminus aE_1. \tag{12.17}$$

Therefore,

$$\begin{aligned}(\Phi_* m)((a,b]\times E) &= m(bE_1\setminus aE_1) = m(bE_1) - m(aE_1)\\ &= b^d m(E_1) - a^d m(E_1)\\ &= d\cdot m(E_1)\int_a^b r^{d-1}dr. \qquad (12.18)\end{aligned}$$

Let ρ denote the unique measure on $\mathcal{B}_{(0,\infty)}$ such that

$$\rho(J) = \int_J r^{d-1}dr \qquad (12.19)$$

for all $J\in\mathcal{B}_{(0,\infty)}$. Symbolically, we will abbreviate this by writing $\rho(dr) = r^{d-1}dr$.

Definition 12.7. For $E\in\mathcal{B}_{S^{d-1}}$, let $\sigma(E) := d\cdot m(E_1)$. We call σ the surface measure on S.

It is easy to check that σ is a measure. Indeed if $E\in\mathcal{B}_{S^{d-1}}$, then $E_1 = \Phi^{-1}((0,1]\times E)\in\mathcal{B}_{\mathbb{R}^d}$ so that $m(E_1)$ is defined. Moreover if $E=\coprod_{i=1}^\infty E_i$, then $E_1 = \coprod_{i=1}^\infty (E_i)_1$ and

$$\sigma(E) = d\cdot m(E_1) = \sum_{i=1}^\infty m((E_i)_1) = \sum_{i=1}^\infty \sigma(E_i).$$

The intuition behind this definition is as follows. If $E\subset S^{d-1}$ is a set and $\epsilon>0$ is a small number, then the volume of

$$(1,1+\epsilon]\cdot E = \{r\omega : r\in(1,1+\epsilon] \text{ and } \omega\in E\}$$

should be approximately given by $m((1,1+\epsilon]\cdot E)\cong\sigma(E)\epsilon$. On the other hand

$$m((1,1+\epsilon]E) = m(E_{1+\epsilon}\setminus E_1) = \left\{(1+\epsilon)^d - 1\right\} m(E_1).$$

Therefore we expect the area of E should be given by

$$\sigma(E) = \lim_{\epsilon\downarrow 0}\frac{\left\{(1+\epsilon)^d-1\right\} m(E_1)}{\epsilon} = dm(E_1).$$

According to these definitions and Eq. (12.18) we have shown that

$$\Phi_* m((a,b]\times E) = \rho((a,b])\cdot\sigma(E). \qquad (12.20)$$

Let

$$\mathcal{E} = \{(a,b]\times E : 0<a<b, E\in\mathcal{B}_{S^{d-1}}\},$$

then $\mathcal{E}$ is an elementary class. Since $\sigma(\mathcal{E}) = \mathcal{B}_{(0,\infty)}\otimes\mathcal{B}_{S^{d-1}}$, we conclude from Eq. (12.20) that

$$\Phi_* m = \rho\times\sigma$$

and this implies the following theorem.

Theorem 12.8. *If $f:\mathbb{R}^d\to[0,\infty]$ is a $(\mathcal{B}_{R^d},\mathcal{B})$–measurable function then*

$$\int f(x)dm(x) = \int_{[0,\infty)\times S} f(r\,\omega)\;d\sigma(\omega) r^{d-1}dr. \qquad (12.21)$$

Let us now work out some integrals using Eq. (12.21).

Lemma 12.9. *Let $a > 0$ and*

$$I_d(a) := \int_{\mathbb{R}^d} e^{-a|x|^2} dm(x).$$

Then $I_d(a) = (\pi/a)^{d/2}$.

Proof. By Tonelli's theorem and induction,

$$\begin{aligned} I_d(a) &= \int_{\mathbb{R}^{d-1}\times\mathbb{R}} e^{-a|y|^2} e^{-at^2} m_{d-1}(dy)\ dt \\ &= I_d(a) I_1(a) = I_1^d(a). \end{aligned} \tag{12.22}$$

So it suffices to compute:

$$I_2(a) = \int_{\mathbb{R}^2} e^{-a|x|^2} dm(x) = \int_{\mathbb{R}^2\setminus\{0\}} e^{-a(x_1^2+x_2^2)} dx_1 dx_2.$$

We now make the change of variables, .

$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta \text{ for } 0 < r < \infty \text{ and } 0 < \theta < 2\pi.$$

In vector form this transform is

$$x = T(r,\Theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

and the differential is given by

$$T'(r,\Theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and the Jacobian determinant is

$$\det T'(r,\Theta) = r\cos^2\theta + r\sin^2\theta = r.$$

Hence by the change of variables formula,

$$\begin{aligned} I_2(a) &= \int_0^\infty dr\ r \int_0^{2\pi} d\theta\ e^{-ar^2} \\ &= 2\pi \int_0^\infty r\ e^{-ar^2}\ dr \\ &= 2\pi \lim_{M\to\infty} \int_0^M r\ e^{-ar^2}\ dr \\ &= 2\pi \lim_{M\to\infty} \frac{e^{-ar^2}}{-2a} \int_0^M = \frac{2\pi}{2a} = \pi/a. \end{aligned}$$

This shows that $I_2(a) = \pi/a$ and the result now follows from Eq. (12.22). ■

Corollary 12.10. *The surface area $\sigma(S^{d-1})$ of the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is*

$$\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{12.23}$$

where Γ is the gamma function given by

$$\Gamma(x) := \int_0^\infty r^{x-1} e^{-r} dr \tag{12.24}$$

Moreover, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1) = 1$.

Proof. We may alternatively compute $I_d(1) = \pi^{d/2}$ using Theorem 12.8;

$$I_d(1) = \int_0^\infty dr\ r^{d-1}e^{-r^2} \int_{S^{d-1}} d\sigma = \sigma(S^{d-1}) \int_0^\infty r^{d-1}e^{-r^2} dr.$$

We simplify this last integral by making the change of variables $u = r^2$ so that $r = u^{1/2}$ and $dr = \frac{1}{2}u^{-1/2}du$. The result is

$$\begin{aligned} \int_0^\infty r^{d-1}e^{-r^2}dr &= \int_0^\infty u^{\frac{d-1}{2}}e^{-u}\frac{1}{2}u^{-1/2}du \\ &= \frac{1}{2}\int_0^\infty u^{\frac{d}{2}-1}e^{-u}du \\ &= \frac{1}{2}\Gamma(d/2). \qquad (12.25) \end{aligned}$$

Collecting these observations implies that

$$\pi^{d/2} = I_d(1) = \frac{1}{2}\sigma(S^{d-1})\Gamma(d/2)$$

which proves Eq. (12.23).

The computation of $\Gamma(1)$ is easy and is left to the reader. By Eq. (12.25),

$$\begin{aligned} \Gamma(1/2) &= 2\int_0^\infty e^{-r^2}dr = \int_{-\infty}^\infty e^{-r^2}dr \\ &= I_1(1) = \sqrt{\pi}. \end{aligned}$$

■

13. Signed Measures

Definition 13.1. A signed measure ν on a measurable space $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ such that

(1) Either $\nu(\mathcal{M}) \subseteq (-\infty, \infty]$ or $\nu(\mathcal{M}) \subseteq [-\infty, \infty)$.
(2) ν is countably additive, this is to say if $E = \coprod_{j=1}^\infty E_j$ with $E_j \in \mathcal{M}$, then $\nu(E) = \sum_{j=1}^\infty \nu(E_j)$.[5]
(3) $\nu(\emptyset) = 0$

Example 13.2. Suppose that μ_+ and μ_- are two positive measures on $\mathcal{M}$ such that either $\mu_+(X) < \infty$ or $\mu_-(X) < \infty$, then $\nu = \mu_+ - \mu_-$ is a signed measure.

Example 13.3. Suppose that $f : X \to \overline{\mathbb{R}}$ measurable and either $\int_E f^+ d\mu$ or $\int_E f^- d\mu < \infty$, then

$$\nu(E) = \int_E f\ d\mu$$

is a signed measure. This is actually a special case of the last example with $\mu_\pm(E) \equiv \int_E f^\pm d\mu$. Notice that the measure $\mu_\pm$ in this example have the property that they

[5] If $\nu(E) \in \mathbb{R}$ then the series $\sum_{j=1}^\infty \nu(E_j)$ is absolutely convergent since it is independent of rearrangements.

are concentrated on disjoint sets, namely μ_+ "lives" on $\{f > 0\}$ and μ_- "lives" on the set $\{f < 0\}$.

The main theorems of this section assert that these are the only examples of signed measures.

13.1. Hahn Decomposition Theorem.

Definition 13.4. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$, then

(1) E is **positive** if for all $A \in \mathcal{M}$ such that $A \subset E$, $\nu(A) \geq 0$, i.e. $\nu|_{\mathcal{M}_E} \geq 0$.
(2) E is **negative** if for all $A \in \mathcal{M}$ such that $A \subset E$, $\nu(A) \leq 0$, i.e. $\nu|_{\mathcal{M}_E} \leq 0$.
(3) E is **null** if for all $A \in \mathcal{M}$ such that $A \subset E$, i.e. $\nu|_{\mathcal{M}_E} = 0$.

Here $\mathcal{M}_E \equiv \{A \cap E : A \in \mathcal{M}\}$ = trace of M on E.

Lemma 13.5. *Suppose that ν is a signed measure on $(X, \mathcal{M})$. Then*

(1) *Any subset of a positive set is positive.*
(2) *The countable union of positive (negative or null) sets is still positive (negative or null).*
(3) *Let us now further assume that $\nu(\mathcal{M}) \subset [-\infty, \infty)$ and $E \in \mathcal{M}$ is a set such that $\nu(E) \in (0, \infty)$. Then there exists a positive set $P \subseteq E$ such that $\nu(P) \geq \nu(E)$.*

Proof. The first assertion is obvious. If $P_j \in \mathcal{M}$ are positive sets, let $P = \bigcup_{n=1}^{\infty} P_n$. By replacing P_n by the positive set $P_n \setminus \left(\bigcup_{j=1}^{n-1} P_j \right)$ we may assume that the $\{P_n\}_{n=1}^{\infty}$ are pairwise disjoint so that $P = \coprod_{n=1}^{\infty} P_n$. Now if $E \subset P$ and $E \in \mathcal{M}$, $E = \coprod_{n=1}^{\infty} (E \cap P_n)$ so

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap P_n) \geq 0.$$

which shows that P is positive. The proof for the negative and the null case is analogous.

The idea for proving the third assertion is to keep removing "big" sets of negative measure from E. The set remaining from this procedure will be P. We now begin the formal proof.

For all $A \in \mathcal{M}$ let $n(A) = 1 \wedge \sup\{-\nu(B) : B \subseteq A\}$. Since $\nu(\emptyset) = 0$, $n(A) \geq 0$ and $n(A) = 0$ iff A is positive. Choose $A_0 \subseteq E$ such that $-\nu(A_0) \geq \frac{1}{2} n(E)$ and set $E_1 = E \setminus A_0$, then choose $A_1 \subseteq E_1$ such that $-\nu(A_1) \geq \frac{1}{2} n(E_1)$ and set $E_2 = E \setminus (A_0 \cup A_1)$. Continue this procedure inductively so that if $A_0, \ldots, A_{k-1}$ have been chosen let

$$E_k = E \setminus \left(\coprod_{i=0}^{k-1} A_i \right)$$

and choose $A_k \subset E_k$ such that $-\nu(A_k) \geq \frac{1}{2} n(E_k)$. We will now show that

$$P := E \setminus \coprod_{k=0}^{\infty} A_k = \bigcap_{k=0}^{\infty} E_k$$

is a positive set such that $\nu(P) \geq \nu(E)$.

Since

$$E = P \cup \coprod_{k=0}^{\infty} A_k,$$

$$\nu(E) - \nu(P) = \nu(E \setminus P) = \sum_{k=0}^{\infty} \nu(A_k) \leq -\frac{1}{2}\sum_{k=0}^{\infty} n(E_k) \tag{13.1}$$

and hence $\nu(E) \leq \nu(P)$. Moreover, $\nu(E) - \nu(P) > -\infty$ since $\nu(E) \geq 0$ and $\nu(P) \neq \infty$ by the assumption $\nu(\mathcal{M}) \subset [-\infty, \infty)$. Therefore we may conclude from Eq. (13.1) that $\sum_{k=0}^{\infty} n(E_k) < \infty$ and in particular $\lim_{k\to\infty} n(E_k) = 0$. Now if $A \subset P$ then $A \subset E_k$ for all k and this implies that $\nu(A) \geq 0$ since by definition of $n(E_k)$,

$$-\nu(A) \leq n(E_k) \to 0 \text{ as } k \to \infty.$$

■

Definition 13.6. Suppose that ν is a signed measure on $(X, \mathcal{M})$. A **Hahn decomposition** for ν is a partition $\{P, N\}$ of X such that P is positive and N is negative.

Theorem 13.7 (Hahn Decomposition Theorem). *Every signed measure space $(X, \mathcal{M}, \nu)$ has a Hahn decomposition, $\{P, N\}$. Moreover, if $\{\tilde{P}, \tilde{N}\}$ is another Hahn decomposition, then $P\Delta\tilde{P} = N\Delta\tilde{N}$ is a null set, so the decomposition is unique modulo null sets.*

Proof. With out loss of generality we may assume that $\nu(\mathcal{M}) \subset [-\infty, \infty)$. If not just consider $-\nu$ instead. Let us begin with the uniqueness assertion. Suppose that $A \in \mathcal{M}$, then

$$\nu(A) = \nu(A \cap P) + \nu(A \cap N) \leq \nu(A \cap P) \leq \nu(P)$$

and similarly

$$\nu(A) \leq \nu(\tilde{P}) \text{ for all } A \in \mathcal{M}.$$

and in particular we have

$$\begin{aligned} \nu(P) &\leq \nu(P \cup \tilde{P}) \leq \nu(\tilde{P}) \text{ and} \\ \nu(\tilde{P}) &\leq \nu(P \cup \tilde{P}) \leq \nu(P) \end{aligned}$$

which shows that

$$s := \nu(\tilde{P}) = \nu(P \cup \tilde{P}) = \nu(P).$$

Since

$$\begin{aligned} s &= \nu(P \cup \tilde{P}) = \nu(P) + \nu(\tilde{P}) - \nu(P \cap \tilde{P}) = 2s - \nu(P \cap \tilde{P}) \\ s &= \nu(P \cup \tilde{P}) = \nu(P) + \nu(\tilde{P}) \end{aligned}$$

we see that $\nu(P \cap \tilde{P}) = s$ and since

$$s = \nu(P \cup \tilde{P}) = \nu(P \cap \tilde{P}) + \nu(\tilde{P}\Delta P)$$

it follows that $\nu(\tilde{P}\Delta P) = 0$. Thus $N\Delta\tilde{N} = \tilde{P}\Delta P$ is a positive set with zero measure, i.e. $N\Delta\tilde{N} = \tilde{P}\Delta P$ is a null set and this proves the uniqueness assertion.

