## 11. Introduction to the Spectral Theorem

The following spectral theorem is a minor variant of the usual spectral theorem for matrices. This reformulation has the virtue of carrying over to general (unbounded) self adjoint operators on infinite dimensional Hilbert spaces.

**Theorem 11.1.** *Suppose $A$ is an $n \times n$ complex self adjoint matrix, i.e. $A^* = A$ or equivalently $A_{ji} = \bar{A}_{ij}$ and let $\mu$ be counting measure on $\{1, 2, \dots, n\}$. Then there exists a unitary map $U : \mathbb{C}^n \to L^2(\{1, 2, \dots, n\}, d\mu)$ and a real function $\lambda : \{1, 2, \dots, n\} \to \mathbb{R}$ such that $UA\xi = \lambda \cdot U\xi$ for all $\xi \in \mathbb{C}^n$. We summarize this equation by writing $UAU^{-1} = M_\lambda$ where*

$$M_\lambda : L^2(\{1, 2, \dots, n\}, d\mu) \to L^2(\{1, 2, \dots, n\}, d\mu)$$

*is the linear operator, $g \in L^2(\{1, 2, \dots, n\}, d\mu) \to \lambda \cdot g \in L^2(\{1, 2, \dots, n\}, d\mu)$.*

**Proof.** By the usual form of the spectral theorem for self-adjoint matrices, there exists an orthonormal basis $\{e_i\}_{i=1}^n$ of eigenvectors of $A$, say $Ae_i = \lambda_i e_i$ with $\lambda_i \in \mathbb{R}$. Define $U : \mathbb{C}^n \to L^2(\{1, 2, \dots, n\}, d\mu)$ to be the unique (unitary) map determined by $Ue_i = \delta_i$ where

$$\delta_i(j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

and let $\lambda : \{1, 2, \dots, n\} \to \mathbb{R}$ be defined by $\lambda(i) := \lambda_i$. ■

**Definition 11.2.** Let $A : H \to H$ be a possibly unbounded operator on $H$. We let

$$D(A^*) = \{y \in H : \ \exists\, z \in H \ \ni \ (Ax, y) = (x, z) \ \forall\, x \in D(A)\}$$

and for $y \in D(A^*)$ set $A^* y = z$.

**Definition 11.3.** If $A = A^*$ the $A$ is self adjoint.

**Proposition 11.4.** *Let $(X, \mu)$ be $\sigma$ – finite measure space, $H = L^2(X, d\mu)$ and $f : X \to \mathbb{C}$ be a measurable function. Set $Ag = fg = M_f g$ for all*

$$g \in D(M_f) = \{g \in H : fg \in H\}.$$

*Then $D(M_f)$ is a dense subspace of $H$ and $M_f^* = M_{\bar{f}}$.*

**Proof.** For any $g \in H = L^2(X, d\mu)$ and $m \in \mathbb{N}$, let $g_m := g 1_{|f| \le m}$. Since $|fg_m| \le m\,|g|$ it follows that $fg_m \in H$ and hence $g_m \in D(M_f)$. By the dominated convergence theorem, it follows that $g_m \to g$ in $H$ as $m \to \infty$, hence $D(M_f)$ is dense in $H$.

Suppose $h \in \mathcal{D}(M_f^*)$ then there exists $k \in L^2$ such that $(M_f g, h) = (g, k)$ for all $g \in D(M_f)$, i.e.

$$\int_X fg\overline{h}\; d\mu = \int_X g\overline{k}\; d\mu \text{ for all } g \in D(M_f)$$

or equivalently

$$\int_X g(\overline{\overline{f}h - k}) d\mu = 0 \text{ for all } g \in D(M_f). \tag{11.1}$$

Choose $X_n \subset X$ such that $X_n \uparrow X$ and $\mu(X_n) < \infty$ for all $n$. It is easily checked that

$$g_n := 1_{X_n} \frac{\overline{f}h - k}{|\overline{f}h - k|} 1_{|f| \le n}$$

is in $D(M_f)$ and putting this function into Eq. (11.1) shows

$$\int_X \left|\overline{f}h - k\right| 1_{|f|\le n} d\mu = 0 \text{ for all } n.$$

Using the monotone convergence theorem, we may let $n \to \infty$ in this equation to find $\int_X \left|\overline{f}h - k\right| d\mu = 0$ and hence that $\bar{f}h = k \in L^2$. This shows $h \in D(M_{\bar{f}})$ and $M_f^* h = \overline{f}h$. ■

**Theorem 11.5** (Spectral Theorem). *Suppose $A^* = A$ then there exists $(X, \mu)$ a $\sigma$ – finite measure space, $f : X \to \mathbb{R}$ measurable, and $U : H \to L^2(x, \mu)$ unitary such that $UAU^{-1} = M_f$. Note this is a statement about domains as well, i.e. $UD(M_f) = D(A)$.*

I would like to give some examples of computing $A^*$ and Theorem 11.5 as well. We will consider here the case of constant coefficient differential operators on $L^2(\mathbb{R}^n)$. First we need the following definition.

