

## 8. SURFACES, SURFACE INTEGRALS AND INTEGRATION BY PARTS

**Definition 8.1.** A subset  $M \subset \mathbb{R}^n$  is a  $n-1$  dimensional  $C^k$ -**Hypersurface** if for all  $x_0 \in M$  there exists  $\epsilon > 0$  an open set  $0 \in D \subset \mathbb{R}^n$  and a  $C^k$ -diffeomorphism  $\psi : D \rightarrow B(x_0, \epsilon)$  such that  $\psi(D \cap \{x_n = 0\}) = B(x_0, \epsilon) \cap M$ . See Figure 16 below.

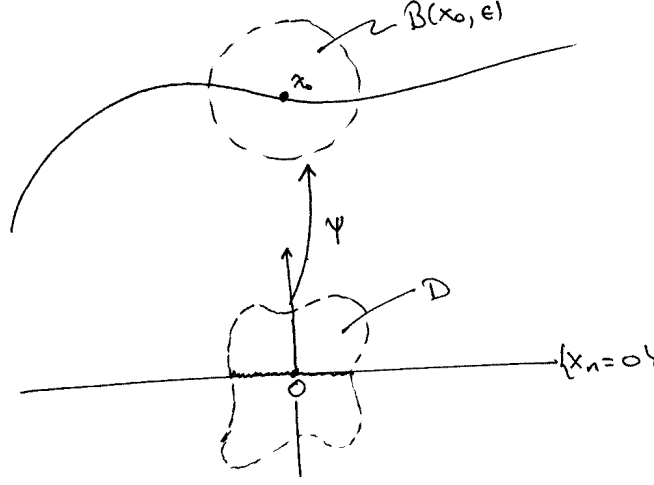


FIGURE 16. An embedded submanifold of  $\mathbb{R}^2$ .

**Example 8.2.** Suppose  $V \subset_0 \mathbb{R}^{n-1}$  and  $g : V \xrightarrow{C^k} \mathbb{R}$ . Then  $M := \Gamma(g) = \{(y, g(y)) : y \in V\}$  is a  $C^k$  hypersurface. To verify this assertion, given  $x_0 = (y_0, g(y_0)) \in \Gamma(g)$  define

$$\psi(y, z) := (y + y_0, g(y + y_0) - z).$$

Then  $\psi : \{V - y_0\} \times \mathbb{R} \xrightarrow{C^k} V \times \mathbb{R}$  diffeomorphism

$$\psi(\{V - y_0\} \times \{0\}) = \{(y + y_0, g(y + y_0)) : y \in V - y_0\} = \Gamma(g).$$

**Proposition 8.3** (Parametrized Surfaces). Let  $k \geq 1$ ,  $D \subset_0 \mathbb{R}^{n-1}$  and  $\Sigma \in C^k(D, \mathbb{R}^n)$  satisfy

- (1)  $\Sigma : D \rightarrow M := \Sigma(D)$  is a homeomorphism and
- (2)  $\Sigma'(y) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$  is injective for all  $y \in D$ . (We will call  $M$  a  $C^k$ -**parametrized surface** and  $\Sigma : D \rightarrow M$  a **parametrization of  $M$** .)

Then  $M$  is a  $C^k$ -hypersurface in  $\mathbb{R}^n$ . Moreover if  $f \in C(W \subset_0 \mathbb{R}^d, \mathbb{R}^n)$  is a continuous function such that  $f(W) \subset M$ , then  $f \in C^k(W, \mathbb{R}^n)$  iff  $\Sigma^{-1} \circ f \in C^k(U, D)$ .

**Proof.** Let  $y_0 \in D$  and  $x_0 = \Sigma(y_0)$  and  $n_0$  be a normal vector to  $M$  at  $x_0$ , i.e.  $n_0 \perp \text{Ran}(\Sigma'(y_0))$ , and let

$$\psi(t, y) := \Sigma(y_0 + y) + t n_0 \text{ for } t \in \mathbb{R} \text{ and } y \in D - y_0,$$

see Figure 17 below. Since  $D_y \psi(0, 0) = \Sigma'(y_0)$  and  $\frac{\partial \psi}{\partial t}(0, 0) = n_0 \notin \text{Ran}(\Sigma'(y_0))$ ,  $\psi'(0, 0)$  is invertible. so by the inverse function theorem there exists a neighborhood  $V$  of  $(0, 0) \in \mathbb{R}^n$  such that  $\psi|_V : V \rightarrow \mathbb{R}^n$  is a  $C^k$ -diffeomorphism.

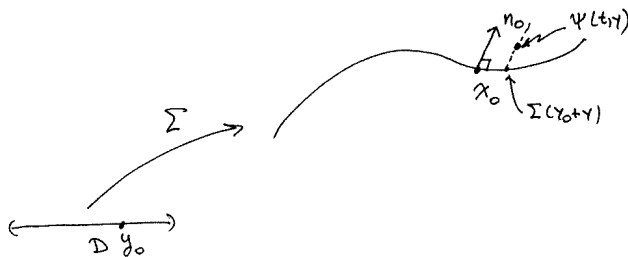


FIGURE 17. Showing a parametrized surface is an embedded hyper-surface.

Choose an  $\epsilon > 0$  such that  $B(x_0, \epsilon) \cap M \subset \Sigma(V \cap \{t = 0\})$  and  $B(x_0, \epsilon) \subset \psi(V)$ . Then set  $U := \psi^{-1}(B(x_0, \epsilon))$ . One finds  $\psi|_U : U \rightarrow B(x_0, \epsilon)$  has the desired properties.