Let

$$s \equiv \sup\{\nu(A) : A \in \mathcal{M}\}$$

which is non-negative since $\nu(\emptyset) = 0$. If $s = 0$, we are done since $P = \emptyset$ and $N = X$ is the desired decomposition. So assume $s > 0$ and choose $A_n \in \mathcal{M}$ such that $\nu(A_n) > 0$ and $\lim_{n\to\infty} \nu(A_n) = s$. By Lemma 13.5, there exists positive sets $P_n \subseteq A_n$ such that $\nu(P_n) \geq \nu(A_n)$. Then clearly $s \geq \nu(P_n) \geq \nu(A_n) \to s$ as $n \to \infty$ implies that $s = \lim_{n\to\infty} \nu(P_n)$. The set $P \equiv \cup P_n$ is a positive set being the union of positive sets and since $P_n \subset P$ for all n,

$$\nu(P) \geq \nu(P_n) \to s \text{ as } n \to \infty.$$

This shows that $\nu(P) \geq s$ and hence by the definition of s, $s = \nu(P)$. In particular this shows that $s < \infty$.

We now claim that $N = P^c$ is a negative set and therefore, $\{P, N\}$ is the desired Hahn decomposition. If N were not negative, we could find $E \subset N = P^c$ such that $\nu(E) > 0$. We then would have

$$\nu(P \cup E) = \nu(P) + \nu(E) = s + \nu(E) > s$$

which contradicts the definition of s. ∎

13.2. Jordan Decomposition.

Definition 13.8. Let μ and ν be two signed measures on $(X, \mathcal{M})$. We say μ and ν are **mutually singular** and write $\mu \perp \nu$ if there exists $A \in \mathcal{M}$ such that A is ν-null and A^c is μ-null. i.e. ν and μ "live" on disjoint sets.

Remark 13.9. If μ_1, μ_2 and ν are signed measures on $(X, \mathcal{M})$ such that $\mu_1 \perp \nu$ and $\mu_2 \perp \nu$ and $\mu_1 + \mu_2$ is well defined, then $(\mu_1 + \mu_2) \perp \nu$. If $\{\mu_i\}_{i=1}^\infty$ is a sequence of positive measures such that $\mu_i \perp \nu$ for all i then $\mu = \sum_{i=1}^\infty \mu_i \perp \nu$ as well.

Proof. In both cases, choose $A_i \in \mathcal{M}$ such that A_i is ν-null and A_i^c is μ_i-null for all i. Then by Lemma 13.5, $A := \cup_i A_i$ is still a ν –null set. Since

$$A^c = \cap_i A_i^c \subset A_m^c \text{ for all } m$$

we see that A^c is a μ_i - null set for all i and is therefore a null set for $\mu = \sum_{i=1}^\infty \mu_i$. This shows that $\mu \perp \nu$. ∎

Definition 13.10. Let $X = P \cup N$ be a Hahn Decomposition of ν and define

$$\nu_+(E) = \nu(P \cap E)$$
$$\nu_-(E) = -\nu(N \cap E)$$

Suppose $X = \widetilde{P} \cup \widetilde{N}$ is another Hahn Decomposition and $\widetilde{\nu}_\pm$ are define as above with P and N replaced by $\widetilde{P}$ and $\widetilde{N}$ respectively. Then

$$\begin{aligned}\widetilde{\nu}_+(E) &= \nu(E \cap \widetilde{P}) = \nu(E \cap \widetilde{P} \cap P) + \nu((E \cap \widetilde{P} \cap N) \\ &= \nu(E \cap \widetilde{P} \cap P)\end{aligned}$$

since $N \cap \tilde{P}$ is both positive and negative and hence null. Similarly $\nu_+(E) = \nu(E \cap \widetilde{P} \cap P)$ showing that $\nu_+ = \widetilde{\nu}_+$ and therefore also that $\nu_- = \widetilde{\nu}_-$.

Theorem 13.11. ***Jordan Decomposition***: *There exists unique positive measure $\nu_\pm$ such that $\nu_+ \perp \nu_-$ and $\nu = \nu_+ - \nu_-$.*

Proof. Existence has been proved. For uniqueness. Suppose $\nu = \nu_+ - \nu_-$ is a Jordan Decomposition. Since $\nu_+ \perp \nu_-$ there exists $P, N = P^c \in \mathcal{M}$ such that $\nu_+(N) = 0$ and $\nu_-(P) = 0$. Then clearly P is positive for ν and N is negative for ν. Now $\nu(E \cap P) = \nu_+(E)$ and $\nu(E \cap N) = \nu_-(E)$. The uniqueness now follows from the remarks after Definition 13.10. ∎

Definition 13.12. $|\nu|(E) = \nu_+(E) + \nu_-(E)$ is called the total variation of ν. A signed measure is called $\sigma -$ **finite** provided that $|\nu| := \nu_+ + \nu_-$ is a σ finite measure.

Lemma 13.13. *Let ν be a signed measure on $(X, \mathcal{M})$ and $A \in \mathcal{M}$. Then $|\nu|(A) < \infty$ iff $|\nu(A)| < \infty$ so that ν is σ finite iff there exists $X_n \in \mathcal{M}$ such that $|\nu(X_n)| < \infty$ and $X = \cup_{n=1}^{\infty} X_n$. Also suppose that $P, N \in \mathcal{M}$ is a Hahn decomposition for ν and let $g = 1_P - 1_N$, then $d\nu = gd|\nu|$, i.e.*

$$\nu(A) = \int_A gd|\nu| \text{ for all } A \in \mathcal{M}.$$

Proof. Let $P, N \in \mathcal{M}$ be a Hahn decomposition for ν, then

$$\begin{aligned} \nu(A) &= \nu(A \cap P) + \nu(A \cap N) \\ &= |\nu(A \cap P)| - |\nu(A \cap N)| \text{ and} \\ |\nu|(A) &= \nu(A \cap P) - \nu(A \cap N) \\ &= |\nu(A \cap P)| + |\nu(A \cap N)|. \end{aligned} \tag{13.2}$$

Therefore $|\nu(A)| < \infty$ iff $|\nu(A \cap P)|$ and $|\nu(A \cap N)|$ are finite iff $|\nu|(A) < \infty$ proving the first assertion. The second assertion is a consequence of the following identity,

$$\begin{aligned} \nu(A) &= \nu(A \cap P) + \nu(A \cap N) \\ &= |\nu|(A \cap P) - |\nu|(A \cap N) \\ &= \int_A (1_P - 1_N) d|\nu|. \end{aligned}$$

■

Remark 13.14. Suppose that μ is a positive measure on $(X, \mathcal{M})$ and $g : X \to \mathbb{R}$ is an extended integrable function. If ν is the signed measure $d\nu = gd\mu$, then $d\nu_+ = g_+ d\mu$, $d\nu_- = g_- d\mu$ and $d|\nu| = |g| d\mu$.

Proof. The pair, $P = \{g > 0\}$ and $N = \{g \leq 0\} = P^c$ is a Hahn decomposition for ν. Therefore

$$\nu_+(A) = \nu(A \cap P) = \int_{A \cap P} gd\mu = \int_A 1_{\{g>0\}} gd\mu = \int_A g_+ d\mu,$$

$$\nu_-(A) = -\nu(A \cap N) = -\int_{A \cap N} gd\mu = -\int_A 1_{\{g \leq 0\}} gd\mu = -\int_A g_- d\mu.$$

and

$$\begin{aligned} |\nu|(A) &= \nu_+(A) + \nu_-(A) \\ &= \int_A g_+ d\mu - \int_A g_- d\mu = \int_A (g_+ - g_-) d\mu \\ &= \int_A |g| d\mu. \end{aligned}$$

■

Definition 13.15. Let ν be a signed measure on $(X, \mathcal{M})$, let

$$L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$$

and for $f \in L^1(\nu)$ we define

$$\int_X f d\nu = \int_X f d\nu_+ - \int_X f d\nu_-.$$

Lemma 13.16. *Let μ be a positive measure on $(X, \mathcal{M})$, g be an extended integrable function on $(X, \mathcal{M}, \mu)$ and $d\nu = g d\mu$. Then $L^1(\nu) = L^1(|g|\, d\mu)$ and for $f \in L^1(\nu)$,*

$$\int_X f d\nu = \int_X f g d\mu.$$

Proof. We have already seen that $d\nu_+ = g_+ d\mu$, $d\nu_- = g_- d\mu$, and $d\,|\nu| = |g|\, d\mu$ so that $L^1(\nu) = L^1(|\nu|) = L^1(|g|\, d\mu)$ and for $f \in L^1(\nu)$,

$$\begin{aligned}
\int_X f d\nu &= \int_X f d\nu_+ - \int_X f d\nu_- \\
&= \int_X f g_+ d\mu - \int_X f g_- d\mu \\
&= \int_X f\,(g_+ - g_-)\, d\mu \\
&= \int_X f g d\mu.
\end{aligned}$$

■

Remark 13.17. Let ν be a signed measure on $(X, \mathcal{M})$, then $|\nu|(E) = \sup\{\int_E f\ d\nu : |f| \leq 1\}$.

Proof. If $E \in \mathcal{M}$ and $|f| \leq 1$, then

$$\begin{aligned}
\left|\int_E f\ d\nu\right| &= \left|\int_E f\ d\nu_+ - \int_E f\ d\nu^-\right| \leq \left|\int_E f\ d\nu_+\right| + \left|\int_E f\ d\nu^-\right| \\
&\leq \int_E |f| d\nu_+ + \int_E |f| d\nu_- = \int_E |f|\ d|\nu| \\
&\leq |\nu|\,(E)
\end{aligned}$$

which shows that

$$\sup\{\int_E f\ d\nu : |f| \leq 1\} \leq |\nu|(E).$$

For the reverse inequality, let $X = P \cup N$ be a Hahn decomposition of ν and define $f \equiv 1_P - 1_N$ then

$$\begin{aligned}
\int_E f\ d\nu &= \nu(E \cap P) - \nu(E \cap N) \\
&= \nu^+(E) + \nu^-(E) = |\nu|(E).
\end{aligned}$$

■

14. Radon-Nikodym Theorem

Definition 14.1. Let ν be a signed measure and μ be a positive measure on $(X, \mathcal{M})$. We say the ν is **absolutely continuous relative to** μ and write $\nu \ll \mu$ provided that $\nu(A) = 0$ whenever $\mu(A) = 0$. That is to say if A is a null set for μ then A is also a null set for ν.

Remark 14.2. Since $|\nu(A)| \le |\nu|(A)$ for all $A \in \mathcal{M}$, if $|\nu| \ll \mu$ then $\nu \ll \mu$. The converse is true as well. To see this let $P \in \mathcal{M}$ be chosen so that $\{P, N = P^c\}$ is a Hahn decomposition for ν. If $A \in \mathcal{M}$ and $\mu(A) = 0$ then $\nu(A \cap P) = 0$ and $\nu(A \cap N) = 0$ since $\mu(A \cap P) = 0$ and $\mu(A \cap N) = 0$. Therefore

$$|\nu|(A) = \nu(A \cap P) - \nu(A \cap N) = 0.$$

Lemma 14.3. *1) Let ν be a signed measure and μ be a positive measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu \equiv 0$. 2) Suppose that $\nu = \sum_{i=1}^{\infty} \nu_i$ where ν_i are positive measures on $(X, \mathcal{M})$ such that $\nu_i \ll \mu$, then $\nu \ll \mu$. Also if ν_1 and ν_2 are two signed measure such that $\nu_i \ll \mu$ for $i = 1, 2$ and $\nu = \nu_1 + \nu_2$ is well defined, then $\nu \ll \mu$.*

Proof. (1) Because $\nu \perp \mu$, there exists $A \in \mathcal{M}$ such that A is a ν-null set and $B = A^c$ is a μ - null set. Since B is μ – null and $\nu \ll \mu$, B is also ν – null. This shows by Lemma 13.5 that $X = A \cup B$ is also ν – null, i.e. ν is the zero measure. The proof of (2) is easy and is left to the reader. ■

Theorem 14.4. *Let ν be a finite signed measure and μ be a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(A)| < \epsilon$ whenever $A \in \mathcal{M}$ and $\mu(A) < \delta$.*

Proof. ($\Longrightarrow$) If $\mu(A) = 0$ then $|\nu(A)| < \epsilon$ for all $\epsilon > 0$ which shows that $\nu(A) = 0$, i.e. $\nu \ll \mu$.

($\Longleftarrow$)Since $|\nu(A)| \le |\nu|(A)$ it suffices to assume $\nu \ge 0$ and $\nu(X) < \infty$. Suppose for the sake of contradiction there exists $\epsilon > 0$ and $A_n \in \mathcal{M}$ such that $\nu(A_n) \ge \epsilon > 0$ while $\mu(A_n) \le \frac{1}{2^n}$. Let

$$A = \{A_n \text{ i.o.}\} = \bigcap_{N=1}^{\infty} \bigcup_{n \ge N} A_n$$

so that

$$\mu(A) = \lim_{N\to\infty} \mu\left(\cup_{n\ge N} A_n\right) \le \sum_{n=N}^{\infty} \mu(A_n) \le 2^{-(N-1)}$$

for all N and hence $\mu(A) = 0$. On the other hand,

$$\begin{aligned}\nu(A) &= \lim_{N\to\infty} \nu\left(\cup_{n\ge N} A_n\right) \\ &\ge \lim_{n\to\infty} \inf \nu(A_n) \ge \epsilon > 0\end{aligned}$$

showing that ν is not absolutely continuous relative to μ ($\nu \not\ll \mu$). ■

Corollary 14.5. *Let μ be a positive measure on $(X, \mathcal{M})$ and $f \in L^1(d\mu)$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_A f \, d\mu\right| < \epsilon$ for all $A \in \mathcal{M}$ such that $\mu(A) < \delta$.*

Proof. Apply theorem 14.4 to the signed measure

$$\nu(A) = \int_A f \, d\mu \text{ for all } A \in \mathcal{M}.$$

■

Lemma 14.6. *Let μ and λ be positive finite measures. Then either $\mu \perp \lambda$ or there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\lambda \geq \epsilon\mu$ on E, i.e. $\lambda - \epsilon\mu \geq 0$ on $\mathcal{M}_E$.*

Proof. Let $X = P_n \cup N_n$ be the Hahn decomposition of $\lambda_n \equiv \lambda - \frac{1}{n}\mu$ so $\lambda_n \geq 0$ on P_n and $\lambda_n \leq 0$ on N_n, i.e. $\lambda \geq \frac{1}{n}\mu$ on P_n and $\lambda \leq \frac{1}{n}\mu$ on N_n. Let $N \equiv \bigcap_{n=1}^{\infty} N_n$ then $\lambda \leq 0$ on N which implies that $\lambda = 0$ on N. Set $P = N^c = \bigcup_{n=1}^{\infty} P_n$. Either $\mu(P) = 0$ in which case $\mu \perp \lambda$ or $\mu(P) > 0$ then $\mu(P_n) > 0$ for some n. Now $\lambda - \frac{1}{n}\mu \geq 0$ on P_n and $\mu(P_n) > 0$. ■

Theorem 14.7 (Lebesgue-Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure and μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then there exists unique σ-finite signed measures λ and δ such that 1) $\nu = \lambda + \delta$, 2) $\lambda \perp \mu$ and 3) $\delta \ll \mu$. Moreover there exists an extended μ – integrable function $f : X \to \mathbb{R}$ such that $\delta = \mu_f$ where $d\mu_f := f d\mu$ and this function is unique up to μ-null sets. In short*

$$d\nu = f d\mu + d\lambda$$

with $\lambda \perp \mu$.