**Definition 11.6.** Let $a_\alpha \in C^\infty(U)$, $L = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ – a $m^{\text{th}}$ order linear differential operator on $\mathcal{D}(U)$ and

$$L^\dagger \phi = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \partial^\alpha \left[a_\alpha \phi\right]$$

denote the **formal adjoint** of $L$ as in Lemma 5.4 above. For $f \in L^p(U)$ we say $Lf \in L^p(U)$ or $L^p_{loc}(U)$ if the generalized function $Lf$ may be represented by an element of $L^p(U)$ or $L^p_{loc}(U)$ respectively, i.e. $Lf = g \in L^p_{loc}(U)$ iff

$$\int_U f \cdot L^\dagger \phi \, dm = \int_U g\phi dm \text{ for all } \phi \in C_c^\infty(U). \tag{11.2}$$

In terms of the complex inner product,

$$(f, g) := \int_U f(x)\bar{g}(x) dm(x)$$

Eq. (11.2) is equivalent to

$$\left(f \cdot L^\circledast \phi\right) = (g, \phi) \text{ for all } \phi \in C_c^\infty(U)$$

where

$$L^\circledast \phi := \sum_{|\alpha|\le m} (-1)^{|\alpha|} \partial^\alpha \left[\bar{a}_\alpha \phi\right].$$

Notice that $L^\circledast$ satisfies $L^\circledast \bar{\phi} = \overline{L^\dagger \phi}$. (We do not write $L^*$ here since $L^\circledast$ is to be considered an operator on the space on $\mathcal{D}'(U)$.)

*Remark* 11.7. Recall that if $f, h \in L^2(\mathbb{R}^n)$, then the following are equivalent

(1) $\hat{f} = h$.
(2) $(h, g) = \left(f, \mathcal{F}^{-1} g\right)$ for all $g \in C_c^\infty(\mathbb{R}^n)$.
(3) $(h, g) = \left(f, \mathcal{F}^{-1} g\right)$ for all $g \in \mathcal{S}(\mathbb{R}^n)$.
(4) $(h, g) = \left(f, \mathcal{F}^{-1} g\right)$ for all $g \in L^2(\mathbb{R}^n)$.

Indeed if $\hat{f} = h$ and $g \in L^2(\mathbb{R}^n)$, the unitarity of $\mathcal{F}$ implies

$$(h, g) = \left(\hat{f}, g\right) = (\mathcal{F}f, g) = \left(f, \mathcal{F}^{-1} g\right).$$

Hence $1 \implies 4$ and it is clear that $4 \implies 3 \implies 2$. If 2 holds, then again since $\mathcal{F}$ is unitary we have

$$(h, g) = \left(f, \mathcal{F}^{-1} g\right) = \left(\hat{f}, g\right) \text{ for all } g \in C_c^\infty(\mathbb{R}^n)$$

which implies $h = \hat{f}$ a.e., i.e. $h = \hat{f}$ in $L^2(\mathbb{R}^n)$.

**Proposition 11.8.** *Let $p(x) = \sum_{|\alpha| \le m} a_\alpha x^\alpha$ be a polynomial on $\mathbb{C}^n$,*

$$L := p(\partial) := \sum_{|\alpha| \le m} a_\alpha \partial^\alpha \tag{11.3}$$

*and $f \in L^2(\mathbb{R}^n)$. Then $Lf \in L^2(\mathbb{R}^n)$ iff $p(i\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n)$ and in which case*

$$(Lf)\hat{}\,(\xi) = p(i\xi)\hat{f}(\xi). \tag{11.4}$$

*Put more concisely, letting*

$$D(B) = \left\{ f \in L^2(\mathbb{R}^n) : Lf \in L^2(\mathbb{R}^n) \right\}$$

*with $Bf = Lf$ for all $f \in D(B)$, we have*

$$\mathcal{F} B \mathcal{F}^{-1} = M_{p(i\xi)}.$$

**Proof.** As above, let

$$L^\dagger := \sum_{|\alpha| \le m} a_\alpha (-\partial)^\alpha \text{ and } L^{\circledast} := \sum_{|\alpha| \le m} \bar{a}_\alpha (-\partial)^\alpha. \tag{11.5}$$

For $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} L^{\circledast} \phi^\vee(x) &= L^{\circledast} \int \phi(\xi) e^{ix\cdot\xi} d\lambda(\xi) = \sum_{|\alpha| \le m} \bar{a}_\alpha (-\partial_x)^\alpha \int \phi(\xi) e^{ix\cdot\xi} d\lambda(\xi) \\ &= \int \overline{p(i\xi)} \phi(\xi) e^{ix\cdot\xi} d\lambda(\xi) = \mathcal{F}^{-1}\left[\overline{p(i\xi)}\phi(\xi)\right](x) \end{aligned}$$

So if $f \in L^2(\mathbb{R}^n)$ such that $Lf \in L^2(\mathbb{R}^n)$. Then by Remark 11.7,

$$\begin{aligned} (\widehat{Lf}, \phi) &= (Lf, \phi^\vee) = \langle f, L^{\circledast}\phi^\vee \rangle = \langle f(x), \mathcal{F}^{-1}\left[\overline{p(i\xi)}\phi(\xi)\right](x)\rangle \\ &= \langle \hat{f}(\xi), \left[\overline{p(i\xi)}\phi(\xi)\right]\rangle = \langle p(i\xi)\hat{f}(\xi), \phi(\xi)\rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^n) \end{aligned}$$

from which it follows that Eq. (11.4) holds and that $p(i\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n)$.