Now suppose  $f \in C(W \subset_0 \mathbb{R}^d, \mathbb{R}^n)$  such that  $f(W) \subset M$ ,  $a \in W$  and  $x_0 = f(a) \in M$ . By shrinking  $W$  if necessary we may assume  $f(W) \subset B(x_0, \epsilon)$  where  $B(x_0, \epsilon)$  is the ball used previously. (This is where we used the continuity of  $f$ .) Then

$$\Sigma^{-1} \circ f = \pi \circ \psi^{-1} \circ f$$

where  $\pi$  is projection onto  $\{t = 0\}$ . From this identity it clearly follows  $\Sigma^{-1} \circ f$  is  $C^k$  if  $f$  is  $C^k$ . The converse is easier since if  $\Sigma^{-1} \circ f$  is  $C^k$  then  $f = \Sigma \circ (\Sigma^{-1} \circ f)$  is  $C^k$  as well. ■

### 8.1. Surface Integrals.

**Definition 8.4.** Suppose  $\Sigma : D \subset_0 \mathbb{R}^{n-1} \rightarrow M \subset \mathbb{R}^n$  is a  $C^1$ - parameterized hypersurface of  $\mathbb{R}^n$  and  $f \in C_c(M, \mathbb{R})$ . Then the surface integral of  $f$  over  $M$ ,  $\int_M f \, d\sigma$ , is defined by

$$\begin{aligned} \int_M f \, d\sigma &= \int_D f \circ \Sigma(y) \left| \det \left[ \frac{\partial \Sigma(y)}{\partial y_1}, \dots, \frac{\partial \Sigma(y)}{\partial y_{n-1}} \mid n(y) \right] \right| dy \\ &= \int_D f \circ \Sigma(y) \left| \det [\Sigma'(y)e_1 \mid \dots \mid \Sigma'(y)e_{n-1} \mid n(y)] \right| dy \end{aligned}$$

where  $n(y) \in \mathbb{R}^n$  is a unit normal vector perpendicular of  $\text{ran}(\Sigma'(y))$  for each  $y \in D$ . We will abbreviate this formula by writing

$$(8.1) \quad d\sigma = \left| \det \left[ \frac{\partial \Sigma(y)}{\partial y_1}, \dots, \frac{\partial \Sigma(y)}{\partial y_{n-1}} \mid n(y) \right] \right| dy,$$

see Figure 18 below for the motivation.

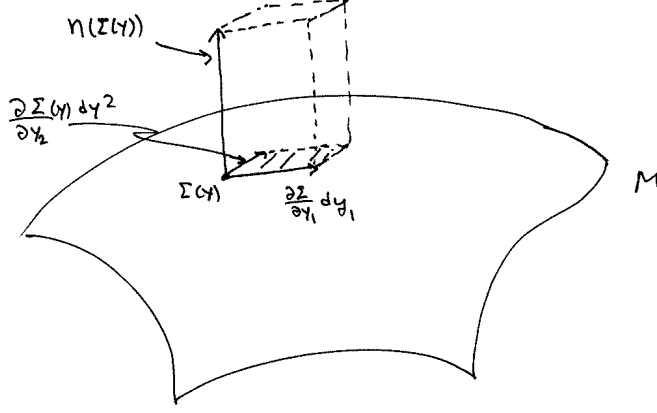


FIGURE 18. The approximate area spanned by  $\Sigma([y, y+dy])$  should be equal to the area spanned by  $\frac{\partial \Sigma(y)}{\partial y_1} dy_1$  and  $\frac{\partial \Sigma(y)}{\partial y_2} dy_2$  which is equal to the volume of the parallelepiped spanned by  $\frac{\partial \Sigma(y)}{\partial y_1} dy_1$ ,  $\frac{\partial \Sigma(y)}{\partial y_2} dy_2$  and  $n(\Sigma(y))$  and hence the formula in Eq. (8.1).

*Remark 8.5.* Let  $A = A(y) := [\Sigma'(y)e_1, \dots, \Sigma'(y)e_{n-1}, n(y)]$ . Then

$$\begin{aligned}
 A^{\text{tr}} A &= \begin{bmatrix} \partial_1 \Sigma^t \\ \partial_2 \Sigma^t \\ \vdots \\ \partial_{n-1} \Sigma^t \\ n^t \end{bmatrix} \begin{bmatrix} |\partial_1 \Sigma| & \dots & |\partial_{n-1} \Sigma| & |n| \end{bmatrix} \\
 &= \begin{bmatrix} \partial_1 \Sigma \cdot \partial_1 \Sigma & \partial_1 \Sigma \cdot \partial_2 \Sigma & \dots & \partial_1 \Sigma \cdot \partial_{n-1} \Sigma & 0 \\ \partial_2 \Sigma \cdot \partial_1 \Sigma & \partial_2 \Sigma \cdot \partial_2 \Sigma & \dots & \partial_2 \Sigma \cdot \partial_{n-1} \Sigma & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_{n-1} \Sigma \cdot \partial_1 \Sigma & \partial_{n-1} \Sigma \cdot \partial_2 \Sigma & \dots & \partial_{n-1} \Sigma \cdot \partial_{n-1} \Sigma & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \left| \det \left[ \frac{\partial \Sigma(y)}{\partial y_1}, \dots, \frac{\partial \Sigma(y)}{\partial y_{n-1}} | n(y) \right] \right| &= |\det(A)| dy = \sqrt{\det(A^{\text{tr}} A)} dy \\
 &= \sqrt{\det \left[ (\partial_i \Sigma \cdot \partial_j \Sigma)_{i,j=1}^{n-1} \right]} = \sqrt{\det \left[ (\Sigma')^{\text{tr}} \Sigma' \right]}.
 \end{aligned}$$

This implies  $d\sigma = \rho^\Sigma(y) dy$  or more precisely that

$$\int_M f d\sigma = \int_D f \circ \Sigma(y) \rho^\Sigma(y) dy$$

where

$$\rho^\Sigma(y) := \sqrt{\det \left[ (\partial_i \Sigma \cdot \partial_j \Sigma)_{i,j=1}^{n-1} \right]} = \sqrt{\det \left[ (\Sigma')^{\text{tr}} \Sigma' \right]}.$$

The next lemma shows that  $\int_M f \, d\sigma$  is well defined, i.e. independent of how  $M$  is parametrized.