Proof. We will first prove the existence assertions of the theorem. The uniqueness assertions will be proved at the end.

Case 1. Assume that μ, ν are *finite positive* measures. Set

$$\mathcal{F} = \{f : X \to [0, \infty] | \mu_f \leq \nu\}$$

and notice that if $f, g \in \mathcal{F}$ then $h = f \vee g \in \mathcal{F}$. Indeed, if $A \in \mathcal{M}$,

$$\begin{aligned} \mu_f(A) &= \int_A h d\mu = \int_{A \cap \{f > g\}} f d\mu + \int_{A \cap \{f \leq g\}} g d\mu \\ &\leq \nu(A \cap \{f > g\}) + \nu(A \cap \{f \leq g\}) \\ &= \nu(A) \quad \text{for all } A \in \mathcal{M}. \end{aligned}$$

The idea now is to take $f \in \mathcal{F}$ to be a maximal element. Heuristically we would like to define $f(x) = \sup\{h(x) : h \in \mathcal{F}\}$. However, since $\mathcal{F}$ is not a countable set in general, we must find another way to construct f. We do this in the next paragraph.

Choose $h_n \in \mathcal{F}$ such that

$$\mu_{h_n}(X) \uparrow M := \sup\{\mu_h(X) : h \in \mathcal{F}\} \text{ as } n \to \infty$$

and define $f_n = h_1 \vee \cdots \vee h_n \in \mathcal{F}$. Clearly f_n is an increasing sequence and $f := \lim_{n\to\infty} f_n$ is a measurable function. Since $\mu_{f_n}(X) \geq \mu_{h_n}(X)$, we must have by the monotone convergence theorem that

$$\mu_f(X) = \lim \mu_{f_n}(X) \geq M$$

and

$$\mu_f(A) = \lim_{n\to\infty} \mu_{f_n}(A) \leq \nu(A) \text{ for all } A \in \mathcal{M}.$$

This shows that $f \in \mathcal{F}$ and $M = \mu_f(X) \leq \nu(X) < \infty$ from which it follows that $\mu(\{f = \infty\}) = 0$ so we may assume $f : X \to [0, \infty]$ has range $[0, \infty)$.

To finish the existence proof in Case 1. we will show that $\lambda \equiv \nu - \mu_f$ is singular relative to μ. If not, by Lemma 14.6 there exists $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\lambda \geq \epsilon\mu$ on E. In this case we would have

$$\nu = \mu_f + \lambda \geq \mu_f + \mu_{\epsilon 1_E} = \mu_{f+\epsilon 1_E}$$

and therefore $f + \epsilon 1_E \in \mathcal{F}$. But this contradicts the definition of M since now

$$\mu_{f+\epsilon 1_E}(X) = \mu_f(X) + \epsilon\mu(E) = M + \epsilon\mu(E) > M.$$

This finishes the proof of Case 1.

Case 2. Assume μ and ν are σ-finite positive measures on $(X, \mathcal{M})$. Choose $X'_n, X''_n \in \mathcal{M}$ such that $X = \coprod_{n=1}^\infty X'_n = \coprod_{n=1}^\infty X''_n$, $\mu(X'_n) < \infty$ and $\nu(X''_n) < \infty$. Let $\{X_n\}_{n=1}^\infty$ be an enumeration of $\{X'_n \cap X''_m\}_{m,n=1}^\infty$. Then $X = \coprod_{n=1}^\infty X_n$, $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$. Define finite measures μ_n and ν_n on $(X, \mathcal{M})$ by $\mu_n(A) = \mu(X_n \cap A)$ and $\nu_n(A) = \nu(X_n \cap A)$ for all $A \in \mathcal{M}$ and notice that

$$\mu = \sum_{n=1}^\infty \mu_n \text{ and } \nu = \sum_{n=1}^\infty \nu_n.$$

By the finite case already proved, there exists measurable functions $f_n : X \to [0, \infty)$ and measures λ_n such that $\nu_n = \mu_{f_n} + \lambda_n$ with $\lambda_n \perp \mu_n$. Define $f := \sum_{n=1}^\infty f_n$ and $\lambda := \sum_{n=1}^\infty \lambda_n$ so that

$$\nu = \sum_{n=1}^\infty \nu_n = \sum_{n=1}^\infty \mu_{f_n} + \sum_{n=1}^\infty \lambda_n = \mu_f + \lambda.$$

To finish the proof of this case it suffices to show that $\lambda \perp \mu$ which will follow by Remark 13.9 if we show $\lambda_n \perp \mu$ for all n. Because

$$0 = \nu_n(X_n^c) = \mu_{f_n}(X_n^c) + \lambda_n(X_n^c)$$

we see that $\lambda_n(X_n^c) = 0$. Since $\lambda_n \perp \mu_n$ there exists $A \in \mathcal{M}$ such that

$$\mu_n(A) = \mu(X_n \cap A) = 0 \text{ and } \lambda_n(A^c) = 0.$$

Let $B = X_n \cap A$, then $\mu(B) = 0$ while

$$\lambda_n(B^c) = \lambda_n(X_n^c \cup A^c) \leq \lambda_n(X_n^c) + \lambda_n(A^c) = 0.$$

This shows that μ lives on B^c and λ_n lives on B so that $\mu \perp \lambda_n$ finishing the proof of case 2.

Case 3. The general case where now $\nu = \nu_+ - \nu_-$ is a signed σ finite measure μ is a positive σ - finite measure. Assume without loss of generality that $\nu_+(X) < \infty$, i.e $\nu(A) < \infty$ for all $A \in \mathcal{M}$. By Case 2, there exist functions $f_\pm : X \to [0, \infty)$ and measure $\lambda_\pm$ such that $\nu_\pm = \mu_{f_\pm} + \lambda_\pm$ with $\lambda_\pm \perp \mu$. Since

$$\infty > \nu_+(X) = \mu_{f_+}(X) + \lambda_+(X),$$

$f_+ \in L^1(\mu)$ and $\lambda_+(X) < \infty$ so that $f = f_+ - f_-$ is an extended integrable function and $\lambda = \lambda_+ - \lambda_-$ is a signed measure. This finishes the existence proof since

$$\nu = \nu_+ - \nu_- = \mu_{f_+} + \lambda_+ - \left(\mu_{f_-} + \lambda_-\right) = \mu_f + \lambda$$

and $\lambda = (\lambda_+ - \lambda_-) \perp \mu$ by Remark 13.9.

Uniqueness. Suppose that $\nu = \delta + \lambda$ and $\nu = \tilde{\delta} + \tilde{\lambda}$ with $\lambda \perp \mu$, $\delta \ll \mu$, $\tilde{\lambda} \perp \tilde{\mu}$ and $\tilde{\delta} \ll \tilde{\mu}$. Since $\lambda \perp \mu$ and $\tilde{\lambda} \perp \mu$ there exists $A, \tilde{A} \in \mathcal{M}$ such that $\mu(A) = \mu(\tilde{A}) = 0$ and A^c is λ – null and $\tilde{A}^c$ is $\tilde{\lambda}$ – null. Let $B = A \cup \tilde{A}$, then

$$\mu(B) \leq \mu(A) + \mu(\tilde{A}) = 0$$

so that B is μ – null. Also $B^c = A^c \cap \tilde{A}^c$ which is λ and $\tilde{\lambda}$ - null set. Therefore if $C \in \mathcal{M}$ then

$$\begin{aligned} \nu(C \cap B^c) &= \delta(C \cap B^c) + \lambda(C \cap B^c) = \delta(C \cap B^c) \\ &= \delta((C \cap B^c) \cup (C \cap B)) = \delta(C) \end{aligned} \tag{14.1}$$

where we have used $\delta \ll \mu$ and $\lambda(C \cap B^c) = 0$. Similarly $\tilde{\delta}(C) = \nu(C \cap B^c)$ which shows that $\delta = \tilde{\delta}$ and hence $\lambda = \tilde{\lambda}$.

Let $X = P \cup N$ be the Hahn decomposition of δ and $C \in \mathcal{M}$, then

$$\begin{aligned} |\delta|(C) &= \delta(C \cap P) - \delta(C \cap N) \\ &= \nu(C \cap P \cap B^c) - \nu(C \cap N \cap B^c) \\ &\leq |\nu|(C \cap P \cap B^c) + |\nu|(C \cap N \cap B^c) \\ &= |\nu|(C \cap B^c) \leq |\nu|(C). \end{aligned}$$

This shows that $|\delta| \leq |\nu|$ and in particular that δ is σ finite since ν is σ finite.

Suppose that f and g are extended integrable functions such that $\delta = \mu_f = \mu_g$, i.e.

$$\int_A f d\mu = \int_A g d\mu \text{ for all } A \in \mathcal{M}. \tag{14.2}$$

Choose $X_n \uparrow X$ such that $|\delta|(X_n) < \infty$. Since

$$\int_{X_n} |f|\, d\mu = |\delta|(X_n) = \int_{X_n} |g|\, d\mu,$$

we have $1_{X_n} f$ and $1_{X_n} g$ are in $L^1(\mu)$ for all n. Applying Eq. (14.2) to $X_n \cap A$ we learn that

$$\int_A 1_{X_n} f d\mu = \int_{A \cap X_n} f d\mu = \int_{A \cap X_n} g d\mu = \int_A 1_{X_n} g d\mu$$

for all $A \in \mathcal{M}$ which implies that $1_{X_n} f = 1_{X_n} g$, μ – a.e. Letting $n \to \infty$ then shows that $f = g$, μ – a.e. ■

Remark 14.8. Suppose that f and g are two positive measurable functions on $(X, \mathcal{M}, \mu)$ such that

$$\int_A f d\mu = \int_A g d\mu \tag{14.3}$$

for all $A \in \mathcal{M}$. It is not in general true that $f = g$, μ – a.e. A trivial counter example is to take $\mathcal{M} = \mathcal{P}(X)$, $\mu(A) = \infty$ for all non-empty $A \in \mathcal{M}$, $f = 1_X$ and $g = 2 \cdot 1_X$. Then Eq. (14.3) holds yet $f \neq g$.

Corollary 14.9. *Let ν be a σ-finite signed measure and μ be a σ-finite positive measure on $(X, \mathcal{M})$. If $\nu \ll \mu$ then $d\nu = f d\mu$ (that is $\nu = \mu_f$) for some extended integrable function $f : X \to \mathbb{R}$ which is unique modulo sets of measure zero.*

Notation 14.10. The function f is called the Radon-Nikodym derivative of ν relative to μ and we will denote this function by $\frac{d\nu}{d\mu}$.

Proof. By Theorem 14.7, there exists an extended integrable function $f : X \to \mathbb{R}$; and a signed measure λ such that $\nu = \mu_f + \lambda$ and $\lambda \perp \mu$. From Lemma 14.3, $\lambda = \nu - \mu_f \ll \mu$ and again by Lemma 14.3, $\lambda = 0$. Alternatively, choose $B \in \mathcal{M}$ such that $\mu(B^c) = 0$ and B is a λ – null set. Since $\nu \ll \mu$, B^c is also a ν – null set so that

$$\begin{aligned}\nu(A) &= \nu(A \cap B) = \mu_f(A \cap B) + \lambda(A \cap B) \\ &= \mu_f(A \cap B) = \mu_f(A)\end{aligned}$$

for all $A \in \mathcal{M}$. ■

15. Complex Measures

Definition 15.1. A complex measure ν on a measurable space $(X, \mathcal{M})$ is a countably additive set function $\nu : \mathcal{M} \to \mathbb{C}$ such that $\nu(\emptyset) = 0$.

Given a complex measure ν, let $\nu_r = \operatorname{Re} \nu$ and $\nu_i = \operatorname{Im} \nu$ so that ν_r and ν_i are finite signed measures such that

$$\nu(A) = \nu_r(A) + i\nu_i(A) \text{ for all } A \in \mathcal{M}.$$

Definition 15.2. Let $L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$ and for $f \in L^1(\nu)$ define

$$\int_X f d\nu := \int_X f d\nu_r + i \int_X f d\nu_i.$$

Example 15.3. Suppose that μ is a positive measure on $(X, \mathcal{M})$ and $g \in L^1(\mu)$, then

$$\nu(A) = \int_A g d\mu \, \forall A \in \mathcal{M} \tag{15.1}$$

μ is a complex measure on $(X, \mathcal{M})$. Moreover, $L^1(\nu) = L^1(|g| \, d\mu)$ and for $f \in L^1(\nu)$

$$\int_X f d\nu = \int_X f g d\mu. \tag{15.2}$$

To check Eq. (15.2), notice that $d\nu_r = \operatorname{Re} g d\mu$ and $d\nu_i = \operatorname{Im} g d\mu$ so that (using Lemma 13.16)

$$L^1(\nu) = L^1(\operatorname{Re} g d\mu) \cap L^1(\operatorname{Im} g d\mu) = L^1(|\operatorname{Re} g| \, d\mu) \cap L^1(|\operatorname{Im} g| \, d\mu) = L^1(|g| \, d\mu).$$

If $f \in L^1(\nu)$, then

$$\int_X f d\nu := \int_X f \operatorname{Re} g d\mu + i \int_X f \operatorname{Im} g d\mu = \int_X f g d\mu.$$

We will now show that any complex measure ν may be represented as in Eq. (15.1). To do this let μ be the finite positive measure on $\mathcal{M}$ defined by

$$\mu = |\nu_r| + |\nu_i|.$$

Then $\nu_r \ll \mu$ and $\nu_i \ll \mu$ and hence by the Radon-Nikodym theorem, there exists real functions $h, k \in L^1(\mu)$ such that

$$d\nu_r = h \, d\mu \text{ and } d\nu_i = k \, d\mu.$$

Therefore if $g := h + ik \in L^1(\mu)$ then $d\nu = (h + ik)d\mu = g d\mu$.