Conversely, if $f \in L^2(\mathbb{R}^n)$ is such that $p(i\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n)$ then for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\left(f, L^{\circledast}\phi\right) = \left(\hat{f}, \mathcal{F}L^{\circledast}\phi\right). \tag{11.6}$$

Since

$$\begin{aligned} \mathcal{F}\left(L^{\circledast}\phi\right)(\xi) &= \int L^{\circledast}\phi(x) e^{-ix\cdot\xi} d\lambda(x) = \int \phi(x) \overline{L_x} e^{-ix\cdot\xi} d\lambda(x) \\ &= \int \phi(x) \overline{a_\alpha} \partial_x^\alpha e^{-ix\cdot\xi} d\lambda(x) = \int \phi(x) \overline{a_\alpha} (-i\xi)^\alpha e^{-ix\cdot\xi} d\lambda(x) \\ &= \overline{p(i\xi)}\hat{\phi}(\xi), \end{aligned}$$

Eq. (11.6) becomes

$$\left(f, L^{\circledast}\phi\right) = \left(\hat{f}(\xi), \overline{p(i\xi)}\hat{\phi}(\xi)\right) = \left(p(i\xi)\hat{f}(\xi), \hat{\phi}(\xi)\right) = \left(\mathcal{F}^{-1}\left[p(i\xi)\hat{f}(\xi)\right](x), \phi(x)\right).$$

This shows $Lf = \mathcal{F}^{-1}\left[p\left(i\xi\right)\hat{f}(\xi)\right] \in L^2\left(\mathbb{R}^n\right)$. ■

**Lemma 11.9.** *Suppose $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ and $L = p(\partial)$ is the constant coefficient differential operator $B = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$ with $D(B) := \mathcal{S}\left(\mathbb{R}^n\right) \subset L^2\left(\mathbb{R}^n\right)$. Then*

$$\mathcal{F}B\mathcal{F}^{-1} = M_{p(i\xi)}|_{\mathcal{S}(\mathbb{R}^n)}.$$

**Proof.** This is result of the fact that $\mathcal{F}\left(\mathcal{S}\left(\mathbb{R}^n\right)\right) = \mathcal{S}\left(\mathbb{R}^n\right)$ and for $f \in \mathcal{S}\left(\mathbb{R}^n\right)$ we have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\lambda(\xi)$$

so that

$$Bf(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) L_x e^{i\xi\cdot x} d\lambda(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi) p(i\xi) e^{i\xi\cdot x} d\lambda(\xi)$$

so that

$$\left(Bf\right)^{\hat{}}\left(\xi\right) = p(i\xi)\hat{f}(\xi) \text{ for all } f \in \mathcal{S}\left(\mathbb{R}^n\right).$$

■

**Lemma 11.10.** *Suppose $g : \mathbb{R}^n \to \mathbb{C}$ is a measurable function such that $|g(x)| \le C\left(1 + |x|^M\right)$ for some constants $C$ and $M$. Let $A$ be the unbounded operator on $L^2\left(\mathbb{R}^n\right)$ defined by $D(A) = \mathcal{S}\left(\mathbb{R}^n\right)$ and for $f \in \mathcal{S}\left(\mathbb{R}^n\right)$, $Af = gf$. Then $A^* = M_{\bar{g}}$.*

**Proof.** If $h \in D\left(M_{\bar{g}}\right)$ and $f \in D(A)$, we have

$$(Af, h) = \int_{\mathbb{R}^n} gf\bar{h}dm = \int_{\mathbb{R}^n} f\overline{\bar{g}h}dm = (f, M_{\bar{g}}h)$$

which shows $M_{\bar{g}} \subset A^*$, i.e. $h \in D(A^*)$ and $A^*h = M_{\bar{g}}h$. Now suppose $h \in D\left(A^*\right)$ and $A^*h = k$, i.e.

$$\int_{\mathbb{R}^n} gf\bar{h}dm = (Af, h) = (f, k) = \int_{\mathbb{R}^n} f\bar{k}dm \text{ for all } f \in \mathcal{S}\left(\mathbb{R}^n\right)$$

or equivalently that

$$\int_{\mathbb{R}^n} \left(g\bar{h} - \bar{k}\right) f dm = 0 \text{ for all } f \in \mathcal{S}\left(\mathbb{R}^n\right).$$

Since the last equality (even just for $f \in C_c^\infty\left(\mathbb{R}^n\right)$) implies $g\bar{h} - \bar{k} = 0$ a.e. we may conclude that $h \in D(M_{\bar{g}})$ and $k = M_{\bar{g}}h$, i.e. $A^* \subset M_{\bar{g}}$. ■

**Theorem 11.11.** *Suppose $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ and $A = p(\partial)$ is the constant coefficient differential operator with $D(A) := C_c^\infty\left(\mathbb{R}^n\right) \subset L^2\left(\mathbb{R}^n\right)$ such that $A = L = p(\partial)$ on $D(A)$, see Eq. (11.3). Then $A^*$ is the operator described by*

$$\begin{aligned} D(A^*) &= \left\{f \in L^2\left(\mathbb{R}^n\right) : L^\dagger f \in L^2\left(\mathbb{R}^n\right)\right\} \\ &= \left\{f \in L^2\left(\mathbb{R}^n\right) : p(i\xi)\hat{f}(\xi) \in L^2\left(\mathbb{R}^n\right)\right\} \end{aligned}$$

*and $A^*f = L^\dagger f$ for $f \in D(A^*)$ where $L^\dagger$ is defined in Eq. (11.5) above. Moreover we have $\mathcal{F}A^*\mathcal{F}^{-1} = M_{\overline{p(i\xi)}}$.*

**Proof.** Let $D(B) = \mathcal{S}(\mathbb{R}^n)$ and $B := L$ on $D(B)$ so that $A \subset B$. We are first going to show $A^* = B^*$. As is easily verified, in general if $A \subset B$ then $B^* \subset A^*$. So we need only show $A^* \subset B^*$. Now by definition, if $g \in D(A^*)$ with $k = A^* g$, then

$$(Af, g) = (f, k) \text{ for all } f \in D(A) := C_c^\infty(\mathbb{R}^n).$$

Suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi = 1$ in a neighborhood of 0. Then $f_n(x) := \phi(x/n) f(x)$ is in $\mathcal{S}(\mathbb{R}^n)$ and hence

$$(f_n, k) = (Lf_n, g). \tag{11.7}$$

An exercise in the product rule and the dominated convergence theorem shows $f_n \to f$ and $Lf_n \to Lf$ in $L^2(\mathbb{R}^n)$ as $n \to \infty$. Therefore we may pass to the limit in Eq. (11.7) to learn

$$(f, k) = (Bf, g) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n)$$

which shows $g \in D(B^*)$ and $B^* g = k$.