**Example 8.6.** Suppose  $V \subset_0 \mathbb{R}^{n-1}$  and  $g : V \xrightarrow{C^k} \mathbb{R}$  and  $M := \Gamma(g) = \{(y, g(y)) : y \in V\}$  as in Example 8.2. We now compute  $d\sigma$  in the parametrization  $\Sigma : V \rightarrow M$  defined by  $\Sigma(y) = (y, g(y))$ . To simplify notation, let

$$\nabla g(y) := (\partial_1 g(y), \dots, \partial_{n-1} g(y)).$$

As is standard from multivariable calculus (and is easily verified),

$$n(y) := \frac{(\nabla g(y), -1)}{\sqrt{1 + |\nabla g(y)|^2}}$$

is a normal vector to  $M$  at  $\Sigma(y)$ , i.e.  $n(y) \cdot \partial_k \Sigma(y) = 0$  for all  $k = 1, 2, \dots, n-1$ . Therefore,

$$\begin{aligned} d\sigma &= |\det [\partial_1 \Sigma \dots \partial_{n-1} \Sigma n]| \, dy \\ &= \frac{1}{\sqrt{1 + |\nabla g(y)|^2}} \left| \det \begin{bmatrix} I_{n-1} & (\nabla g)^{tr} \\ \nabla g & -1 \end{bmatrix} \right| \, dy \\ &= \frac{1}{\sqrt{1 + |\nabla g(y)|^2}} \left| \det \begin{bmatrix} I_{n-1} & 0 \\ \nabla g & -1 - |\nabla g|^2 \end{bmatrix} \right| \, dy \\ &= \frac{1}{\sqrt{1 + |\nabla g(y)|^2}} \left( 1 + |\nabla g(y)|^2 \right) \, dy = \sqrt{1 + |\nabla g(y)|^2} \, dy. \end{aligned}$$

Hence if  $g : M \rightarrow \mathbb{R}$ , we have

$$\int_M g \, d\sigma = \int_V g(\Sigma(y)) \sqrt{1 + |\nabla g(y)|^2} \, dy.$$

**Example 8.7.** Keeping the same notation as in Example 8.6, but now taking  $V := B(0, r) \subset \mathbb{R}^{n-1}$  and  $g(y) := \sqrt{r^2 - |y|^2}$ . In this case  $M = S_+^{n-1}$ , the upper-hemisphere of  $S^{n-1}$ ,  $\nabla g(y) = -y/g(y)$ ,

$$d\sigma = \sqrt{1 + |y|^2 / g^2(y)} \, dy = \frac{r}{g(y)} \, dy$$

and so

$$\int_{S_+^{n-1}} g \, d\sigma = \int_{|y| < r} g(y, \sqrt{r^2 - |y|^2}) \frac{r}{\sqrt{r^2 - |y|^2}} \, dy.$$

A similar computation shows, with  $S_-^{n-1}$  being the lower hemisphere, that

$$\int_{S_-^{n-1}} g \, d\sigma = \int_{|y| < r} g(y, -\sqrt{r^2 - |y|^2}) \frac{r}{\sqrt{r^2 - |y|^2}} \, dy.$$

**Lemma 8.8.** If  $\tilde{\Sigma} : \tilde{D} \rightarrow M$  is another  $C^k$ -parametrization of  $M$ , then

$$\int_D f \circ \Sigma(y) \rho^\Sigma(y) \, dy = \int_{\tilde{D}} f \circ \tilde{\Sigma}(y) \rho^{\tilde{\Sigma}}(y) \, dy.$$

**Proof.** By Proposition 8.3,  $\phi := \Sigma^{-1} \circ \tilde{\Sigma} : \tilde{D} \rightarrow D$  is a  $C^k$  - diffeomorphism. By the change of variables theorem on  $\mathbb{R}^{n-1}$  with  $y = \phi(\tilde{y})$  (using  $\tilde{\Sigma} = \Sigma \circ \phi$ , see Figure 19) we find

$$\begin{aligned}
\int_{\tilde{D}} f \circ \tilde{\Sigma}(\tilde{y}) \rho^{\tilde{\Sigma}}(\tilde{y}) d\tilde{y} &= \int_{\tilde{D}} f \circ \tilde{\Sigma} \sqrt{\det \tilde{\Sigma}'^{\text{tr}} \tilde{\Sigma}'} d\tilde{y} \\
&= \int_{\tilde{D}} f \circ \Sigma \circ \phi \sqrt{\det (\Sigma \circ \phi)'^{\text{tr}} (\Sigma \circ \phi)'} d\tilde{y} \\
&= \int_{\tilde{D}} f \circ \Sigma \circ \phi \sqrt{\det [(\Sigma'(\phi)\phi')^{\text{tr}} \Sigma'(\phi)\phi']} d\tilde{y} \\
&= \int_{\tilde{D}} f \circ \Sigma \circ \phi \sqrt{\det [\phi'^{\text{tr}} [\Sigma'(\phi)^{\text{tr}} \Sigma'(\phi)] \phi']} d\tilde{y} \\
&= \int_{\tilde{D}} (f \circ \Sigma \circ \phi) \cdot \left( \sqrt{\det \Sigma'^{\text{tr}} \Sigma'} \right) \circ \phi \cdot |\det \phi'| d\tilde{y} \\
&= \int_D f \circ \Sigma \sqrt{\det \Sigma'^{\text{tr}} \Sigma'} dy.
\end{aligned}$$

■

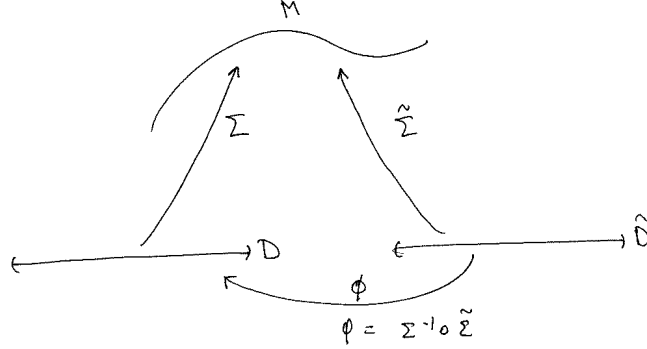


FIGURE 19. Verifying surface integrals are independent of parametrization.