Lemma 15.4. *Suppose that ν is a complex measure on $(X,\mathcal{M})$, μ_i is a finite positive measure on $(X,\mathcal{M})$ and $g_i \in L^1(\mu_i)$ such that $d\nu = g_i d\mu_i$ for $i = 1,2.$ Then*

$$\int_A |g_1|\, d\mu_1 = \int_A |g_2|\, d\mu_2 \text{ for all } A \in \mathcal{M}.$$

In particular, we may define a positive measure $|\nu|$ on $(X,\mathcal{M})$ by

$$|\nu|(A) = \int_A |g_1|\, d\mu_1 \text{ for all } A \in \mathcal{M}.$$

The finite positive measure $|\nu|$ is the total variation measure of ν.

Proof. Let $\lambda = \mu_1 + \mu_2$ so that $\mu_i \ll \lambda$. Let $\rho_i = d\mu_i/d\lambda \geq 0$ and $h_i = \rho_i g_i$. Since

$$\nu(A) = \int_A g_i d\mu_i = \int_A g_i \rho_i d\lambda = \int_A h_i d\lambda \text{ for all } A \in \mathcal{M},$$

$h_1 = h_2$, λ –a.e. Therefore

$$\int_A |g_1|\, d\mu_1 = \int_A |g_1|\, \rho d\lambda = \int_A |h_1|\, d\lambda = \int_A |h_2|\, d\lambda = \int_A |g_2|\, \rho d\lambda = \int_A |g_2|\, d\mu_2.$$

■

Remark 15.5. Suppose that ν is a complex measure on $(X,\mathcal{M})$ such that $d\nu = g d\mu$ and as above $d|\nu| = |g|\, d\mu$. Letting

$$\rho = \begin{cases} \frac{g}{|g|} & \text{if} \quad |g| \neq 0 \\ 1 & \text{if} \quad |g| = 0 \end{cases}$$

we see that

$$d\nu = g d\mu = \rho\, |g|\, d\mu = \rho d\, |\nu|$$

and $|\rho| = 1$ and ρ is uniquely defined modulo $|\nu|$ – null sets. We will denote ρ by $d\nu/d|\nu|$.

With this notation and Example 15.3 we have $L^1(\nu) := L^1(|\nu|)$ and for $f \in L^1(\nu)$,

$$\int_X f d\nu = \int_X f \frac{d\nu}{d|\nu|} d|\nu|.$$

Proposition 15.6. *Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, $\mathcal{M} = \sigma(\mathcal{A})$, ν is a complex measure on $(X,\mathcal{M})$ and for $E \in \mathcal{M}$ let*

$$\begin{aligned}
\mu_0(E) &= \sup\left\{\sum_1^n |\nu(E_j)| : E_j \in \mathcal{A}_E \ \ni E_i \cap E_j = \delta_{ij} E_i,\ n = 1,2,\dots\right\} \\
\mu_1(E) &= \sup\left\{\sum_1^n |\nu(E_j)| : E_j \in \mathcal{M}_E \ \ni E_i \cap E_j = \delta_{ij} E_i,\ n = 1,2,\dots\right\} \\
\mu_2(E) &= \sup\left\{\sum_1^\infty |\nu(E_j)| : E_j \in \mathcal{M}_E \ \ni E_i \cap E_j = \delta_{ij} E_i\right\} \\
\mu_3(E) &= \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}.
\end{aligned}$$

then $\mu_0 = \mu_1 = \mu_2 = \mu_3 = |\nu|$.

Proof. Let $\rho = d\nu/d|\nu|$ and recall that $|\rho| = 1$ for $|\nu|$ – a.e. We will start by showing $|\nu| = \mu_3$. If $|f| \le 1$,

$$\left|\int_E f\ d\nu\right| = \left|\int_E f\ \rho d|\nu|\right| \le \int_E |f|\ d|\nu| \le \int_E 1 d|\nu| = |\nu|(E)$$

which shows that $\mu_3 \le |\nu|$. By taking $f = \bar{\rho}$,

$$\left|\int_E f\ d\nu\right| = \int_E \bar{\rho}\ \rho\ d|\nu| = \int_E 1\ d|\nu| = |\nu|(E)$$

which shows that $|\nu| \le \mu_3$ and hence $|\nu| = \mu_3$.

We will now show $\mu_0 = \mu_1 = \mu_2 = |\nu|$. Clearly $\mu_0 \le \mu_1 \le \mu_2$. Suppose $E_j \in \mathcal{M}_E$ such that $E_i \cap E_j = \delta_{ij} E_i$, then

$$\sum |\nu(E_j)| = \sum |\int_{E_j} \rho d|\nu| \le \sum |\nu|(E_j) = |\nu|(\cup E_j) \le |\nu|(E)$$

which shows that $\mu_2 \le |\nu|$. So to finish the proof it suffices to show that $|\nu| \le \mu_0$.

By Theorem 10.6, there exists simple functions ρ_n on X such that $\rho_n \to \rho$ in $L^1(|\nu|)$ and each ρ_n may be written in the form

$$\rho_n = \sum_{k=1}^{N} z_k 1_{A_k} \tag{15.3}$$

where $z_k \in \mathbb{C}$ and $A_k \in \mathcal{A}$ and $A_k \cap A_j = \emptyset$ if $k \ne j$. I claim that we may assume that $|z_k| \le 1$ in Eq. (15.3) for if $|z_k| > 1$ and $x \in A_k$,

$$|\rho(x) - z_k| \ge \left|\rho(x) - |z_k|^{-1} z_k\right|.$$

This is obvious from a picture and formally follows from the fact that

$$\frac{d}{dt}\left|\rho(x) - t|z_k|^{-1} z_k\right|^2 = 2\left[t - \mathrm{Re}(|z_k|^{-1} z_k \overline{\rho(x)})\right] \ge 0$$

when $t \ge 1$. Therefore if we define

$$w_k := \begin{cases} |z_k|^{-1} z_k & \text{if} \quad |z_k| > 1 \\ z_k & \text{if} \quad |z_k| \le 1 \end{cases}$$

and

$$\tilde{\rho}_n = \sum_{k=1}^{N} w_k 1_{A_k}$$

then

$$|\rho(x) - \rho_n(x)| \ge |\rho(x) - \tilde{\rho}_n(x)|$$

and therefore $\tilde{\rho}_n \to \rho$ in $L^1(|\nu|)$.

So we now assume that ρ_n is as in Eq. (15.3) with $|z_k| \le 1$. Because $E_k = A_k \cap E \in \mathcal{A}_E$ and $E_k \cap E_j = \emptyset$ if $j \ne k$,

$$\begin{aligned} \left|\int_E \bar{\rho}_n\ d\nu\right| &= \left|\sum z_k \nu(A_k \cap E)\right| \le \sum |z_k|\ |\nu(A_k \cap E)| \\ &\le \sum |\nu(A_k \cap E)| \le \mu_0(E). \end{aligned} \tag{15.4}$$

Since

$$\left|\int_E \bar{\rho}_n\ d\nu - \int_E \bar{\rho}\ d\nu\right| = \left|\int_E (\bar{\rho}_n - \bar{\rho})\, \rho d|\nu|\right| \le \int_E |\rho_n - \rho|\, d|\nu| \to 0 \text{ as } n \to \infty$$

we may let $n \to \infty$ in Eq. (15.4) to conclude

$$|\nu|(E) = \int_E \bar{\rho}\, d\nu = \lim_{n\to\infty} \left| \int_E \bar{\rho}_n \, d\nu \right| \le \mu_0(E).$$

■

15.1. **Absolute Continuity on an Algebra.**

Definition 15.7. Suppose that ν is a complex measure and μ is a positive measure on $(X, \mathcal{M})$. We say the ν is absolutely continuous relative to μ and write $\nu \ll \mu$ iff $\nu(A) = 0$ for all $A \in \mathcal{M}$ such that $\mu(A) = 0$.

The following theorems will be useful in Section 17 below.

Lemma 15.8. *Let ν be a complex measure and μ be a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff $|\nu| \ll \mu$.*

Proof. Since $|\nu(A)| \le |\nu|(A)$ it is clear that $|\nu| \ll \mu$ implies $\nu \ll \mu$. For the opposite direction, let $\rho = \frac{d\nu}{d|\nu|}$. If $A \in \mathcal{M}$ and $\mu(A) = 0$ then by assumption

$$0 = \nu(B) = \int_B \rho d|\nu|$$

for all $B \in \mathcal{M}_A$. This shows that $\rho 1_A = 0$ for $|\nu|$ – a.e. and hence

$$|\nu|(A) = \int_A |\rho|\, d|\nu| = \int_X 1_A |\rho|\, d|\nu| = 0,$$

i.e. $\mu(A) = 0$ implies $|\nu|(A) = 0$.

Alternatively one may prove this lemma by arguing that $\nu \ll \mu$ implies that $\nu_r \ll \mu$ and $\nu_i \ll \mu$ which implies that $|\nu_r| \ll \mu$ and $|\nu_i| \ll \mu$. Since $|\nu| \le |\nu_r| + |\nu_i|$, this shows that $|\nu| \ll \mu$. ■

Theorem 15.9. *Let ν be a complex measure and μ be a positive measure on $(X, \mathcal{M})$. Suppose that $\mathcal{A} \subset \mathcal{M}$ is an algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and that μ is σ-finite on $\mathcal{A}$. Then $\nu \ll \mu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(A)| < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$.*

Proof. By Remark 14.2, Lemma 13.13 and Lemma 15.8, it suffices to prove the theorem in the case that ν is a positive measure. The implication $(\Longrightarrow)$ is a consequence of Theorem 14.4.

$(\Longleftarrow)$ Let $\epsilon > 0$ and $\delta > 0$ be such that $\nu(A) < \epsilon$ for all $A \in \mathcal{A}$ with $\mu(A) < \delta$. Suppose that $B \in \mathcal{M}$ with $\mu(B) < \delta$. Use the regularity Theorem 4.13 to find $A \in \mathcal{A}_\sigma$ such that $B \subset A$ and $\mu(B) \le \mu(A) < \delta$. Write $A = \cup_n A_n$ with $A_n \in \mathcal{A}$. By replacing A_n by $\cup_{j=1}^n A_j$ if necessary we may assume that A_n is increasing in n. Then $\mu(A_n) \le \mu(A) < \delta$ for each n and hence by assumption $\nu(A_n) < \epsilon$. Since $B \subset A = \cup_n A_n$ it follows that $\nu(B) \le \nu(A) = \lim_{n\to\infty} \nu(A_n) \le \epsilon$. Thus we have shown that $\nu(B) \le \epsilon$ for all $B \in \mathcal{M}$ such that $\mu(B) < \delta$. ■

16. Measure Differentiation Theorems on $\mathbb{R}^n$

In this chapter, let $\mathcal{B} = \mathcal{B}_{\mathbb{R}^n}$ denote the Borel σ – algebra on $\mathbb{R}^n$ and m be Lebesgue measure on $\mathcal{B}$.

Definition 16.1. A collection of measurable sets $\{E_r\}_{r>0} \subset \mathcal{B}$ is said to shrink nicely to $x \in \mathbb{R}^n$ if (i) $E_r \subseteq \overline{B_r(x)}$ for all $r > 0$ and (ii) there exists $\alpha > 0$ such that $m(E_r) \ge \alpha m(B_r(x))$. We will abbreviate this by writing $E_r \downarrow \{x\}$ nicely.

The main result of this chapter is the following theorem.

Theorem 16.2. *Suppose that ν is a complex measure on $(\mathbb{R}^n, \mathcal{B})$, then there exists $g \in L^1(\mathbb{R}^n, m)$ and a complex measure λ such that $\lambda \perp m$, $d\nu = gdm + d\lambda$, and for m - a.e. x,*

$$g(x) = \lim_{r \downarrow 0} \frac{\nu(E_r)}{m(E_r)} \tag{16.1}$$

for any collection of $\{E_r\}_{r>0} \subset \mathcal{B}$ which shrink nicely to $\{x\}$.

Proof. The existence of g and λ such that $\lambda \perp m$ and $d\nu = gdm + d\lambda$ is a consequence of the Radon-Nikodym theorem. Since

$$\frac{\nu(E_r)}{m(E_r)} = \frac{1}{m(E_r)} \int_{E_r} g(x) dm(x) + \frac{\lambda(E_r)}{m(E_r)}$$

Eq. (16.1) is a consequence of Theorem 16.12 and Corollary 16.14 below. ■

The rest of this chapter will be devoted to filling in the details of the proof of this theorem.

16.1. A Covering Lemma and Averaging Operators.

Lemma 16.3 (Covering Lemma)**.** *Let $\mathcal{E}$ be a collection of open balls in $\mathbb{R}^n$ and $U = \cup_{B \in \mathcal{E}} B$. If $c < m(U)$, then there exists disjoint balls $B_1, \ldots, B_k \in \mathcal{E}$ such that $\sum_{j=1}^{k} m(B_j) > 3^{-n} c$.*

Proof. Choose a compact set $K \subset U$ such that $m(K) > c$ and then let $\mathcal{E}_1 := \{A_i\}_{i=1}^N \subset \mathcal{E}$ be a finite subcover of K. Choose $B_1 \in \mathcal{E}_1$ to be a ball with largest diameter in $\mathcal{E}_1$. Let $\mathcal{E}_2 = \{A \in \mathcal{E}_1 : A \cap B_1 = \emptyset\}$. If $\mathcal{E}_2$ is not empty, choose $B_2 \in \mathcal{E}_2$ to be a ball with largest diameter in $\mathcal{E}_2$. Similarly Let $\mathcal{E}_3 = \{A \in \mathcal{E}_2 : A \cap B_2 = \emptyset\}$ and if $\mathcal{E}_3$ is not empty, choose $B_3 \in \mathcal{E}_3$ to be a ball with largest diameter in $\mathcal{E}_3$. Continue choosing $B_i \in \mathcal{E}$ for $i = 1, 2, \ldots, k$ this way until $\mathcal{E}_{k+1}$ is empty.

If $B = B(x_0, r) \subset \mathbb{R}^n$, let $B^* = B(x_0, 3r) \subset \mathbb{R}^n$, that is B^* is the ball concentric with B which has three times the radius of B. We will now show $K \subset \cup_{i=1}^k B_i^*$. For each j there exists a first i such that $B_i \cap A_j \neq \emptyset$. In this case $\operatorname{diam}(A_j) \leq \operatorname{diam}(B_i)$ and $A_j \subseteq B_i^*$. Therefore $A_j \subset \cup_{i=1}^k B_i^*$ for all j and hence $K \subset \cup_{j=1}^k A_j \subset \cup_{i=1}^k B_i^*$. Hence by subadditivity,

$$c < m(K) \leq \sum_{i=1}^{k} m(B_i^*) \leq 3^n \sum_{i=1}^{k} m(B_i)$$

■

Definition 16.4. Let $V \subset \mathbb{R}^n$ be an open set and $f : V \to \mathbb{C}$ be a measurable function. We say that f is locally integrable on V if

$$\int_K |f| dx < \infty$$

for all compact sets $K \subset V$. Let $L^1_{loc}(V, m)$ denote the space of locally integrable functions f on V and write L^1_{loc} for $L^1_{loc}(\mathbb{R}^n, m)$.