By Lemma 11.10, we may conclude that $A^* = B^* = M_{\overline{p(i\xi)}}$ and by Proposition 11.8 we then conclude that

$$\begin{aligned} D(A^*) &= \left\{ f \in L^2(\mathbb{R}^n) : p(i\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n) \right\} \\ &= \left\{ f \in L^2(\mathbb{R}^n) : L^\dagger f \in L^2(\mathbb{R}^n) \right\} \end{aligned}$$

and for $f \in D(A^*)$ we have $A^* f = L^\dagger f$. ■

**Example 11.12.** If we take $L = \Delta$ with $D(L) := C_c^\infty(\mathbb{R}^n)$, then

$$L^* = \bar{\Delta} = \mathcal{F} M_{-|\xi|^2} \mathcal{F}^{-1}$$

where $D(\bar{\Delta}) = \left\{ f \in L^2(\mathbb{R}^n) : \Delta f \in L^2(\mathbb{R}^n) \right\}$ and $\bar{\Delta} f = \Delta f$.

**Theorem 11.13.** *Suppose $A = A^*$ and $A \le 0$. Then for all $u_0 \in D(A)$ there exists a unique solution $u \in C^1([0, \infty))$ such that $u(t) \in D(A)$ for all $t$ and*

$$\dot{u}(t) = Au(t) \text{ with } u(0) = u_0. \tag{11.8}$$

*Writing $u(t) = e^{tA} u_0$, the map $u_0 \to e^{tA} u_0$ is a linear contraction semi-group, i.e.*

$$\|e^{tA} u_0\| \le \|u_0\| \text{ for all } t \ge 0. \tag{11.9}$$

*So $e^{tA}$ extends uniquely to $H$ by continuity. This extension satisfies:*

1. ***Strong Continuity:*** *the map $t \in [0, \infty) \to e^{tA} u_0$ is continuous for all $u_0 \in H$.*
2. ***Smoothing property:*** *$t > 0$*

$$e^{tA} u_0 \in \bigcap_{n=0}^{\infty} D(A^n) =: C^\infty(A)$$

*and*

$$\|A^k e^{tA}\| \le \left(\frac{k}{t}\right)^k e^{-k} \text{ for all } k \in \mathbb{N}. \tag{11.10}$$

**Proof. Uniqueness.** Suppose $u$ solves Eq. (11.8), then

$$\frac{d}{dt}(u(t), u(t)) = 2\mathrm{Re}(\dot{u}, u) = 2\mathrm{Re}(Au, u) \le 0.$$

Hence $\|u(t)\|$ is decreasing so that $\|u(t)\| \le \|u_0\|$.This implies the uniqueness assertion in the theorem and the norm estimate in Eq. (11.9).

**Existence:** By the spectral theorem we may assume $A = M_f$ acting on $L^2(X, \mu)$ for some $\sigma -$ finite measure space $(X, \mu)$ and some measurable function $f : X \to [0, \infty)$. We wish to show $u(t) = e^{tf} u_0 \in L^2$ solves

$$\dot{u}(t) = fu(t) \text{ with } u(0) = u_0 \in D(M_f) \subset L^2.$$

Let $t > 0$ and $|\Delta| < t$. Then by the mean value inequality

$$\left| \frac{e^{(t+\triangle)f} - e^{tf}}{\triangle} u_0 \right| = \max \left\{ |f e^{(t+\tilde{\triangle})f} u_0| : \tilde{\Delta} \text{ between } 0 \text{ and } \Delta \right\} \le |f u_0| \in L^2.$$

This estimated along with the fact that

$$\frac{u(t+\Delta) - u(t)}{\triangle} = \frac{e^{(t+\triangle)f} - e^{tf}}{\triangle} u_0 \overset{\text{point wise}}{\to} f e^{tf} u_0 \text{ as } \Delta \to 0$$

enables us to use the dominated convergence theorem to conclude

$$\dot{u}(t) = L^2\text{-} \lim_{\Delta \to 0} \frac{u(t+\triangle) - u(t)}{\triangle} = e^{tf} f u_0 = f u(t)$$

as desired. i.e. $\dot{u}(t) = fu(t)$.

The extension of $e^{tA}$ to $H$ is given by $M_{e^{tf}}$. For $g \in L^2$, $\left|e^{tf} g\right| \le |g| \in L^2$ and $e^{tf} g \to e^{\tau g} f$ pointwise as $t \to \tau$, so the Dominated convergence theorem shows $t \in [0, \infty) \to e^{tA} g \in H$ is continuous. For the last two assertions, let $t > 0$ and $f(x) = x^k e^{tx}$. Then $(\ln f)'(x) = \frac{k}{x} + t$ which is zero when $x = -k/t$ and therefore

$$\max_{x \le 0} \left| x^k e^{tx} \right| = |f(-k/t)| = \left( \frac{k}{t} \right)^k e^{-k}.$$

Hence

$$\|A^k e^{tA}\|_{op} \le \max_{x \le 0} \left| x^k e^{tx} \right| \le \left( \frac{k}{t} \right)^k e^{-k} < \infty.$$

■

**Theorem 11.14.** *Take* $A = \mathcal{F} M_{-|\xi|^2} \mathcal{F}^{-1}$ *so* $A|_{\mathcal{S}} = \Delta$ *then*

$$C^\infty(A) := \bigcap_{n=1}^\infty D(A^n) \subset C^\infty\left(\mathbb{R}^d\right)$$

*i.e. for all* $f \in C^\infty(A)$ *there exists a version* $\tilde{f}$ *of* $f$ *such that* $\tilde{f} \in C^\infty(\mathbb{R}^d)$.