**Definition 8.9.** Let  $M$  be a  $C^1$ -embedded hypersurface and  $f \in C_c(M)$ . Then we define the **surface integral of  $f$  over  $M$**  as

$$\int_M f d\sigma = \sum_{i=1}^n \int_{M_i} \phi_i f d\sigma$$

where  $\phi_i \in C_c^1(M, [0, 1])$  are chosen so that  $\sum_i \phi_i \leq 1$  with equality on  $\text{supp}(f)$  and the  $\text{supp}(\phi_i f) \subset M_i \subset M$  where  $M_i$  is a subregion of  $M$  which may be viewed as a parametrized surface.

*Remark 8.10.* The integral  $\int_M f d\sigma$  is well defined for if  $\psi_j \in C_c^1(M, [0, 1])$  is another sequence satisfying the properties of  $\{\phi_i\}$  with  $\text{supp}(\psi_j) \subset M'_j \subset M$  then (using

Lemma 8.8 implicitly)

$$\sum_i \int_{M_i} \phi_i f \, d\sigma = \sum_i \int_{M_i} \sum_j \psi_j \phi_i f \, d\sigma = \sum_{ij} \int_{M_i \cap M'_j} \psi_j \phi_i f \, d\sigma$$

with a similar computation showing

$$\sum_j \int_{M'_j} \psi_j f \, d\sigma = \sum_{ji} \int_{M_i \cap M'_j} \psi_j \phi_i f \, d\sigma = \sum_{ij} \int_{M_i \cap M'_j} \psi_j \phi_i f \, d\sigma.$$

*Remark 8.11.* By the Reisz theorem, there exists a unique Radon measure  $\sigma$  on  $M$  such that

$$\int_M f \, d\sigma = \int_M f \, d\sigma.$$

This  $\sigma$  is called surface measure on  $M$ .

**Lemma 8.12** (Surface Measure). *Let  $M$  be a  $C^2$  - embedded hypersurface in  $\mathbb{R}^n$  and  $B \subset M$  be a measurable set such that  $\bar{B}$  is compact and contained inside  $\Sigma(D)$  where  $\Sigma : D \rightarrow M \subset \mathbb{R}^n$  is a parametrization. Then*

$$\sigma(B) = \lim_{\epsilon \downarrow 0} m(B^\epsilon) = \frac{d}{d\epsilon} \Big|_{0+} m(B^\epsilon)$$

where

$$B^\epsilon := \{x + t n(x) : x \in B, 0 \leq t \leq \epsilon\}$$

and  $n(x)$  is a unit normal to  $M$  at  $x \in M$ , see Figure 20.

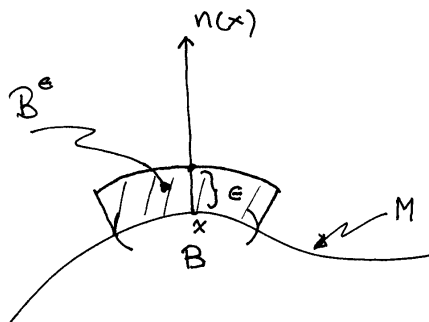


FIGURE 20. Computing the surface area of  $B$  as the volume of an  $\epsilon$  - fattened neighborhood of  $B$ .

**Proof.** Let  $A := \Sigma^{-1}(B)$  and  $\nu(y) := n(\Sigma(y))$  so that  $\nu \in C^{k-1}(D, \mathbb{R}^n)$  if  $\Sigma \in C^k(D, \mathbb{R}^n)$ . Define

$$\psi(y, t) = \Sigma(y) + t\nu(\Sigma(y)) = \Sigma(y) + t\nu(y)$$

so that  $B^\epsilon = \psi(A \times [0, \epsilon])$ . Hence by the change of variables formula

$$(8.2) \quad m(B^\epsilon) = \int_{A \times [0, \epsilon]} |\det \psi'(y, t)| \, dy \, dt = \int_0^\epsilon dt \int_A dy |\det \psi'(y, t)|$$

so that by the fundamental theorem of calculus,

$$\frac{d}{d\epsilon}|_{0+} m(B^\epsilon) = \frac{d}{d\epsilon}|_{0+} \int_0^\epsilon dt \int_A dy |\det \psi'(y, t)| = \int_A |\det \psi'(y, 0)| dy.$$

But

$$|\det \psi'(y, 0)| = |\det[\Sigma'(y)|n(\Sigma(y))]| = \rho_\Sigma(y)$$

which shows

$$\frac{d}{d\epsilon}|_{0+} m(B^\epsilon) = \int_A \rho_\Sigma(y) dy = \int_D 1_B(\Sigma(y)) \rho_\Sigma(y) dy =: \sigma(B).$$

■

**Example 8.13.** Let  $\Sigma = rS^{n-1}$  be the sphere of radius  $r > 0$  contained in  $\mathbb{R}^n$  and for  $B \subset \Sigma$  and  $\alpha > 0$  let

$$B_\alpha := \{t\omega : \omega \in B \text{ and } 0 \leq t \leq \alpha\} = \alpha B_1.$$

Assuming  $N(\omega) = \omega/r$  is the outward pointing normal to  $S^{n-1}$ , we have

$$B^\epsilon = B_{(1+\epsilon/r)} \setminus B_1 = [(1 + \epsilon/r)B_1] \setminus B_1$$

and hence

$$m(B^\epsilon) = m([(1 + \epsilon/r)B_1] \setminus B_1) = m([(1 + \epsilon/r)B_1]) - m(B_1) = [(1 + \epsilon/r)^n - 1] m(B_1).$$