Definition 16.5. For $f \in L^1_{loc}$, $x \in \mathbb{R}^n$ and $r > 0$ let

$$(A_r f)(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f dm \tag{16.2}$$

where $B_r(x) = B(x,r) \subset \mathbb{R}^n$, and $|A| := m(A)$.

Lemma 16.6. *Let $f \in L^1_{loc}$, then for each $x \in \mathbb{R}^n$, $(0,\infty) \ni r \to (A_r f)(x)$ is continuous and for each $r > 0$ $\mathbb{R}^n \ni x \to (A_r f)(x)$ is measurable.*

Proof. Recall that $|B_r(x)| = m(E_1)r^n$ which is continuous in r. Also $\lim_{r\to r_0} 1_{B_r(x)}(y) = 1_{B_{r_0}(x)}(y)$ if $|y| \neq r_0$ and since $m(\{y : |y| \neq r_0\}) = 0$ (you prove!), $\lim_{r\to r_0} 1_{B_r(x)}(y) = 1_{B_{r_0}(x)}(y)$ for m -a.e. y. So by the dominated convergence theorem,

$$\lim_{r\to r_0} \int_{B_r(x)} f dm = \int_{B_{r_0}(x)} f dm$$

and therefore

$$(A_r f)(x) = \frac{1}{m(E_1)r^n} \int_{B_r(x)} f dm$$

is continuous in r. Let $g_r(x,y) := 1_{B_r(x)}(y) = 1_{|x-y|<r}$. Then g_r is $\mathcal{B} \otimes \mathcal{B}$ – measurable (for example write it as a limit of continuous functions) so that by Fubini's theorem

$$x \to \int_{B_r(x)} f dm = \int_{B_r(x)} g_r(x,y)f(y)dm(y)$$

is $\mathcal{B}$ – measurable and hence so is $x \to (A_r f)(x)$. ■

16.2. **Maximal Functions.**

Definition 16.7. For $f \in L^1(m)$, the Hardy - Littlewood maximal function Hf is defined by

$$(Hf)(x) = \sup_{r>0} A_r|f|(x)$$

Lemma 16.6 allows us to write

$$(Hf)(x) = \sup_{r\in\mathbb{Q},\ r>0} A_r|f|(x)$$

and then to concluded that Hf is measurable.

Theorem 16.8 (Maximal Inequality)**.** *If $f \in L^1(m)$ and $\alpha > 0$, then*

$$m(Hf > \alpha) \leq \frac{3^n}{\alpha}\|f\|_{L^1}.$$

This should be compared with Chebyshcev's inequality which states that

$$m(|f| > \alpha) \leq \frac{\|f\|_{L^1}}{\alpha}.$$

Proof. Let $E_\alpha \equiv \{Hf > \alpha\}$. For all $x \in E_\alpha$ there exists r_x such that $A_{r_x}|f|(x) > \alpha$. Hence $E_\alpha \subseteq \cup_{x\in E_\alpha} B_x(r_x)$. By Lemma 16.3, if $c < m(E_\alpha) \leq$

$m(\cup_{x\in E_\alpha} B_x(r_x))$ then there exists $x_1,\dots,x_k \in E_\alpha$ and disjoint balls $B_i = B_{x_i}(r_{x_i})$ for $i = 1,2,\dots,k$ such that $\sum |B_i| > 3^{-n}c$. Since

$$\frac{\int_{B_i} |f|dm}{|B_i|} = A_{r_{x_i}}|f|(x_i) > \alpha,$$

$$|B_i| < \alpha^{-1}\int_{B_i} |f|dm$$

and hence

$$3^{-n}c < \frac{1}{\alpha}\sum \int_{B_i} |f|dm \le \frac{1}{\alpha}\int_{\mathbb{R}^n} |f|dm = \frac{1}{\alpha}\|f\|_{L^1}.$$

This shows that $c < 3^n\alpha^{-1}\|f\|_{L^1}$ for all $c < m(E_\alpha)$ which proves $m(E_\alpha) \le 3^n\alpha^{-1}\|f\|$ ■

Theorem 16.9. *If $f \in L^1_{loc}$ then $\lim_{r\downarrow 0}(A_r f)(x) = f(x)$ for m – a.e. $x \in \mathbb{R}^n$.*

Proof. With out loss of generality we may assume $f \in L^1(m)$. We now begin with the special case where $f = g \in L^1(m)$ is also continuous. In this case we find:

$$\begin{aligned} |(A_r g)(x) - g(x)| &\le \frac{1}{|B_r(x)|}\int_{B_r(x)} |g(y) - g(x)|dm(y) \\ &\le \sup_{y\in B_r(x)} |g(y) - g(x)| \to 0 \text{ as } r \to 0. \end{aligned}$$

In fact we have shown that $(A_r g)(x) \to g(x)$ as $r \to 0$ uniformly for x in compact subsets of $\mathbb{R}^n$.

For general $f \in L^1(m)$,

$$\begin{aligned} |A_r f(x) - f(x)| &\le |A_r f(x) - A_r g(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\ &= |A_r(f-g)(x)| + |A_r g(x) - g(x)| + |g(x) - f(x)| \\ &\le H(f-g)(x) + |A_r g(x) - g(x)| + |g(x) - f(x)| \end{aligned}$$

and therefore,

$$\overline{\lim_{r\downarrow 0}}|A_r f(x) - f(x)| \le H(f-g)(x) + |g(x) - f(x)|.$$

So if $\alpha > 0$, then

$$E_\alpha \equiv \left\{\overline{\lim_{r\downarrow 0}}|A_r f(x) - f(x)| > \alpha\right\} \subseteq \left\{H(f-g) > \frac{\alpha}{2}\right\} \cup \left\{|g-f| > \frac{\alpha}{2}\right\}$$

and thus

$$\begin{aligned} m(E_\alpha) &\le m\left(H(f-g) > \frac{\alpha}{2}\right) + m\left(|g-f| > \frac{\alpha}{2}\right) \\ &\le \frac{3^n}{\alpha/2}\|f-g\|_{L^1} + \frac{1}{\alpha/2}\|f-g\|_{L^1} \\ &\le 2(3^n+1)\alpha^{-1}\|f-g\|_{L^1}, \end{aligned}$$

where in the second inequality we have used the Maximal inequality and Chebyshcev's inequality. Since this is true for all continuous $g \in C(\mathbb{R}^n) \cap L^1(m)$ and this set is dense in $L^1(m)$, we may make $\|f-g\|_{L^1}$ as small as we please. This shows that

$$m\left(\left\{x : \overline{\lim_{r\downarrow 0}}|A_r f(x) - f(x)| > 0\right\}\right) = m(\cup_{n=1}^\infty E_{1/n}) \le \sum_{n=1}^\infty m(E_{1/n}) = 0.$$

■

16.3. Lebesque Set.

Definition 16.10. For $f \in L^1_{loc}(m)$, the Lebesgue set of f is

$$L_f \equiv \left\{ x \in \mathbb{R}^n : \lim_{r \downarrow 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y) - f(x)| dy = 0 \right\}.$$

Theorem 16.11. *For all* $f \in L^1_{loc}(m)$, $0 = m(L_f^c) = m(\mathbb{R}^n \setminus L_f)$.

Proof. For $w \in \mathbb{C}$ define $g_w(x) = |f(x) - w|$ and $E_w \equiv \{x : \lim_{r\downarrow 0} (A_r g_w)(x) \neq g_w(x)\}$. Then by Theorem 16.9 $m(E_w) = 0$ for all $w \in \mathbb{C}$ and therefore $m(E) = 0$ where

$$E = \bigcup_{w \in \mathbb{Q} + i\mathbb{Q}} E_w.$$

By definition of E, if $x \notin E$ then.

$$\lim_{r \downarrow 0} (A_r |f(\cdot) - w|)(x) = |f(x) - w|$$

for all $w \in \mathbb{Q} + i\mathbb{Q}$. Since

$$|f(\cdot) - f(x)| \le |f(\cdot) - w| + |w - f(x)|,$$

$$\begin{aligned} (A_r |f(\cdot) - f(x)|)(x) &\le (A_r |f(\cdot) - w|)(x) + (A_r |w - f(x)|)(x) \\ &= (A_r |f(\cdot) - w|)(x) + |w - f(x)| \end{aligned}$$

and hence for $x \notin E$,

$$\begin{aligned} \overline{\lim_{r \downarrow 0}} (A_r |f(\cdot) - f(x)|)(x) &\le |f(x) - w| + |w - f(x)| \\ &\le 2|f(x) - w|. \end{aligned}$$

Since this is true for all $w \in \mathbb{Q} + i\mathbb{Q}$, we see that

$$\overline{\lim_{r \downarrow 0}} (A_r |f(\cdot) - f(x)|)(x) = 0 \text{ for all } x \notin E,$$

i.e. $E^c \subseteq L_f$ or equivalently $L_f^c \subseteq E$. So $m(L_f^c) \le m(E) = 0$. ■

Theorem 16.12 (Lebesque Differentiation Theorem)**.** *Suppose* $f \in L^1_{loc}$ *for all* $x \in L_f$ *(so in particular for* m *– a.e.* x*)*

$$\lim_{r \downarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$$

and

$$\lim_{r \downarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$$

when $E_r \downarrow \{x\}$ *nicely.*

Proof. For all $x \in \mathcal{L}_f$,

$$\begin{aligned}\left|\frac{1}{m(E_r)}\int_{E_r} f(y)dy - f(x)\right| &= \left|\frac{1}{m(E_r)}\int_{E_r} (f(y)-f(x))\,dy\right| \\ &\le \frac{1}{m(E_r)}\int_{E_r} |f(y)-f(x)|dy \\ &\le \frac{1}{\alpha m(B_r(x))}\int\limits_{B_r(x)} |f(y)-f(x)|dy\end{aligned}$$

which tends to zero as $r \downarrow 0$ by Theorem 16.11. In the second inequality we have used the fact that $m(\overline{B_r(x)} \setminus B_r(x)) = 0$. ■

Lemma 16.13. *Suppose $\lambda \ge 0$ is a σ-finite measure on $\mathcal{B} \equiv \mathcal{B}_{\mathbb{R}^n}$ such that $\lambda \perp m$. Then for m – a.e. x,*

$$\lim_{r\downarrow 0}\frac{\lambda(B_r(x))}{m(B_r(x))} = 0.$$

Proof. Let $A \in \mathcal{B}$ such that $\lambda(A) = 0$ and $m(A^c) = 0$. By the regularity theorem (Exercise 4.21), for all $\epsilon > 0$ there exists an open set $V_\epsilon \subset \mathbb{R}^n$ such that $A \subseteq V_\epsilon$ and $\lambda(V_\epsilon) < \epsilon$. Let

$$F_k \equiv \left\{x \in A : \overline{\lim_{r\downarrow 0}}\frac{\lambda(B_r(x))}{m(B_r(x))} > \frac{1}{k}\right\}$$

the for $x \in F_k$ choose $r_x > 0$ such that $B_{r_x}(x) \subseteq V_\epsilon$ and $\frac{\lambda(B_{r_x}(x))}{m(B_{r_x}(x))} > \frac{1}{k}$, i.e.

$$m(B_{r_x}(x)) < k\ \lambda(B_{r_x}(x)).$$

Let $\mathcal{E} = \{B_{r_x}(x)\}_{x\in F_k}$ and $U \equiv \bigcup\limits_{x\in F_k} B_{r_x}(x)$. Heuristically if all the balls in $\mathcal{E}$ were disjoint and $\mathcal{E}$ were countable, then

$$\begin{aligned} m(F_k) &\le \sum_{x\in F_k} m(B_{r_x}(x)) < k\sum_{x\in F_k}\lambda(B_{r_x}(x)) \\ &= k\lambda(U) \le k\ \lambda(V_\epsilon) \le k\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary this would imply that $m(F_k) = 0$.

To fix the above argument, suppose that $c < m(U)$ and use the covering lemma to find disjoint balls $B_1, \dots, B_N \in \mathcal{E}$ such that

$$\begin{aligned} c &< 3^n\sum_{i=1}^N m(B_i) < k3^n\sum_{i=1}^n \lambda(B_i) \\ &\le k3^n\lambda(U) \le k3^n\lambda(V_\epsilon) \le k3^n\epsilon.\end{aligned}$$

Since $c < m(U)$ is arbitrary we learn that $m(F_k) \le m(U) \le k3^n\epsilon$ and in particular that $m(F_k) \le k3^n\epsilon$. Since $\epsilon > 0$ is arbitrary, this shows that $m(F_k) = 0$. This implies, letting

$$F_\infty \equiv \left\{x \in A : \overline{\lim_{r\downarrow 0}}\frac{\lambda(B_r(x))}{m(B_r(x))} > 0\right\},$$

that $m(F_\infty) = \lim_{k\to\infty} m(F_k) = 0$. Since $m(A^c) = 0$, this shows that

$$m(\{x \in \mathbb{R}^n : \overline{\lim_{r\downarrow 0}}\frac{\lambda(B_r(x))}{m(B_r(x))} > 0\}) = 0.$$

■

Corollary 16.14. *Let λ be a complex or a σ-finite signed measure such that $\lambda \perp m$. Then for m – a.e. x,*

$$\lim_{r\downarrow 0} \frac{\lambda(E_r)}{m(E_r)} = 0$$

whenever $E_r \downarrow \{x\}$ nicely.

Proof. Recalling the $\lambda \perp m$ implies $|\lambda| \perp m$, Lemma 16.13 and the inequalities,

$$\frac{|\lambda(E_r)|}{m(E_r)} \le \frac{|\lambda|(E_r)}{\alpha m(B_r(x))} \le \frac{|\lambda|(\overline{B_r(x)})}{\alpha m(B_r(x))} \le \frac{|\lambda|(B_{2r}(x))}{\alpha 2^{-n} m(B_{2r}(x))}$$

proves the result. ■

17. The Fundamental Theorem of Calculus

Notation 17.1. Given a function $F : \mathbb{R} \to \bar{\mathbb{R}}$ or $F : \mathbb{R} \to \mathbb{C}$, let $F(x-) = \lim_{y\uparrow x} F(y)$, $F(x+) = \lim_{y\downarrow x} F(y)$ and $F(\pm\infty) = \lim_{x\to\pm\infty} F(x)$ whenever the limits exist. Notice that if F is a monotone functions then $F(\pm\infty)$ and $F(x\pm)$ exist for all x.

Theorem 17.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and define $G(x) = F(x+)$. Then*
(a) $\{x \in \mathbb{R} : F(x+) > F(x-)\}$ is countable.
(b) $F'(x)$ and $G'(x)$ exists for m – a.e. x and $F' = G'$ m – a.e.