**Proof.** By assumption $|\xi|^{2n} \hat{f}(\xi) \in L^2$ for all $n$. Therefore $\hat{f}(\xi) = \frac{g_n(\xi)}{1+|\xi|^{2n}}$ for some $g_n \in L^2$ for all $n$. Therefore for $n$ chosen so that $2n > m + d$, we have

$$\int_{\mathbb{R}^d} |\xi|^m |\hat{f}(\xi)| d\xi \le \|g_n\|_{L^2} \left\| \frac{|\xi|^m}{1 + |\xi|^{2n}} \right\|_2 < \infty$$

which shows $|\xi|^m |\hat{f}(\xi)| \in L^1$ for all $m = 0, 1, 2, \ldots$ We may now differentiate the inversion formula, $f(x) = \int \hat{f}(\xi) e^{ix \cdot \xi} d\xi$ to find

$$D^\alpha f(x) = \int (i\xi)^\alpha \hat{f}(\xi) e^{ix \cdot \xi} d\xi \text{ for any } \alpha$$

and thus conclude $f \in C^\infty$. ■

**Exercise 11.1.** Some Exercises: Section 2.5 4, 5, 6, 8, 9, 11, 12, 17.

### 11.1. Du Hammel's principle again.

**Lemma 11.15.** *Suppose $A$ is an operator on $H$ such that $A^*$ is densely defined then $A^*$ is closed.*

**Proof.** If $f_n \in D(A^*) \to f \in H$ and $A^* f_n \to g$ then for all $h \in D(A)$

$$(g, h) = \lim_{n\to\infty} (A^* f_n, h)$$

while

$$\lim_{n\to\infty} (A^* f_n, h) = \lim_{n\to\infty} (f_n, Ah) = (f, Ah),$$

i.e. $(Ah, f) = (h, g)$ for all $h \in D(A)$. Thus $f \in D(A^*)$ and $A^* f = g$. ■

**Corollary 11.16.** *If $A^* = A$ then $A$ is closed.*

**Corollary 11.17.** *Suppose $A$ is closed and $u(t) \in D(A)$ is a path such that $u(t)$ and $Au(t)$ are continuous in $t$. Then*

$$A \int_0^T u(\tau) d\tau = \int_0^T Au(\tau) d\tau.$$

**Proof.** Let $\pi_n$ be a sequence of partitions of $[0, T]$ such that $mesh(\pi_n) \to 0$ as $n \to \infty$ and set

$$f_n = \sum_{\pi_n} u(\tau_i)(\tau_{i+1} - \tau_i) \in D(A).$$

Then $f_n \to \int_0^T u(\tau) d\tau$ and

$$A f_n = \sum_{\pi_n} Au(\tau_i)(\tau_{i+1} - \tau_i) \to \int_0^T Au(\tau) d\tau.$$

Therefore $\int_0^T u(\tau) d\tau \in D(A)$ and $A \int_0^T u(\tau) d\tau = \int_0^T Au(\tau) d\tau$. ■

**Lemma 11.18.** *Suppose $A = A^*$, $A \le 0$, and $h : [0, \infty] \to H$ is continuous. Then*

$$(s, t) \in [0, \infty) \times [0, \infty) \to e^{sA} h(t)$$

$$(s, t) \in (0, \infty) \times [0, \infty) \to A^k e^{sA} h(t)$$

*are continuous maps into $H$.*

**Proof.** Let $k \ge 0$, then if $s \ge \sigma$,

$$\begin{aligned}
\left\| A^k \left( e^{sA} h(t) - e^{\sigma A} h(\tau) \right) \right\| &= \left\| A^k e^{\sigma A} \left( e^{(s-\sigma)A} h(t) - h(\tau) \right) \right\| \\
&\le \left\| A^k e^{\sigma A} \right\| \left\| e^{(s-\sigma)A} \left[ h(t) - h(\tau) \right] + e^{(s-\sigma)A} h(\tau) - h(\tau) \right\| \\
&\le \left( \frac{k}{\sigma} \right)^k e^{-k} \cdot \left[ \left\| h(t) - h(\tau) \right\| + \left\| e^{(s-\sigma)A} h(\tau) - h(\tau) \right\| \right].
\end{aligned}$$

So

$$\lim_{s \downarrow \sigma \text{ and } t \to \tau} \left\| A^k \left( e^{sA} h(t) - e^{\sigma A} h(\tau) \right) \right\| = 0$$

and we may take $\sigma = 0$ if $k = 0$. Similarly, if $s \le \sigma$,

$$\begin{aligned}\left\|A^k\left(e^{sA}h(t) - e^{\sigma A}h(\tau)\right)\right\| &= \left\|A^k e^{sA}\left(h(t) - e^{(\sigma-s)A}h(\tau)\right)\right\| \\ &\le \left\|A^k e^{sA}\right\|\left[\|h(t)-h(\tau)\| + \left\|h(\tau) - e^{(\sigma-s)A}h(\tau)\right\|\right] \\ &\le \left(\frac{k}{s}\right)^k e^{-k}\left[\|h(t)-h(\tau)\| + \left\|h(\tau) - e^{(\sigma-s)A}h(\tau)\right\|\right]\end{aligned}$$

and the latter expression tends to zero as $s \uparrow \sigma$ and $t \to \tau$. ■