Therefore,

$$\sigma(B) = \frac{d}{d\epsilon}|_0 [(1 + \epsilon/r)^n - 1] m(B_1) = \frac{n}{r} m(B_1) = nr^{n-1} m(r^{-1}B_1) = r^{n-1} \sigma(r^{-1}B),$$

i.e.

$$\sigma(B) = \frac{n}{r} m(B_1) = nr^{n-1} m(r^{-1}B_1) = r^{n-1} \sigma(r^{-1}B).$$

**Theorem 8.14.** If  $f : \mathbb{R}^n \rightarrow [0, \infty]$  is a  $(\mathcal{B}_{\mathbb{R}^n}, \mathcal{B})$ -measurable function then

$$(8.3) \quad \int_{\mathbb{R}^n} f(x) dm(x) = \int_{[0, \infty) \times S^{n-1}} f(r \omega) r^{n-1} dr d\sigma(\omega).$$

In particular if  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is measurable then

$$(8.4) \quad \int_{\mathbb{R}^n} f(|x|) dx = \int_0^\infty f(r) dV(r)$$

where  $V(r) = m(B(0, r)) = r^n m(B(0, 1)) = n^{-1} \sigma(S^{n-1}) r^n$ .

**Proof.** Let  $B \subset S^{n-1}$ ,  $0 < a < b$  and let  $f(x) = 1_{B_b \setminus B_a}(x)$ . Then

$$\begin{aligned} \int_{[0, \infty) \times S^{n-1}} f(r \omega) r^{n-1} dr d\sigma(\omega) &= \int_{[0, \infty) \times S^{n-1}} 1_B(\omega) 1_{[a, b]}(r) r^{n-1} dr d\sigma(\omega) \\ &= \sigma(B) \int_a^b r^{n-1} dr = n^{-1} \sigma(B) (b^n - a^n) \\ &= m(B_1) (b^n - a^n) = m(B_b \setminus B_a) = \int_{\mathbb{R}^n} f(x) dm(x). \end{aligned}$$

Since sets of the form  $B_b \setminus B_a$  generate  $\mathcal{B}_{\mathbb{R}^n}$  and are closed under intersections, this suffices to prove the theorem.

Alternatively one may show that any  $f \in C_c(\mathbb{R}^n)$  may be uniformly approximated by linear combinations of characteristic functions of the form  $1_{B_b \setminus B_a}$ . Indeed,

let  $S^{n-1} = \bigcup_{i=1}^K B_i$  be a partition of  $S^{n-1}$  with  $B_i$  small and choose  $w_i \in B_i$ . Let  $0 < r_1 < r_2 < r_3 < \dots < r_n = R < \infty$ . Assume  $\text{supp}(f) \subset B(0, R)$ . Then  $\{(B_i)_{r_{j+1}} \setminus (B_i)_{r_j}\}_{i,j}$  partitions  $\mathbb{R}^n$  into small regions. Therefore

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) dx &\cong \sum f(r_j \omega_i) m((B_i)_{r_{j+1}} \setminus (B_i)_{r_j}) \\ &= \sum f(r_j \omega_i) (r_{j+1}^n - r_j^n) m((B_i)_1) \\ &= \sum f(r_j \omega_i) \int_{r_j}^{r_{j+1}} r^{n-1} dr n m((B_i)_1) \\ &= \sum \int_{r_j}^{r_{j+1}} f(r_j \omega_i) r^{n-1} dr \sigma(B_i) \\ &\cong \sum_{ij} \int_{r_j}^{r_{j+1}} \left( \int_{S^{n-1}} f(r_j \omega) d\sigma(\omega) \right) r^{n-1} dr \\ &\cong \int_0^\infty \left( \int_{S^{n-1}} f(r\omega) d\sigma(\omega) \right) r^{n-1} dr. \end{aligned}$$

Eq. (8.4) is a simple special case of Eq. (8.3). It can also be proved directly as follows. Suppose first  $f \in C_c^1([0, \infty))$  then

$$\begin{aligned} \int_{\mathbb{R}^n} f(|x|) dx &= - \int_{\mathbb{R}^n} dx \int_{|x|}^\infty dr f'(r) = - \int_{\mathbb{R}^n} dx \int_{\mathbb{R}} 1_{|x| \leq r} f'(r) dr \\ &= - \int_0^\infty V(r) f'(r) dr = \int_0^\infty V'(r) f(r) dr. \end{aligned}$$

The result now extends to general  $f$  by a density argument. ■

We are now going to work out some integrals using Eq. (8.3). The first we leave as an exercise.

**Exercise 8.1.** Use the results of Example 8.7 and Theorem 8.14 to show,

$$\sigma(S^{n-1}) = 2\sigma(S^{n-2}) \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \rho^{n-2} d\rho.$$

The result in Exercise 8.1 may be used to compute the volume of spheres in any dimension. This method will be left to the reader. We will do this in another way. The first step will be to directly compute the following Gaussian integrals. The result will also be needed for later purposes.

**Lemma 8.15.** Let  $a > 0$  and

$$(8.5) \quad I_n(a) := \int_{\mathbb{R}^n} e^{-a|x|^2} dm(x).$$

Then  $I_n(a) = (\pi/a)^{n/2}$ .