Proof. (a) Suppose $x, y \in (-N, N)$ and $x < y$. Then $F(x+) \le F(y-)$ so that

$$(F(x-), F(x+)) \cap (F(y-), F(y+)) = \emptyset.$$

Therefore, $\{(F(x-), F(x+))\}_{x\in\mathbb{R}}$ are disjoint possible empty intervals in $\mathbb{R}$. Let $\alpha \subseteq (-N, N)$ be a finite set then

$$\coprod_{x\in\alpha} (F(x-), F(+)) \subseteq [F(-N), F(N)]$$

and therefore,

$$\begin{aligned} \sum_{x\in\alpha} [F(x+) - F(x-)] &= \sum_{x\in\alpha} m(F(x-), F(+)) = m\left(\coprod_{x\in\alpha} (F(x-), F(+))\right) \\ &\le m([F(-N), F(N)]) = F(N) - F(-N). \end{aligned}$$

Since this is true for all $\alpha \subset\subset (-N, N)$,

$$\sum_{x\in(-N,N)} [F(x+) - F(x-)] \le F(N) - F(-N) < \infty \tag{17.1}$$

and in particular,

$$\Gamma_N := \{x \in (-N, N) | F(x+) - F(x-) > 0\}$$

is countable and therefore

$$\Gamma := \{x \in \mathbb{R} | F(x+) - F(x-) > 0\} = \cup_{N=1}^{\infty} \Gamma_N$$

is also countable.

(b) The function G is increasing since if $x < y$ then

$$F(x+) \le F(y) \le F(y+).$$

If $z > y > x$ then

$$F(x+) \le F(y+) \le F(z)$$

and therefore

$$F(x+) \le \lim_{y \downarrow x} F(y+) \le F(z).$$

Hence

$$G(x) = F(x+) \le \lim_{y \downarrow x} F(y+) \le \lim_{z \downarrow x} F(z) = F(x+) = G(x)$$

which shows that G is right continuous.

Let μ_G denote the unique measure on $\mathcal{B}$ such that $\mu_G((a,b]) = G(b) - G(a)$ for all $a < b$. By Theorem 16.2, for m - a.e. x, for all sequences $\{E_r\}_{r>0}$ which shrink nicely to $\{x\}$, $\lim_{r \downarrow 0}(\mu_G(E_r)/m(E_r))$ exists and is independent of the choice of sequence $\{E_r\}_{r>0}$ shrinking to $\{x\}$. Since $(x, x+r] \downarrow \{x\}$ and $(x-r, x] \downarrow \{x\}$ nicely,

$$\lim_{r \downarrow 0} \frac{\mu_G(x, x+r])}{m((x, x+r])} = \lim_{r \downarrow 0} \frac{G(x+r) - G(x)}{r} = \frac{d}{dx^+} G(x)$$

and

$$\lim_{r \downarrow 0} \frac{\mu_G((x-r, x])}{m((x-r, x])} = \lim_{r \downarrow 0} \frac{G(x) - G(x-r)}{r} = \lim_{r \downarrow 0} \frac{G(x-r) - G(x)}{-r} = \frac{d}{dx^-} G(x)$$

exist and are equal for m - a.e. x, i.e. $G'(x)$ exists for m -a.e. x. Alternatively we have

$$\left| \frac{G(x+r) - G(x)}{r} \right| = \begin{cases} \frac{\mu_G(x, x+r])}{r} & \text{if} \quad r > 0 \\ \frac{\mu_G(x+r, x])}{|r|} & \text{if} \quad r < 0 \end{cases}$$

so that

$$\begin{aligned} \left| \frac{G(x+r) - G(x)}{r} \right| &\le \frac{\mu_G(x - |r|, x + |r|])}{|r|} \\ &= 2 \frac{\mu_G(x - |r|, x + |r|])}{m(B_{|r|}(x))} \to 0 \text{ as } r \to 0 \end{aligned}$$

by Theorem 16.2.

For $x \in \mathbb{R}$, let

$$H(x) \equiv G(x) - F(x) = F(x+) - F(x) \ge 0.$$

The proof will be completed by showing that $H'(x) = 0$ for m – a.e. x. Let

$$\Lambda \equiv \{x \in \mathbb{R} : F(x+) > F(x)\} \subset \Gamma.$$

Then $\Lambda \subset \mathbb{R}$ is a countable set and $H(x) = 0$ if $x \notin \Lambda$. Let λ be the measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\lambda = \sum_{x \in \mathbb{R}} H(x) \delta_x = \sum_{x \in \Lambda} H(x) \delta_x.$$

Since

$$\begin{aligned} \lambda((-N, N)) &= \sum_{x \in (-N,N)} H(x) = \sum_{x \in \Lambda \cap (-N,N)} (F(x+) - F(x)) \\ &\le \sum_{x \in (-N,N)} (F(x+) - F(x-)) < \infty \end{aligned}$$

by Eq. (17.1), λ is finite on bounded sets. Since $\lambda(\Lambda^c) = 0$ and $m(\Lambda) = 0$, $\lambda \perp m$ and so by Corollary 16.14 for m - a.e. x,

$$\left| \frac{H(x+r) - H(x)}{r} \right| \le 2 \frac{H(x + |r|) + H(x - |r|) + H(x)}{|r|} \le 2 \frac{\lambda([x - |r|, x + |r|])}{|r|}$$

and the last term goes to zero as $r \to 0$ because $\{[x-r, x+r]\}_{r>0}$ shrinks nicely to $\{x\}$. Hence we conclude for m – a.e. x that $H'(x) = 0$. ■

Definition 17.3. For $-\infty \le a < b < \infty$, a partition $\mathbb{P}$ of $(a,b]$ is a finite subset of $[a,b] \cap \mathbb{R}$ such that $\{a,b\} \cap \mathbb{R} \subset \mathbb{P}$. For $x \in \mathbb{P} \setminus \{b\}$, let $x_+ = \min\{y \in \mathbb{P} : y > x\}$ and if $x = b$ let $x_+ = b$.

Proposition 17.4. *Let μ be a complex measure on $\mathcal{B}$ and set $F(x) = \mu((-\infty, x])$. Then $F(-\infty) = 0$, F is right continuous and for $-\infty < a < b < \infty$,*

$$|\mu|(a,b] = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |\mu(x, x_+]| = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)| \tag{17.2}$$

where supremum is over all partitions $\mathbb{P}$ of $(a,b]$. Moreover $\mu \ll m$ iff for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon \tag{17.3}$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Proof. Eq. (17.2) follows from Proposition 15.6 and the fact that $\mathcal{B} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the algebra generated by $(a,b] \cap \mathbb{R}$ with $a, b \in \bar{\mathbb{R}}$. Suppose that Eq. (17.3) holds under the stronger condition that $\{(a_i, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$. If $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then for all $\rho > 0$, $\{(a_i + \rho, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$ and $\sum_{i=1}^{n}(b_i - (a_i + \rho)) < \delta$ so that by assumption,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i + \rho)| < \epsilon.$$

Since $\rho > 0$ is arbitrary in this equation and F is right continuous, we conclude that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| \le \epsilon$$

whenever $\{(a_i, b_i)\}_{i=1}^n$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$. So it suffices to prove Eq. (17.3) under the stronger condition that $\{(a_i, b_i]\}_{i=1}^n$ are disjoint intervals in $(a,b]$. But this last assertion follows directly from Theorem 15.9 and the fact that $\mathcal{B} = \sigma(\mathcal{A})$. ■

Definition 17.5. Given a function $F : \mathbb{R} \to \mathbb{C}$ and $a \in \mathbb{R}$ define

$$T_F(a) \equiv \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)|$$

where the supremum is taken over all partitions of $(-\infty, a]$. More generally if $-\infty \le a < b$, let

$$T_F(a,b] = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |\mu(x, x_+]| = \sup_{\mathbb{P}} \sum_{x\in\mathbb{P}} |F(x_+) - F(x)|$$

where supremum is over all partitions $\mathbb{P}$ of $(a,b]$. A function $F : \mathbb{R} \to \mathbb{C}$ is said to be of bounded variation if $T_F(\infty) < \infty$ and we write $F \in BV$. More generally we will let $BV((a,b])$ denote the function $F : [a,b] \cap \mathbb{R} \to \mathbb{C}$ such that $T_F(a,b] < \infty$.

Definition 17.6. A function $F:\mathbb{R}\to\mathbb{C}$ is absolutely continuous if for all $\epsilon>0$ there exists $\delta>0$ such that

$$\sum_{i=1}^{n}|F(b_i)-F(a_i)|<\epsilon$$

whenever $\{(a_i,b_i)\}_{i=1}^{n}$ are disjoint open intervals in $(a,b]$ such that $\sum_{i=1}^{n}(b_i-a_i)<\delta$.

Lemma 17.7. *Let $F:\mathbb{R}\to\mathbb{C}$ and $-\infty\leq a<b<c$, then 1)*

$$T_F(a,c]=T_F(a,b]+T_F(b,c]. \tag{17.4}$$

2) Letting $a=-\infty$ in this expression implies

$$T_F(c)=T_F(b)+T_F(b,c] \tag{17.5}$$

and in particular T_F is monotone increasing. 3) If $T_F(b)<\infty$ for some $b\in\mathbb{R}$ then $T_F(-\infty)=0$ and

$$T_F(a+)-T_F(a)\leq\limsup_{y\downarrow a}|F(y)-F(a)| \tag{17.6}$$

for all $a\in(-\infty,b)$. In particular T_F is right continuous if F is right continuous.

Proof. By the triangle inequality, if $\mathbb{P}$ and $\mathbb{P}'$ are partition of $(a,c]$ such that $\mathbb{P}\subset\mathbb{P}'$, then

$$\sum_{x\in\mathbb{P}}|F(x_+)-F(x)|\leq\sum_{x\in\mathbb{P}'}|F(x_+)-F(x)|.$$

So if $\mathbb{P}$ is a partition of $(a,c]$, then $\mathbb{P}\subset\mathbb{P}':=\mathbb{P}\cup\{b\}$ implies

$$\begin{aligned}\sum_{x\in\mathbb{P}}|F(x_+)-F(x)| &\leq \sum_{x\in\mathbb{P}'}|F(x_+)-F(x)|\\ &= \sum_{x\in\mathbb{P}'\cap[a,b]}|F(x_+)-F(x)|+\sum_{x\in\mathbb{P}'\cap[b,c]}|F(x_+)-F(x)|\\ &\leq T_F(a,b]+T_F(b,c].\end{aligned}$$

Thus we see that $T_F(a,c]\leq T_F(a,b]+T_F(b,c]$. Similarly if $\mathbb{P}_1$ is a partition of $(a,b]$ and $\mathbb{P}_2$ is a partition of $(b,c]$, then $\mathbb{P}=\mathbb{P}_1\cup\mathbb{P}_2$ is a partition of $(a,c]$ and

$$\begin{aligned}\sum_{x\in\mathbb{P}_1}|F(x_+)-F(x)|+\sum_{x\in\mathbb{P}_2}|F(x_+)-F(x)| &= \sum_{x\in\mathbb{P}}|F(x_+)-F(x)|\\ &\leq T_F(a,c].\end{aligned}$$

From this we conclude $T_F(a,b]+T_F(b,c]\leq T_F(a,c]$ which finishes the proof of Eqs. (17.4) and (17.5).

Suppose that $T_F(b)<\infty$ and given $\epsilon>0$ let $\mathbb{P}$ be a partition of $(-\infty,b]$ such that

$$T_F(b)\leq\sum_{x\in\mathbb{P}}|F(x_+)-F(x)|+\epsilon.$$

Let $x_0=\min\mathbb{P}$ then $T_F(b)=T_F(x_0)+T_F(x_0,b]$ and by the previous equation

$$T_F(x_0)+T_F(x_0,b]\leq\sum_{x\in\mathbb{P}}|F(x_+)-F(x)|+\epsilon\leq T_F(x_0,b]+\epsilon$$

which shows that $T_F(x_0)\leq\epsilon$. Since T_F is monotone increasing and $\epsilon>0$, we conclude that $T_F(-\infty)=0$.

Finally let $a \in (-\infty, b)$ and given $\epsilon > 0$ let $\mathbb{P}$ be a partition of $(a, b]$ such that

$$T_F(b) - T_F(a) = T_F(a, b] \le \sum_{x\in\mathbb{P}} |F(x_+) - F(x)| + \epsilon. \tag{17.7}$$

Let $y \in (a, a_+)$, then

$$\begin{aligned}\sum_{x\in\mathbb{P}} |F(x_+) - F(x)| + \epsilon &\le \sum_{x\in\mathbb{P}\cup\{y\}} |F(x_+) - F(x)| + \epsilon \\ &= |F(y) - F(a)| + \sum_{x\in\mathbb{P}\setminus\{y\}} |F(x_+) - F(x)| + \epsilon \\ &\le |F(y) - F(a)| + T_F(y, b] + \epsilon. \qquad (17.8)\end{aligned}$$

Combining Eqs. (17.7) and (17.8) shows

$$\begin{aligned}T_F(y) - T_F(a) + T_F(y, b] &= T_F(b) - T_F(a) \\ &\le |F(y) - F(a)| + T_F(y, b] + \epsilon.\end{aligned}$$

Since $y \in (a, a_+)$ is arbitrary we conclude that

$$T_F(a+) - T_F(a) = \limsup_{y\downarrow a} T_F(y) - T_F(a) \le \limsup_{y\downarrow a} |F(y) - F(a)| + \epsilon$$

which proves Eq. (17.6) since $\epsilon > 0$ is arbitrary. ■

Lemma 17.8. *1) Monotone functions $F : \mathbb{R} \to \mathbb{R}$ are in $BV(a, b]$ for all $-\infty < a < b < \infty$. 2) Linear combinations of functions in BV are in BV, i.e. BV is a vector space. 3) If $F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous then F is continuous and $F \in BV(a, b]$ for all $-\infty < a < b < \infty$. 4) If $F : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $\sup_{x\in\mathbb{R}} |F'(x)| = M < \infty$, then F is absolutely continuous and $T_F(a, b] \le M(b - a)$ for all $-\infty < a < b < \infty$. 5) Let $f \in L^1((a, b], m)$ and set*

$$F(x) = \int_{(a,x]} f dm \tag{17.9}$$

for $x \in (a, b]$. Then F is absolutely continuous.

Proof. 1) If F is monotone increasing and $\mathbb{P}$ is a partition of $(a, b]$ then

$$\sum_{x\in\mathbb{P}} |F(x_+) - F(x)| = \sum_{x\in\mathbb{P}} (F(x_+) - F(x)) = F(b) - F(a)$$

so that $T_F(a, b] = F(b) - F(a)$. Also note that $F \in BV$ iff $F(\infty) - F(-\infty) < \infty$. Item 2) follows from the triangle inequality. 3) Since F is absolutely continuous, there exists $\delta > 0$ such that whenever $a < b < a + \delta$ and $\mathbb{P}$ is a partition of $(a, b]$,

$$\sum_{x\in\mathbb{P}} |F(x_+) - F(x)| \le 1.$$

This shows that $T_F(a, b] \le 1$ for all $a < b$ with $b - a < \delta$. Thus using Eq. (17.4), it follows that $T_F(a, b] \le N < \infty$ if $b - a < N\delta$ for an $N \in \mathbb{N}$.