**Lemma 11.19.** *Let $h \in C([0,\infty), H)$, $D := \left\{(s,t) \in \mathbb{R}^2 : s > t \ge 0\right\}$ and $F(s,t) := \int_0^t e^{(s-\tau)A}h(\tau)d\tau$ for $(s,t) \in D$. Then*

(1) *$F \in C^1(D,H)$ (in fact $F \in C^\infty(D,H)$),*

$$\frac{\partial}{\partial t}F(s,t) = e^{(s-t)A}h(t) \tag{11.11}$$

*and*

$$\frac{\partial F(s,t)}{\partial s} = \int_0^t A e^{(s-\tau)A}h(\tau)d\tau. \tag{11.12}$$

(2) *Given $\epsilon > 0$ let*

$$u_\epsilon(t) := F(t+\epsilon, t) = \int_0^t e^{(t+\epsilon-\tau)A}h(\tau)d\tau.$$

*Then $u_\epsilon \in C^1((-\epsilon,\infty), H)$, $u_\epsilon(t) \in D(A)$ for all $t > -\epsilon$ and*

$$\dot{u}_\epsilon(t) = e^{\epsilon A}h(t) + A u_\epsilon(t). \tag{11.13}$$

**Proof.** We claim the function

$$(s,t) \in D \to F(s,t) := \int_0^t e^{(s-\tau)A}h(\tau)d\tau$$

is continuous. Indeed if $(s',t') \in D$ and $(s,t) \in D$ is sufficiently close to $(s',t')$ so that $s > t'$, we have

$$\begin{aligned}F(s,t) - F(s',t') &= \int_0^t e^{(s-\tau)A}h(\tau)d\tau - \int_0^{t'} e^{(s'-\tau)A}h(\tau)d\tau \\ &= \int_0^t e^{(s-\tau)A}h(\tau)d\tau - \int_0^{t'} e^{(s-\tau)A}h(\tau)d\tau \\ &\quad + \int_0^{t'}\left[e^{(s-\tau)A} - e^{(s'-\tau)A}\right]h(\tau)d\tau\end{aligned}$$

so that

$$\begin{aligned}\|F(s,t) - F(s',t')\| &\le \left|\int_{t'}^t \left\|e^{(s-\tau)A}h(\tau)\right\| d\tau\right| + \int_0^{t'}\left\|\left[e^{(s-\tau)A} - e^{(s'-\tau)A}\right]h(\tau)\right\| d\tau \\ &\le \left|\int_{t'}^t \|h(\tau)\| d\tau\right| + \int_0^{t'}\left\|\left[e^{(s-\tau)A} - e^{(s'-\tau)A}\right]h(\tau)\right\| d\tau. \qquad (11.14)\end{aligned}$$

By the dominated convergence theorem,

$$\lim_{(s,t)\to(s',t')}\left|\int_{t'}^t \|h(\tau)\| d\tau\right| = 0$$

and

$$\lim_{(s,t)\to(s',t')}\int_0^{t'}\left\|\left[e^{(s-\tau)A}-e^{(s'-\tau)A}\right]h(\tau)\right\|d\tau=0$$

which along with Eq. (11.14) shows $F$ is continuous.

By the fundamental theorem of calculus,

$$\frac{\partial}{\partial t}F(s,t)=e^{(s-t)A}h(t)$$

and as we have seen this expression is continuous on $D$. Moreover, since

$$\frac{\partial}{\partial s}e^{(s-\tau)A}h(\tau)=Ae^{(s-\tau)A}h(\tau)$$

is continuous and bounded for on $s>t>\tau$, we may differentiate under the integral to find

$$\frac{\partial F(s,t)}{\partial s}=\int_0^t Ae^{(s-\tau)A}h(\tau)d\tau \text{ for } s>t.$$

A similar argument (making use of Eq. (11.10) with $k=1$) shows $\frac{\partial F(s,t)}{\partial s}$ is continuous for $(s,t)\in D$.

By the chain rule, $u_\epsilon(t):=F(t+\epsilon,t)$ is $C^1$ for $t>-\epsilon$ and

$$\begin{aligned}\dot{u}_\epsilon(t)&=\frac{\partial F(t+\epsilon,t)}{\partial s}+\frac{\partial F(t+\epsilon,t)}{\partial t}\\&=e^{\epsilon A}h(t)+\int_0^t Ae^{(s-\tau+\epsilon)A}h(\tau)d\tau=e^{\epsilon A}h(t)+u_\epsilon(t).\end{aligned}$$

■

**Theorem 11.20.** *Suppose $A=A^*$, $A\le 0$, $u_0\in H$ and $h:[0,\infty)\to H$ is continuous. Assume further that $h(t)\in D(A)$ for all $t\in[0,\infty)$ and $t\to Ah(t)$ is continuous, then*

$$u(t):=e^{tA}u_0+\int_0^t e^{(t-\tau)A}h(\tau)d\tau \tag{11.15}$$

*is the unique function $u\in C^1((0,\infty),H)\cap C([0,\infty),H)$ such that $u(t)\in D(A)$ for all $t>0$ satisfying the differential equation*

$$\dot{u}(t)=Au(t)+h(t) \text{ for } t>0 \text{ and } u(0+)=u_0.$$

**Proof. Uniqueness**: If $v(t)$ is another such solution then $w(t):=u(t)-v(t)$ satisfies,

$$\dot{w}(t)=Aw(t) \text{ with } w(0+)=0$$

which we have already seen implies $w=0$.