**Proof.** By Tonelli's theorem and induction,

$$\begin{aligned} I_n(a) &= \int_{\mathbb{R}^{n-1} \times \mathbb{R}} e^{-a|y|^2} e^{-at^2} m_{n-1}(dy) dt \\ (8.6) \quad &= I_{n-1}(a)I_1(a) = I_1^n(a). \end{aligned}$$

So it suffices to compute:

$$I_2(a) = \int_{\mathbb{R}^2} e^{-a|x|^2} dm(x) = \int_{\mathbb{R}^2 \setminus \{0\}} e^{-a(x_1^2+x_2^2)} dx_1 dx_2.$$

We now make the change of variables,

$$x_1 = r \cos \theta \text{ and } x_2 = r \sin \theta \text{ for } 0 < r < \infty \text{ and } 0 < \theta < 2\pi.$$

In vector form this transform is

$$x = T(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

and the differential and the Jacobian determinant are given by

$$T'(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \text{ and } \det T'(r, \theta) = r \cos^2 \theta + r \sin^2 \theta = r.$$

Notice that  $T : (0, \infty) \times (0, 2\pi) \rightarrow \mathbb{R}^2 \setminus \ell$  where  $\ell$  is the ray,  $\ell := \{(x, 0) : x \geq 0\}$  which is a  $m^2$ -null set. Hence by Tonelli's theorem and the change of variable theorem, for any Borel measurable function  $f : \mathbb{R}^2 \rightarrow [0, \infty]$  we have

$$\int_{\mathbb{R}^2} f(x) dx = \int_0^{2\pi} \int_0^\infty f(r \cos \theta, r \sin \theta) r dr d\theta.$$

In particular,

$$\begin{aligned} I_2(a) &= \int_0^\infty dr r \int_0^{2\pi} d\theta e^{-ar^2} = 2\pi \int_0^\infty r e^{-ar^2} dr \\ &= 2\pi \lim_{M \rightarrow \infty} \int_0^M r e^{-ar^2} dr = 2\pi \lim_{M \rightarrow \infty} \frac{e^{-ar^2}}{-2a} \Big|_0^M = \frac{2\pi}{2a} = \pi/a. \end{aligned}$$

This shows that  $I_2(a) = \pi/a$  and the result now follows from Eq. (8.6). ■

**Corollary 8.16.** Let  $S^{n-1} \subset \mathbb{R}^n$  be the unit sphere in  $\mathbb{R}^n$  and

$$\Gamma(x) := \int_0^\infty u^{x-1} e^{-u} du \text{ for } x > 0$$

be the **gamma function**. Then

(1) The surface area  $\sigma(S^{n-1})$  of the unit sphere  $S^{n-1} \subset \mathbb{R}^n$  is

$$(8.7) \quad \sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

(2) The  $\Gamma$ -function satisfies

- (a)  $\Gamma(1/2) = \sqrt{\pi}$ ,  $\Gamma(1) = 1$  and  $\Gamma(x+1) = x\Gamma(x)$  for  $x > 0$ .
- (b) For  $n \in \mathbb{N}$ ,

$$(8.8) \quad \Gamma(n+1) = n! \text{ and } \Gamma(n+1/2) = \frac{(2n-1)!!}{2^n} \cdot \sqrt{\pi}.$$

(3) For  $n \in \mathbb{N}$ ,

$$(8.9) \quad \sigma(S^{2n+1}) = \frac{2\pi^{n+1}}{n!} \text{ and } \sigma(S^{2n}) = \frac{2(2\pi)^n}{(2n-1)!}.$$

**Proof.** Let  $I_n$  be as in Lemma 8.15. Using Theorem 8.14 we may alternatively compute  $\pi^{n/2} = I_n(1)$  as

$$\pi^{n/2} = I_n(1) = \int_0^\infty dr r^{n-1} e^{-r^2} \int_{S^{n-1}} d\sigma = \sigma(S^{n-1}) \int_0^\infty r^{n-1} e^{-r^2} dr.$$

We simplify this last integral by making the change of variables  $u = r^2$  so that  $r = u^{1/2}$  and  $dr = \frac{1}{2}u^{-1/2}du$ . The result is

$$(8.10) \quad \begin{aligned} \int_0^\infty r^{n-1} e^{-r^2} dr &= \int_0^\infty u^{\frac{n-1}{2}} e^{-u} \frac{1}{2} u^{-1/2} du \\ &= \frac{1}{2} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} du = \frac{1}{2} \Gamma(n/2). \end{aligned}$$

Collecting these observations implies that

$$\pi^{n/2} = I_n(1) = \frac{1}{2} \sigma(S^{n-1}) \Gamma(n/2)$$

which proves Eq. (8.7).

The computation of  $\Gamma(1)$  is easy and is left to the reader. By Eq. (8.10),

$$\begin{aligned} \Gamma(1/2) &= 2 \int_0^\infty e^{-r^2} dr = \int_{-\infty}^\infty e^{-r^2} dr \\ &= I_1(1) = \sqrt{\pi}. \end{aligned}$$

The relation,  $\Gamma(x+1) = x\Gamma(x)$  is the consequence of integration by parts:

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-u} u^{x+1} \frac{du}{u} = \int_0^\infty u^x \left( -\frac{d}{du} e^{-u} \right) du \\ &= x \int_0^\infty u^{x-1} e^{-u} du = x \Gamma(x). \end{aligned}$$

Eq. (8.8) follows by induction from the relations just proved. Eq. (8.9) is a consequence of items 1. and 2. as follows:

$$\sigma(S^{2n+1}) = \frac{2\pi^{(2n+2)/2}}{\Gamma((2n+2)/2)} = \frac{2\pi^{n+1}}{\Gamma(n+1)} = \frac{2\pi^{n+1}}{n!}$$

and

$$\sigma(S^{2n}) = \frac{2\pi^{(2n+1)/2}}{\Gamma((2n+1)/2)} = \frac{2\pi^{n+1/2}}{\Gamma(n+1/2)} = \frac{2\pi^{n+1/2}}{\frac{(2n-1)!}{2^n} \cdot \sqrt{\pi}} = \frac{2(2\pi)^n}{(2n-1)!}.$$

■

**8.2. More spherical coordinates.** In this section we will define spherical coordinates in all dimensions. Along the way we will develop an explicit method for computing surface integrals on spheres. As usual when  $n = 2$  define spherical coordinates  $(r, \theta) \in (0, \infty) \times [0, 2\pi)$  so that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \psi_2(\theta, r).$$