4) Suppose that $\{(a_i, b_i)\}_{i=1}^n \subset (a, b]$ are disjoint intervals, then by the mean value theorem,

$$\begin{aligned}\sum_{i=1}^n |F(b_i) - F(a_i)| &\le \sum_{i=1}^n |F'(c_i)| (b_i - a_i) \le Mm\left(\cup_{i=1}^n (a_i, b_i)\right) \\ &\le M\sum_{i=1}^n (b_i - a_i) \le M(b - a)\end{aligned}$$

form which it clearly follows that F is absolutely continuous. Moreover we may conclude that $T_F(a,b] \le M(b-a)$.

5) Let μ be the positive measure $d\mu = |f|\,dm$ on $(a,b]$. Let $\{(a_i,b_i)\}_{i=1}^n \subset (a,b]$ be disjoint intervals as above, then

$$\begin{aligned}\sum_{i=1}^n |F(b_i)-F(a_i)| &= \sum_{i=1}^n \left|\int_{(a_i,b_i]} f dm\right| \\ &\le \sum_{i=1}^n \int_{(a_i,b_i]} |f|\,dm \\ &= \int_{\cup_{i=1}^n (a_i,b_i]} |f|\,dm = \mu(\cup_{i=1}^n (a_i,b_i]). \end{aligned} \tag{17.10}$$

Since μ is absolutely continuous relative to m for all $\epsilon > 0$ there exist $\delta > 0$ such that $\mu(A) < \epsilon$ if $m(A) < \delta$. Taking $A = \cup_{i=1}^n (a_i,b_i]$ in Eq. (17.10) shows that F is absolutely continuous. It is also easy to see from Eq. (17.10) that $T_F(a,b] \le \int_{(a,b]} |f|\,dm$. ■

Theorem 17.9. *Let $F : \mathbb{R} \to \mathbb{C}$ be a function, then*

(1) *$F \in BV$ iff $\operatorname{Re} F \in BV$ and $\operatorname{Im} F \in BV$.*
(2) *If $F : \mathbb{R} \to \mathbb{R}$ is in BV then the functions $F_\pm := (T_F \pm F)/2$ are bounded and increasing functions.*
(3) *$F : \mathbb{R} \to \mathbb{R}$ is in BV iff $F = F_+ - F_-$ where $F_\pm$ are bounded increasing functions.*
(4) *If $F \in BV$ then $F(x\pm)$ exist for all $x \in \bar{\mathbb{R}}$. Let $G(x) := F(x+)$.*
(5) *$F \in BV$ then $\{x : \lim_{y\to x} F(y) \ne F(x)\}$ is a countable set and in particular $G(x) = F(x+)$ for all but a countable number of $x \in \mathbb{R}$.*
(6) *If $F \in BV$, then for m – a.e. x, $F'(x)$ and $G'(x)$ exist and $F'(x) = G'(x)$.*

Proof. Item 1. is a consequence of the inequalities

$$|F(b)-F(a)| \le |\operatorname{Re} F(b) - \operatorname{Re} F(a)| + |\operatorname{Im} F(b) - \operatorname{Im} F(a)| \le 2\,|F(b)-F(a)|\,.$$

2. By Lemma 17.7, for all $a < b$,

$$T_F(b) - T_F(a) = T_F(a,b] \ge |F(b)-F(a)| \tag{17.11}$$

and therefore

$$T_F(b) \pm F(b) \ge T_F(a) \pm F(a)$$

which shows that $F_\pm$ are increasing. Moreover from Eq. (17.11), for $b \ge 0$ and $a \le 0$,

$$\begin{aligned} |F(b)| &\le |F(b)-F(0)| + |F(0)| \le T_F(0,b] + |F(0)| \\ &\le T_F(0,\infty) + |F(0)| \end{aligned}$$

and similarly

$$|F(a)| \le |F(0)| + T_F(-\infty,0)$$

which shows that F is bounded by $|F(0)| + T_F(\infty)$. Therefore $F_\pm$ is bounded as well.

3. By Lemma 17.8 if $F = F_+ - F_-$, then

$$T_F(a,b] \le T_{F_+}(a,b] + T_{F_-}(a,b] = |F_+(b) - F_+(a)| + |F_-(b) - F_-(a)|$$

which is bounded showing that $F \in BV$. Conversely if F is bounded variation, then $F = F_+ - F_-$ where $F_\pm$ are defined as in Item 2.

Items 4. − 6. follow from Items 1. − 3. and Theorem 17.2. ∎

Theorem 17.10. *Suppose that $F : \mathbb{R} \to \mathbb{C}$ is in BV, then*

$$|T_F(x+) - T_F(x)| \le |F(x+) - F(x)| \tag{17.12}$$

for all $x \in \mathbb{R}$. If we further assume that F is right continuous then there exists a unique measure μ on $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$. such that

$$\mu((-\infty, x]) = F(x) - F(-\infty) \text{ for all } x \in \mathbb{R}. \tag{17.13}$$

Proof. Since $F \in BV$, $F(x+)$ exists for all $x \in \mathbb{R}$ and hence Eq. (17.12) is a consequence of Eq. (17.6). Now assume that F is right continuous. In this case Eq. (17.12) shows that $T_F(x)$ is also right continuous. By considering the real and imaginary parts of F separately it suffices to prove there exists a unique finite signed measure μ satisfying Eq. (17.13) in the case that F is real valued. Now let $F_\pm = (T_F \pm F)/2$, then $F_\pm$ are increasing right continuous bounded functions. Hence there exists unique measure $\mu_\pm$ on $\mathcal{B}$ such that

$$\mu_\pm((-\infty, x]) = F_\pm(x) - F_\pm(-\infty) \; \forall x \in \mathbb{R}.$$

The finite signed measure $\mu \equiv \mu_+ - \mu_-$ satisfies Eq. (17.13). So it only remains to prove that μ is unique.

Suppose that $\tilde{\mu}$ is another such measure such that (17.13) holds with μ replaced by $\tilde{\mu}$. Then for $(a, b]$,

$$|\mu|(a, b] = \sup_{\mathbb{P}} \sum_{x \in \mathbb{P}} |F(x_+) - F(x)| = |\tilde{\mu}|(a, b]$$

where the supremum is over all partition of $(a, b]$. This shows that $|\mu| = |\tilde{\mu}|$ on $\mathcal{A} \subset \mathcal{B}$ – the algebra generated by half open intervals and hence $|\mu| = |\tilde{\mu}|$. It now follows that $|\mu| + \mu$ and $|\tilde{\mu}| + \tilde{\mu}$ are finite positive measure on $\mathcal{B}$ such that

$$\begin{aligned}(|\mu| + \mu)((a, b]) &= |\mu|((a, b]) + (F(b) - F(a)) \\ &= |\tilde{\mu}|((a, b]) + (F(b) - F(a)) \\ &= (|\tilde{\mu}| + \tilde{\mu})((a, b])\end{aligned}$$

from which we infer that $|\mu| + \mu = |\tilde{\mu}| + \tilde{\mu} = |\mu| + \tilde{\mu}$ on $\mathcal{B}$. Thus $\mu = \tilde{\mu}$.

Alternatively, one may prove the uniqueness by showing that $\mathcal{C} := \{A \in \mathcal{B} : \mu(A) = \tilde{\mu}(A)\}$ is a monotone class which contains $\mathcal{A}$. ∎

Definition 17.11. A function $F : \mathbb{R} \to \mathbb{C}$ is said to be of normalized bounded variation if $F \in BV$, F is right continuous and $F(-\infty) = 0$. We will abbreviate this by saying $F \in NBV$. (The condition: $F(-\infty) = 0$ is not essential and plays no role in the discussion below.)

Theorem 17.12. *Suppose that $F \in NBV$ and μ_F is the measure defined by Eq. (17.13), then*

$$d\mu_F = F' dm + d\lambda \tag{17.14}$$

where $\lambda \perp m$ and in particular for $-\infty < a < b < \infty$,

$$F(b) - F(a) = \int_a^b F' dm + \lambda((a, b]). \tag{17.15}$$

Proof. By Theorem 16.2, there exists $f \in L^1(m)$ and a complex measure λ such that for m -a.e. x,

$$f(x) = \lim_{r \downarrow 0} \frac{\mu(E_r)}{m(E_r)}, \tag{17.16}$$

for any collection of $\{E_r\}_{r>0} \subset \mathcal{B}$ which shrink nicely to $\{x\}$, $\lambda \perp m$ and

$$d\mu_F = f dm + d\lambda.$$

From Eq. (17.16) it follows that

$$\begin{aligned} \lim_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} &= \lim_{h \downarrow 0} \frac{\mu_F((x, x+h])}{h} = f(x) \text{ and} \\ \lim_{h \downarrow 0} \frac{F(x-h) - F(x)}{-h} &= \lim_{h \downarrow 0} \frac{\mu_F((x-h, x])}{h} = f(x) \end{aligned}$$

for m – a.e. x, i.e. $\frac{d}{dx^+}F(x) = \frac{d}{dx^-}F(x) = f(x)$ for m –a.e. x. This implies that F is m – a.e. differentiable and $F'(x) = f(x)$ for m – a.e. x. ■

Corollary 17.13. *Let $F : \mathbb{R} \to \mathbb{C}$ be in NBV, then*

1. $\mu_F \perp m$ *iff* $F' = 0 \quad m$ *a.e.*
2. $\mu_F \ll m$ *iff* $\lambda = 0$ *iff*

$$\mu_F((a,b]) = \int_{(a,b]} F'(x) dm(x) \text{ for all } a < b. \tag{17.17}$$

Proof. 1. If $F' = 0 \quad m$ a.e., then by Eq. (17.14), $\mu_F = \lambda \perp m$. If $\mu_F \perp m$, then by Eq. (17.14), $F'dm = d\mu_F - d\lambda \perp dm$ and by Remark 13.9 $F'dm = 0$, i.e. $F' = 0$ m -a.e.

2. If $\mu_F \ll m$, then $d\lambda = d\mu_F - F'dm \ll dm$ which implies by Lemma 14.3 that $\lambda = 0$. Therefore Eq. (17.15) becomes (17.17). Now let

$$\rho(A) := \int_A F'(x) dm(x) \text{ for all } A \in \mathcal{B}.$$

Recall by the Radon - Nikodym theorem that $\int_{\mathbb{R}} |F'(x)|\, dm(x) < \infty$ so that ρ is a complex measure on $\mathcal{B}$. So if Eq. (17.17) holds, then $\rho = \mu_F$ on the algebra generated by half open intervals. Therefore $\rho = \mu_F$ as in the uniqueness part of the proof of Theorem 17.10. Therefore $d\mu_F = F'dm$ and hence $\lambda = 0$. ■

Theorem 17.14. *Suppose that $F : [a,b] \to \mathbb{C}$ is a measurable function. Then the following are equivalent:*

1. *F is absolutely continuous on $[a,b]$.*
2. *There exists $f \in L^1([a,b]), dm)$ such that*

$$F(x) - F(a) = \int_a^x f dm \;\forall x \in [a,b] \tag{17.18}$$

3. *F' exists a.e., $F' \in L^1([a,b], dm)$ and*

$$F(x) - F(a) = \int_a^x F' dm \, \forall x \in [a,b]. \tag{17.19}$$

Proof. In order to apply the previous results, extend F to $\mathbb{R}$ by $F(x) = F(b)$ if $x \geq b$ and $F(x) = F(a)$ if $x \leq a$.

1. $\implies$ 3. If F is absolutely continuous then F is continuous on $[a, b]$ and $F - F(a) = F - F(-\infty) \in NBV$ by Lemma 17.8. By Proposition 17.4, $\mu_F \ll m$ and hence Item 3. is now a consequence of Item 2. of Corollary 17.13. The assertion 3. $\implies$ 2. is trivial.

2. $\implies$ 1. If 2. holds then F is absolutely continuous on $[a, b]$ by Lemma 17.8. ■

18. Appendix: Compactness on metric spaces

Definition 18.1. Let (X, τ) be a topological space and $A \subset X$. An **open cover** of A is a collection of open sets $\{V_\alpha\}_{\alpha \in I}$ such that $A \subset \cup_{\alpha \in I} V_\alpha$.

Definition 18.2. A set $K \subseteq X$ is compact if every open cover of K has a finite subcover, That is to say if $K \subset \cup_{\alpha \in I} V_\alpha$ with the $V_\alpha \in \tau$, then there exists $\{\alpha_i \in I : i = 1, 2, \ldots, n\}$ such that "open covers" $\{V_\alpha\}_{\alpha \in I}$ of K there exists finite subcover $K \subseteq V_{\alpha_1}, \cup V_{\alpha_2} \cup \ldots V_{\alpha_n}$. We will write $K \sqsubset\sqsubset X$ to indicate that K is a compact subset of X.

Example 18.3. Any finite subset $K \subset X$ is compact.

Proposition 18.4. *Let (X, ρ) be a metric space. If $K \subseteq X$ is compact then K is closed.*

Proof. We must show that K^c is open which is equivalent to showing for all $x \notin K$ there is a $\delta > 0$ such that $B(x, \delta) \cap K = \emptyset$. To construct δ, choose for all $k \in K$, $r_k > 0$ such that $d(x, k) > r_k/2$. Then $V_k := B(k, r_k)$ for $k \in K$ forms an open cover of K, hence there exists $k_1, \ldots, k_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(k_n, r_{k_n}).$$

Let $\delta = \min\{r_{k_1}, \ldots, r_{k_n}) > 0$, then $B(x, r) \cap B(k_i, r_{k_i}) = \emptyset$ for all i. Thus

$$B(x, r) \subseteq (\bigcup_{i=1}^{n} B(k_i, r_{k_i}))^c \subseteq K^c$$

which shows that K^c is open hence that K is closed. ■

Definition 18.5. Let (X, ρ) be a metric space. A subset $K \subset X$ is

(1) complete if $(K, \rho|_K)$ is a complete metric space.
(2) totally bounded if for all $\epsilon > 0$ there exists a finite subset $F \subset K$ such that $K \subset \cup_{k \in F} B(k, \epsilon)$.
(3) sequentially compact if every sequence $\{x_n\}_{n=1}^{\infty} \subset K$ has a convergent subsequence in K. That is there should be $y_k = x_{n_k}$ and $x \in K$ such that $\lim_{k \to \infty} y_k = x$.

Remark 18.6. If $K \subset X$ is a totally bounded subset and $F \subset K$, then F is also totally bounded. Indeed, any cover K by $\epsilon -$ balls is also a cover of F.