**Existence:** By linearity and Theorem 11.13 we may assume with out loss of generality that $u_0=0$ in which case

$$u(t)=\int_0^t e^{(t-\tau)A}h(\tau)d\tau.$$

By Lemma 11.18, we know $\tau\in[0,t]\to e^{(t-\tau)A}h(\tau)\in H$ is continuous, so the integral in Eq. (11.15) is well defined. Similarly by Lemma 11.18,

$$\tau\in[0,t]\to e^{(t-\tau)A}Ah(\tau)=Ae^{(t-\tau)A}h(\tau)\in H$$

and so by Corollary 11.17, $u(t) \in D(A)$ for all $t \geq 0$ and

$$Au(t) = \int_0^t Ae^{(t-\tau)A}h(\tau)d\tau = \int_0^t e^{(t-\tau)A}Ah(\tau)d\tau.$$

Let

$$u_\epsilon(t) = \int_0^t e^{(t+\epsilon-\tau)A}h(\tau)d\tau$$

be defined as in Lemma 11.19. Then using the dominated convergence theorem,

$$\begin{aligned}\sup_{t\leq T}\|u_\epsilon(t)-u(t)\| &\leq \sup_{t\leq T}\int_0^t \left\|\left(e^{(t+\epsilon-\tau)A}-e^{(t-\tau)A}\right)h(\tau)\right\|d\tau \\ &\leq \int_0^T \left\|\left(e^{\epsilon A}-I\right)h(\tau)\right\|d\tau \to 0 \text{ as } \epsilon\downarrow 0,\end{aligned}$$

$$\sup_{t\leq T}\|Au_\epsilon(t)-Au(t)\| \leq \int_0^T \left\|\left(e^{\epsilon A}-I\right)Ah(\tau)\right\|d\tau \to 0 \text{ as } \epsilon\downarrow 0$$

and

$$\left\|\int_0^t e^{\epsilon A}h(\tau)d\tau - \int_0^t h(\tau)d\tau\right\| \leq \int_0^t \left\|\left(e^{\epsilon A}-I\right)h(\tau)\right\|d\tau \to 0 \text{ as } \epsilon\downarrow 0.$$

Integrating Eq. (11.13) shows

$$u_\epsilon(t) = \int_0^t e^{\epsilon A}h(\tau)d\tau + \int_0^t Au_\epsilon(\tau)d\tau \tag{11.16}$$

and then passing to the limit as $\epsilon \downarrow 0$ in this equations shows

$$u(t) = \int_0^t h(\tau)d\tau + \int_0^t Au(\tau)d\tau.$$

This shows $u$ is differentiable and $\dot{u}(t) = h(t) + Au(t)$ for all $t > 0$. ∎

**Theorem 11.21.** *Let $\alpha > 0$, $h : [0,\infty) \to H$ be a locally $\alpha$ – Holder continuous function, $A = A^*$, $A \leq 0$ and $u_0 \in H$. The function*

$$u(t) := e^{tA}u_0 + \int_0^t e^{(t-\tau)A}h(\tau)d\tau$$

*is the unique function $u \in C^1((0,\infty),H) \cap C([0,\infty),H)$ such that $u(t) \in D(A)$ for all $t > 0$ satisfying the differential equation*

$$\dot{u}(t) = Au(t) + h(t) \text{ for } t > 0 \text{ and } u(0+) = u_0.$$

*(For more details see Pazy* [2, §5.7]*.)*

**Proof.** The proof of uniqueness is the same as in Theorem 11.20 and for existence we may assume $u_0 = 0$.

With out loss of generality we may assume $u_0 = 0$ so that

$$u(t) = \int_0^t e^{(t-\tau)A}h(\tau)d\tau.$$

By Lemma 11.18, we know $\tau \in [0,t] \to e^{(t-\tau)A}h(\tau) \in H$ is continuous, so the integral defining $u$ is well defined. For $\epsilon > 0$, let

$$u_\epsilon(t) := \int_0^t e^{(t+\epsilon-\tau)A}h(\tau)d\tau = \int_0^t e^{(t-\tau)A}e^{\epsilon A}h(\tau)d\tau.$$

Notice that $v(\tau) := e^{\epsilon A}h(\tau) \in C^{\infty}(A)$ for all $\tau$ and moreover since $Ae^{\epsilon A}$ is a bounded operator, it follows that $\tau \to Av(\tau)$ is continuous. So by Lemma 11.18, it follows that $\tau \in [0,t] \to Ae^{(t-\tau)A}v(\tau) \in H$ is continuous as well. Hence we know $u_\epsilon(t) \in D(A)$ and

$$Au_\epsilon(t) = \int_0^t Ae^{(t-\tau)A}e^{\epsilon A}h(\tau)d\tau.$$

Now

$$Au_\epsilon(t) = \int_0^t Ae^{(t+\epsilon-\tau)A}h(t)d\tau + \int_0^t Ae^{(t+\epsilon-\tau)A}\left[h(\tau)-h(t)\right]d\tau,$$

$$\int_0^t Ae^{(t+\epsilon-\tau)A}h(t)d\tau = -e^{(t+\epsilon-\tau)A}h(t)|_{\tau=0}^{\tau=t} = e^{(t+\epsilon)A}h(t) - e^{\epsilon A}h(t)$$

and

$$\begin{aligned}\left\|Ae^{(t+\epsilon-\tau)A}\left[h(\tau)-h(t)\right]\right\| &\le e^{-1}\frac{1}{(t+\epsilon-\tau)}\left\|h(\tau)-h(t)\right\| \\ &\le Ce^{-1}\frac{1}{(t+\epsilon-\tau)}\left|t-\tau\right|^{\alpha} \le Ce^{-1}\left|t-\tau\right|^{\alpha-1}.\end{aligned}$$