For  $n = 3$  we let  $x_3 = r \cos \phi_1$  and then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \psi_2(\theta, r \sin \phi_1),$$

as can be seen from Figure 21, so that

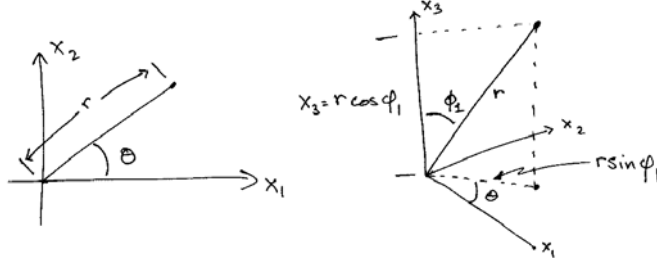


FIGURE 21. Setting up polar coordinates in two and three dimensions.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \psi_2(\theta, r \sin \phi_1) \\ r \cos \phi_1 \end{pmatrix} = \begin{pmatrix} r \sin \phi_1 \cos \theta \\ r \sin \phi_1 \sin \theta \\ r \cos \phi_1 \end{pmatrix} =: \psi_3(\theta, \phi_1, r).$$

We continue to work inductively this way to define

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} \psi_n(\theta, \phi_1, \dots, \phi_{n-2}, r \sin \phi_{n-1},) \\ r \cos \phi_{n-1} \end{pmatrix} = \psi_{n+1}(\theta, \phi_1, \dots, \phi_{n-2}, \phi_{n-1}, r).$$

So for example,

$$\begin{aligned} x_1 &= r \sin \phi_2 \sin \phi_1 \cos \theta \\ x_2 &= r \sin \phi_2 \sin \phi_1 \sin \theta \\ x_3 &= r \sin \phi_2 \cos \phi_1 \\ x_4 &= r \cos \phi_2 \end{aligned}$$

and more generally,

$$\begin{aligned} x_1 &= r \sin \phi_{n-2} \dots \sin \phi_2 \sin \phi_1 \cos \theta \\ x_2 &= r \sin \phi_{n-2} \dots \sin \phi_2 \sin \phi_1 \sin \theta \\ x_3 &= r \sin \phi_{n-2} \dots \sin \phi_2 \cos \phi_1 \\ &\vdots \\ x_{n-2} &= r \sin \phi_{n-2} \sin \phi_{n-3} \cos \phi_{n-4} \\ x_{n-1} &= r \sin \phi_{n-2} \cos \phi_{n-3} \\ x_n &= r \cos \phi_{n-2}. \end{aligned}$$

By the change of variables formula,

(8.11)

$$\int_{\mathbb{R}^n} f(x) dm(x) = \int_0^\infty dr \int_{0 \leq \phi_i \leq \pi, 0 \leq \theta \leq 2\pi} d\phi_1 \dots d\phi_{n-2} d\theta \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r) f(\psi_n(\theta, \phi_1, \dots, \phi_{n-2}, r))$$

where

$$\Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r) := |\det \psi'_n(\theta, \phi_1, \dots, \phi_{n-2}, r)|.$$

**Proposition 8.17.** *The Jacobian,  $\Delta_n$  is given by*

$$(8.12) \quad \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r) = r^{n-1} \sin^{n-2} \phi_{n-2} \dots \sin^2 \phi_2 \sin \phi_1.$$

If  $f$  is a function on  $rS^{n-1}$  – the sphere of radius  $r$  centered at 0 inside of  $\mathbb{R}^n$ , then

$$(8.13) \quad \begin{aligned} \int_{rS^{n-1}} f(x) d\sigma(x) &= r^{n-1} \int_{S^{n-1}} f(r\omega) d\sigma(\omega) \\ &= r^{n-1} \int_{0 \leq \phi_i \leq \pi, 0 \leq \theta \leq 2\pi} f(\psi_n(\theta, \phi_1, \dots, \phi_{n-2}, r)) \sin^{n-2} \phi_{n-2} \dots \sin^2 \phi_2 \sin \phi_1 d\phi_1 \dots d\phi_{n-2} d\theta \end{aligned}$$

**Proof.** We are going to compute  $\Delta_n$  inductively. Letting  $\rho := r \sin \phi_{n-1}$  and writing  $\frac{\partial \psi_n}{\partial \xi}$  for  $\frac{\partial \psi_n}{\partial \xi}(\theta, \phi_1, \dots, \phi_{n-2}, \rho)$  we have

$$\begin{aligned} \Delta_{n+1}(\theta, \phi_1, \dots, \phi_{n-2}, \phi_{n-1}, r) &= \left| \begin{bmatrix} \frac{\partial \psi_n}{\partial \theta} & \frac{\partial \psi_n}{\partial \phi_1} & \dots & \frac{\partial \psi_n}{\partial \phi_{n-2}} & \frac{\partial \psi_n}{\partial \rho} r \cos \phi_{n-1} & \frac{\partial \psi_n}{\partial \rho} \sin \phi_{n-1} \\ 0 & 0 & \dots & 0 & -r \sin \phi_{n-1} & \cos \phi_{n-1} \end{bmatrix} \right| \\ &= r (\cos^2 \phi_{n-1} + \sin^2 \phi_{n-1}) \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, \rho) \\ &= r \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r \sin \phi_{n-1}), \end{aligned}$$

i.e.

$$(8.14) \quad \Delta_{n+1}(\theta, \phi_1, \dots, \phi_{n-2}, \phi_{n-1}, r) = r \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r \sin \phi_{n-1}).$$

To arrive at this result we have expanded the determinant along the bottom row.