Theorem 18.7. *Let (X, ρ) be a metric space and $K \subseteq X$. The following are equivalent*

(a) *K is compact.*
(b) *K is sequentially compact.*

(c) *K is complete and totally bounded.*

Proof. By replacing X by K if necessary we may assume that $K = X$.

($a \Longrightarrow b$) We will show that not $b \Longrightarrow$ not a. Suppose that X is not sequentially compact, then there exists $\{x_n\}_{n=1}^\infty \subset X$ with no convergent subsequence.[6] This implies for all $x \in X$, there exists $\delta_x > 0$ such that $\{n : x_n \in B(x, \delta_x)\}$ is finite. Now $\{B(x, \delta_x)\}_{x\in X}$ is an open cover of X which we claim has not finite subcover. For if there was a finite set $F \subset X$ such that $X = \cup_{x\in F} B(x, \delta_x)$, then at least for one $x \in F$ we would have to have that $\{n : x_n \in B(x, \delta_x)\}$ is infinite, contradicting the definition of δ_x.

($b \Longrightarrow c$) Let $\{x_n\}_{n=1}^\infty \subset X$ be a Cauchy sequence, then assuming b there is an $x \in X$ and a subsequence $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} y_n = x$. By the triangle inequality we have

$$d(x, x_n) \le d(x, y_m) + d(y_m, x_n)$$

and therefore

$$\limsup_{n\to\infty} d(x, x_n) \le \limsup_{m\to\infty} d(x, y_m) + \limsup_{m,n\to\infty} d(y_m, x_n) = 0 + 0,$$

since $\lim_{n\to\infty} y_n = x$ and $\{x_n\}_{n=1}^\infty$ is Cauchy. This shows that K is complete. If K were not totally bounded then there exists $\epsilon > 0$ such that the open cover $\{B(x, \epsilon) : x \in X\}$ has no finite subcover. We now choose a sequence $\{x_n\}_{n=1}^\infty \subset X$ inductively as follows. Let $x_1 \in X$ be chosen arbitrarily. Assume that $x_1, x_2, \ldots, x_k$ have already been chosen. Since $X \neq \cup_{i=1}^k B(x_i, \epsilon)$ we may choose $x_{k+1} \in X \setminus \cup_{i=1}^k B(x_i, \epsilon)$. This defines a sequence $\{x_n\}_{n=1}^\infty$ with the property that $d(x_k, x_i) \ge \epsilon$ for all i and k. It is clear that this sequence is not convergent, in fact it is not even Cauchy. Hence if K is not totally bounded then K is not sequentially compact.

($c \Longrightarrow a$) Let $\mathcal{V} := \{V_\alpha\}_{\alpha\in A}$ be an open cover of X. Since X is totally bounded, there exist a finite set $F_1 \subset X$ such that $X = \cup_{x\in F_1} B(x, 1) = \cup_{x\in F_1} \overline{B(x, 1)}$ where $\overline{B(x, \epsilon)} = \{y \in X : d(x, y) \le \epsilon\}$ which you should check is a closed set. For sake of contradiction, suppose that X is not covered by a finite subset of $\mathcal{V}$. In this case, it follows that there is and $y \in F_1$ such that $X_1 := \overline{B(y_1, 1)}$ is not covered by a finite subset of $\mathcal{V}$. Working in the same with X replaced by X_1 we may find a closed subset $X_2 = \overline{B(y_2, 1/2)} \cap X_1$ which is not covered by a finite subset of $\mathcal{V}$. Working inductively, we may construct closed subsets $\{X_i\}$ of X such that

$$X_1 \supset X_2 \supset X_3 \supset \ldots X_n \supset \ldots,$$

$$\text{diam}(X_i) := \sup\{d(x, y) : x, y \in X_i\} \le 2/i, \tag{18.1}$$

and no X_i may be covered by a finite subset of $\mathcal{V}$. Let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence such that $x_n \in X_n$ for all n. It is clear by Eq. (18.1) that $\{x_n\}_{n=1}^\infty$ is Cauchy and therefore convergent, by assumption c, to some $x \in \cap X_i$. We have use the fact that the X_i 's are closed here. Because $\mathcal{V}$ is a cover of X, there is a $V \in \mathcal{V}$ such that $x \in V$. Since V contains $B(x, \delta)$ for some $\delta > 0$, it is easily seen that by Eq. (18.1)

[6](An alternative argument.) By passing to a subsequence of $E := \{x_n\}_{n=1}^\infty$ if necessary we may assume that $x_n \neq x_m$ if $m \neq n$. Choose $\epsilon_1 > 0$ such that $B(x_1, \epsilon_1) \cap \{x_n\}_{n=1}^\infty = \{x_1\}$ and then choose, inductively, $\epsilon_i > 0$ so that $B(x_i, \epsilon_i) \cap B(x_j, \epsilon_j) = \emptyset$ if $i \neq j$. It is now easily checked that E is closed and that $\mathcal{V} = \{E^c\} \cup \{B(x_i, \epsilon_i) : i = 1, 2, \ldots\}$ is an open cover of X with no finite subcover.

and the triangle inequality that $X_n \subset V$ for all large enough n. But this contradicts the assertion that no X_i could be covered by a finite subset of $\mathcal{V}$. ■

Corollary 18.8. *Compact subsets of $\mathbb{R}^n$ are the closed and bounded sets.*

Proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M, M]^n$ for some positive integer M. For $\delta > 0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M, M]^n := \{\delta x : x \in \mathbb{Z}^n \text{ and } \delta|x_i| \leq M \text{ for } i = 1, 2, \ldots, n\}.$$

We will shows that by choosing $\delta > 0$ sufficiently small, that

$$K \subset [-M, M]^n \subset \cup_{x \in \Lambda_\delta} B(x, \epsilon) \tag{18.2}$$

which shows that K is totally bounded. Hence by Theorem 18.7, K will be compact.

Suppose that $y \in [-M, M]^n$, then there exists $x \in \Lambda_\delta$ such that $|y_i - x_i| \leq \delta$ for $i = 1, 2, \ldots, n$. Hence

$$d^2(x, y) = \sum_{i=1}^{n} (y_i - x_i)^2 \leq n\delta^2$$

which shows that $d(x, y) \leq \sqrt{n}\delta$. Hence if choose $\delta < \epsilon/\sqrt{n}$ we have shows that $d(x, y) < \epsilon$, i.e. Eq. (18.2) holds. ■

19. Appendix: Dynkin's $\pi - \lambda$ Theorem

Definition 19.1 (Lambda Class)**.** As collection $\mathcal{D}$ of subsets of X is a λ–class if $\mathcal{D}$ satisfies the following properties:

1. $X \in \mathcal{D}$
2. If $A, B \in \mathcal{D}$ and $A \cap B = \phi$, then $A \cup B \in \mathcal{D}$. (Closed under disjoint unions.)
3. If $A, B \in \mathcal{D}$ and $A \supset B$, then $A \setminus B \in \mathcal{D}$. (Closed under proper differences.)
4. If $A_n \in \mathcal{D}$ and $A_n \uparrow A$, then $A \in \mathcal{D}$. (Closed under countable increasing unions.)

Definition 19.2 (π – class)**.** A family of sets $\mathcal{C} \subset \mathcal{P}(X)$ is called a π – class if it is closed under finite intersections.

Theorem 19.3 ($\pi - \lambda$ Theorem)**.** *If $\mathcal{D}$ is a λ class which contains a family $\mathcal{C}$ of subsets of X which is closed under intersections (i.e. a π-class) then $\sigma(\mathcal{C}) \subset \mathcal{D}$.*

Proof. (Taken from Billingsley, pp. 33-34.) Let $\mathcal{F}$ be the intersection of all λ–classes which contains $\mathcal{C}$. Then $\mathcal{F}$ is a λ–class and $\mathcal{C} \subset \mathcal{F} \subset \mathcal{D}$.

Let

$$\mathcal{F}_1 := \{A \subset X : A \cap B \in \mathcal{F} \ \forall \ B \in \mathcal{C}\}.$$

Then $\mathcal{F}_1$ is also a λ–class (you check) and since $\mathcal{C} \subset \mathcal{F}_1$, we know that $\mathcal{F} \subset \mathcal{F}_1$. (Recall that $\mathcal{F}$ is the smallest λ–class which contains $\mathcal{C}$.) From this we conclude that if $A \in \mathcal{F}$ and $B \in \mathcal{C}$ then $A \cap B \in \mathcal{F}$.

Let

$$\mathcal{F}_2 := \{B \subset X : A \cap B \in \mathcal{F} \ \forall \ A \in \mathcal{F}\}.$$

Then $\mathcal{F}_2$ is a λ–class which, by the above paragraph, contains $\mathcal{C}$. As above this implies that $\mathcal{F} \subset \mathcal{F}_2$, i.e. we have shown that $\mathcal{F}$ is closed under intersections.

If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, $A \setminus A \cap B \in \mathcal{F}$, and $A \cup B = A \cup (A \setminus A \cap B) \in \mathcal{F}$. Therefore, $\mathcal{F}$ is closed under finite unions. Since $\mathcal{F}$ is a λ–class (in particular closed under countable increasing unions) and closed under finite unions, $\mathcal{F}$ is closed under arbitrary countable unions. Thus, $\mathcal{F}$ is a σ-algebra and since $\mathcal{C} \subset \mathcal{F}$, $\sigma(\mathcal{C}) \subset \mathcal{F} \subset \mathcal{D}$. ■

As an application, let us give another proof of Theorem 4.7 in the case that $\mu(X) = \nu(X) < \infty$.

Proof. (Proof of Theorem 4.7)Let

$$\mathcal{D} = \{A \in \mathcal{M} : \mu(A) = \nu(A)\}.$$

The reader may easily check that $\mathcal{D}$ is a λ class. For example for item 4. we have if $A_n \in \mathcal{D}$ then

$$\mu(A_n) = \nu(A_n).$$

If $A_n \uparrow A$, then by passing to the limit in the previous equation we learn that $\mu(A) = \nu(A)$, i.e. that $A \in \mathcal{D}$. In checking item 3. we make use of the fact that $\mu(X) = \nu(X) < \infty$.) The proof of the finite case now follows from Dynkin's $\pi - \lambda$ Theorem 19.3. ■

20. Appendix: Multiplicative System Theorem

Definition 20.1 (Bounded Convergence)**.** Let Ω be a set. We say that a sequence of functions X_n from Ω to $\mathbb{R}$ or $\mathbb{C}$ converges boundedly to a function X if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in \Omega$ and

$$\sup\{|X_n(\omega)| : \omega \in \Omega \text{ and } n = 1, 2, \ldots\} < \infty.$$

Definition 20.2 (Multiplicative System)**.** A collection of real valued functions (Q) on Ω is a multiplicative system provided $fg \in Q$ whenever $f, g \in Q$.

Theorem 20.3 (Dynkin's Multiplicative System Theorem)**.** *Let $\mathcal{H}$ be a linear space of functions which contains the constant functions and is closed under bounded convergence. If $Q \subset \mathcal{H}$ is multiplicative family of **bounded** functions, then $\mathcal{H}$ contains all **bounded** $\sigma(Q)$-measurable functions.*

Theorem 20.4 (Complex Multiplicative System Theorem)**.** *Let $\mathcal{H}$ be a complex vector space of complex functions on Ω such that: $1 \in \mathcal{H}$, $\mathcal{H}$ is closed under complex conjugation, and $\mathcal{H}$ is closed under bounded convergence. Suppose Q is a multiplicative family of **bounded** functions contained in $\mathcal{H}$ which is also closed under conjugation. Then $\mathcal{H}$ contains all of the **bounded** $\sigma(Q)$ measurable functions.*

The proof of these Theorems is based on Dynkin's π–λ Theorem 19.3.

Proof. (of the Multiplicative System Theorems.) Put $A \in \mathcal{D}$ if $1_A \in \mathcal{H}$ and denote by $\mathcal{C}$ the family of all sets of the form:

$$B := \{\omega \in \Omega : X_1(\omega) \in R_1, \ldots, X_m(\omega) \in R_m\} \tag{20.1}$$

where $m = 1, 2, \ldots,$ $X_k \in Q$ and R_k are open intervals in the real case and R_k are rectangles in $\mathbb{C}$ in the complex case. One may check that $\mathcal{D}$ is a λ–system, $\mathcal{C}$ is closed under intersections, and that $\sigma(Q) = \sigma(\mathcal{C})$. The π–λ Theorem 19.3 will imply that $\sigma(Q) = \sigma(\mathcal{C}) \subset \mathcal{D}$ once we show that $\mathcal{C} \subset \mathcal{D}$. This is our next task.

It is easy to construct, for each k a uniformly bounded sequence of continuous functions f_n^k converging to the characteristic function 1_{R_k}. By Weierstrass' theorem, there exists polynomials $p_m^k(x)$ such that $\left|p_n^k(x) - f_n^k(x)\right| \leq 1/n$ for $|x| \leq \|X_k\|_\infty$ in the real case and polynomials polynomials $p_m^k(z, \bar{z})$ in z and $\bar{z}$

such that $\left|p_n^k(z,\bar{z}) - f_n^k(z)\right| \le 1/n$ for $|z| \le \|X_k\|_\infty$ in the complex case. The functions

$$F_n := p_n^1(X_1)p_n^2(X_2)\dots p_n^m(X_m) \quad \text{(real case)}$$
$$F_n := p_n^1(X_1\bar{X}_1)p_n^2(X_2,\bar{X}_2)\dots p_n^m(X_m,\bar{X}_m) \quad \text{(complex case)}$$

on Ω are uniformly bounded, belong to $\mathcal{H}$ and converge pointwise to 1_B as $n \to \infty$, where B is the set in Eq. (20.1). Hence this set is an element of $\mathcal{D}$ and therefore $\mathcal{C} \subset \mathcal{D}$.

Since $\sigma(Q) \subset \mathcal{D}$, if X is an arbitrary bounded $\sigma(Q)$-measurable function then $\{\frac{k}{n} < X \le \frac{k+1}{n}\} \in \sigma(Q) \subset \mathcal{D}$ in the real case and

$$\left\{\frac{k}{n} < \mathrm{Re}X \le \frac{k+1}{n}, \frac{k'}{n} < \mathrm{Im}X \le \frac{k'+1}{n}\right\} \in \sigma(Q) \subset \mathcal{D}$$

in the complex case. Therefore

$$X_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} 1_{\frac{k}{n} < X \le \frac{k+1}{n}} \in \mathcal{H} \qquad \text{(real case)}$$

and

$$X_n = \sum_k \sum_{k'} \frac{k+ik'}{n} 1_{\left\{\frac{k}{n} < \mathrm{Re}X \le \frac{k+1}{n}, \frac{k'}{n} < \mathrm{Im}X \le \frac{k'+1}{n}\right\}} X_n \in \mathcal{H}. \quad \text{(complex case)}$$

Because $X_n \in \mathcal{H}$ converges to X boundedly and $\mathcal{H}$ is closed under bounded convergence, X is in $\mathcal{H}$. ■