These results along with the dominated convergence theorem shows $\lim_{\epsilon\downarrow 0} Au_\epsilon(t)$ exists and is given by

$$\begin{aligned}\lim_{\epsilon\downarrow 0} Au_\epsilon(t) &= \lim_{\epsilon\downarrow 0}\left[e^{(t+\epsilon)A}h(t) - e^{\epsilon A}h(t)\right] + \lim_{\epsilon\downarrow 0}\int_0^t Ae^{(t+\epsilon-\tau)A}\left[h(\tau)-h(t)\right]d\tau \\ &= e^{tA}h(t) - h(t) + \int_0^t Ae^{(t-\tau)A}\left[h(\tau)-h(t)\right]d\tau.\end{aligned}$$

Because $A$ is a closed operator, it follows that $u(t) \in D(A)$ and

$$Au(t) = e^{tA}h(t) - h(t) + \int_0^t Ae^{(t-\tau)A}\left[h(\tau)-h(t)\right]d\tau.$$

**Claim:** $t \to Au(t)$ is continuous. To prove this it suffices to show

$$v(t) := A\int_0^t e^{(t-\tau)A}(h(\tau)-h(t))d\tau$$

is continuous and for this we have

$$\begin{aligned}v(t+\triangle) - v(t) &= \int_0^{t+\triangle} Ae^{(t+\triangle-\tau)A}(h(\tau)-h(t+\triangle))d\tau - \int_0^t Ae^{(t-\tau)A}(h(\tau)-h(t))d\tau \\ &= I + II\end{aligned}$$

where

$$\begin{aligned}I &= \int_t^{t+\triangle} Ae^{(t+\triangle-\tau)A}(h(\tau)-h(t+\triangle))d\tau \text{ and} \\ II &= \int_0^t \left[Ae^{(t+\triangle-\tau)A}(h(\tau)-h(t+\triangle)) - Ae^{(t-\tau)A}(h(\tau)-h(t))\right]d\tau \\ &= \int_0^t \left[Ae^{(t+\triangle-\tau)A}(h(\tau)-h(t)) - Ae^{(t-\tau)A}(h(\tau)-h(t))\right]d\tau \\ &+ \int_0^t \left[Ae^{(t+\triangle-\tau)A}(h(t)-h(t+\triangle))\right]d\tau \\ &= II_1 + II_2\end{aligned}$$

and

$$II_1 = \int_0^t A\left[e^{(t+\triangle-\tau)A} - e^{(t-\tau)A}\right](h(\tau)-h(t))d\tau \text{ and}$$
$$II_2 = \left[e^{(t+\Delta)A} - e^{\Delta A}\right](h(t)-h(t+\triangle)).$$

We estimate $I$ as

$$\begin{aligned}\|I\| &\leq \left|\int_t^{t+\triangle}\left\|Ae^{(t+\triangle-\tau)A}(h(\tau)-h(t+\triangle))\right\|d\tau\right| \\ &\leq C\left|\int_t^{t+\triangle}\frac{1}{t+\triangle-\tau}\left|t+\triangle-\tau\right|^\alpha d\tau\right| = C\int_0^{|\Delta|}x^{\alpha-1}dx = C\alpha^{-1}\left|\Delta\right|^\alpha \to 0 \text{ as } \Delta\to 0.\end{aligned}$$

It is easily seen that $\|II_2\| \leq 2C\left|\Delta\right|^\alpha \to 0$ as $\Delta \to 0$ and

$$\left\|A\left[e^{(t+\triangle-\tau)A} - e^{(t-\tau)A}\right](h(\tau)-h(t))\right\| \leq C\left|t-\tau\right|^{\alpha-1}$$

which is integrable, so by the dominated convergence theorem,

$$\|II_1\| \leq \int_0^t\left\|A\left[e^{(t+\triangle-\tau)A} - e^{(t-\tau)A}\right](h(\tau)-h(t))\right\|d\tau \to 0 \text{ as } \Delta\to 0.$$

This completes the proof of the claim.

Moreover,

$$\begin{aligned}Au_\epsilon(t) - Au(t) &= e^{(t+\epsilon)A}h(t) - e^{tA}h(t) + h(t) - e^{\epsilon A}h(t) \\ &\quad + \int_0^t A\left(e^{(t+\epsilon-\tau)A} - e^{(t-\tau)A}\right)\left[h(\tau)-h(t)\right]d\tau\end{aligned}$$

so that

$$\begin{aligned}\|Au_\epsilon(t) - Au(t)\| &\leq 2\left\|h(t) - e^{\epsilon A}h(t)\right\| + \int_0^t\left\|Ae^{(t-\tau)A}\left(e^{\epsilon A}-I\right)\left[h(\tau)-h(t)\right]\right\|d\tau \\ &\leq 2\left\|h(t) - e^{\epsilon A}h(t)\right\| + e^{-1}\int_0^t\frac{1}{|t-\tau|}\left\|\left(e^{\epsilon A}-I\right)\left[h(\tau)-h(t)\right]\right\|d\tau\end{aligned}$$

from which it follows $Au_\epsilon(t) \to Au(t)$ boundedly. We may now pass to the limit in Eq. (11.16) to find

$$\begin{aligned}u(t) &= \lim_{\epsilon\downarrow 0}u_\epsilon(t) = \lim_{\epsilon\downarrow 0}\left[\int_0^t e^{\epsilon A}h(\tau)d\tau + \int_0^t Au_\epsilon(\tau)d\tau\right] \\ &= \int_0^t h(\tau)d\tau + \int_0^t Au(\tau)d\tau\end{aligned}$$

from which it follows that $u \in C^1((0,\infty),H)$ and $\dot{u}(t) = h(t) + Au(t)$. ■