Starting with the well known and easy to compute fact that  $\Delta_2(\theta, r) = r$ , Eq. (8.14) implies

$$\begin{aligned} \Delta_3(\theta, \phi_1, r) &= r \Delta_2(\theta, r \sin \phi_1) = r^2 \sin \phi_1 \\ \Delta_4(\theta, \phi_1, \phi_2, r) &= r \Delta_3(\theta, \phi_1, r \sin \phi_2) = r^3 \sin^2 \phi_2 \sin \phi_1 \\ &\vdots \\ \Delta_n(\theta, \phi_1, \dots, \phi_{n-2}, r) &= r^{n-1} \sin^{n-2} \phi_{n-2} \dots \sin^2 \phi_2 \sin \phi_1 \end{aligned}$$

which proves Eq. (8.12). Eq. (8.13) now follows from Eqs. (8.3), (8.11) and (8.12).  $\blacksquare$

As a simple application, Eq. (8.13) implies

$$(8.15) \quad \begin{aligned} \sigma(S^{n-1}) &= \int_{0 \leq \phi_i \leq \pi, 0 \leq \theta \leq 2\pi} \sin^{n-2} \phi_{n-2} \dots \sin^2 \phi_2 \sin \phi_1 d\phi_1 \dots d\phi_{n-2} d\theta \\ &= 2\pi \prod_{k=1}^{n-2} \gamma_k = \sigma(S^{n-2}) \gamma_{n-2} \end{aligned}$$

where  $\gamma_k := \int_0^\pi \sin^k \phi d\phi$ . If  $k \geq 1$ , we have by integration by parts that,

$$\begin{aligned} \gamma_k &= \int_0^\pi \sin^k \phi d\phi = - \int_0^\pi \sin^{k-1} \phi d \cos \phi = 2\delta_{k,1} + (k-1) \int_0^\pi \sin^{k-2} \phi \cos^2 \phi d\phi \\ &= 2\delta_{k,1} + (k-1) \int_0^\pi \sin^{k-2} \phi (1 - \sin^2 \phi) d\phi = 2\delta_{k,1} + (k-1) [\gamma_{k-2} - \gamma_k] \end{aligned}$$

and hence  $\gamma_k$  satisfies  $\gamma_0 = \pi$ ,  $\gamma_1 = 2$  and the recursion relation

$$\gamma_k = \frac{k-1}{k} \gamma_{k-2} \text{ for } k \geq 2.$$

Hence we may conclude

$$\gamma_0 = \pi, \gamma_1 = 2, \gamma_2 = \frac{1}{2}\pi, \gamma_3 = \frac{2}{3}2, \gamma_4 = \frac{3}{4} \frac{1}{2}\pi, \gamma_5 = \frac{4}{5} \frac{2}{3}2, \gamma_6 = \frac{5}{6} \frac{3}{4} \frac{1}{2}\pi$$

and more generally by induction that

$$\gamma_{2k} = \pi \frac{(2k-1)!!}{(2k)!!} \text{ and } \gamma_{2k+1} = 2 \frac{(2k)!!}{(2k+1)!!}.$$

Indeed,

$$\gamma_{2(k+1)+1} = \frac{2k+2}{2k+3} \gamma_{2k+1} = \frac{2k+2}{2k+3} 2 \frac{(2k)!!}{(2k+1)!!} = 2 \frac{[2(k+1)]!!}{(2(k+1)+1)!!}$$

and

$$\gamma_{2(k+1)} = \frac{2k+1}{2k+1} \gamma_{2k} = \frac{2k+1}{2k+2} \pi \frac{(2k-1)!!}{(2k)!!} = \pi \frac{(2k+1)!!}{(2k+2)!!}.$$

The recursion relation in Eq. (8.15) may be written as

$$(8.16) \quad \sigma(S^n) = \sigma(S^{n-1}) \gamma_{n-1}$$

which combined with  $\sigma(S^1) = 2\pi$  implies

$$\begin{aligned} \sigma(S^1) &= 2\pi, \\ \sigma(S^2) &= 2\pi \cdot \gamma_1 = 2\pi \cdot 2, \\ \sigma(S^3) &= 2\pi \cdot 2 \cdot \gamma_2 = 2\pi \cdot 2 \cdot \frac{1}{2}\pi = \frac{2^2\pi^2}{2!!}, \\ \sigma(S^4) &= \frac{2^2\pi^2}{2!!} \cdot \gamma_3 = \frac{2^2\pi^2}{2!!} \cdot 2 \frac{2}{3} = \frac{2^3\pi^2}{3!!}, \\ \sigma(S^5) &= 2\pi \cdot 2 \cdot \frac{1}{2}\pi \cdot \frac{2}{3} \cdot \frac{3}{4} \frac{1}{2}\pi = \frac{2^3\pi^3}{4!!}, \\ \sigma(S^6) &= 2\pi \cdot 2 \cdot \frac{1}{2}\pi \cdot \frac{2}{3} \cdot \frac{3}{4} \frac{1}{2}\pi \cdot \frac{4}{5} \frac{2}{3} = \frac{2^4\pi^3}{5!!} \end{aligned}$$

and more generally that

$$(8.17) \quad \sigma(S^{2n}) = \frac{2(2\pi)^n}{(2n-1)!!} \text{ and } \sigma(S^{2n+1}) = \frac{(2\pi)^{n+1}}{(2n)!!}$$

which is verified inductively using Eq. (8.16). Indeed,

$$\sigma(S^{2n+1}) = \sigma(S^{2n}) \gamma_{2n} = \frac{2(2\pi)^n}{(2n-1)!!} \pi \frac{(2n-1)!!}{(2n)!!} = \frac{(2\pi)^{n+1}}{(2n)!!}$$

and

$$\sigma(S^{(n+1)}) = \sigma(S^{2n+2}) = \sigma(S^{2n+1}) \gamma_{2n+1} = \frac{(2\pi)^{n+1}}{(2n)!!} 2 \frac{(2n)!!}{(2n+1)!!} = \frac{2(2\pi)^{n+1}}{(2n+1)!!}.$$

Using

$$(2n)!! = 2n(2n-1) \dots (2 \cdot 1) = 2^n n!$$

we may write  $\sigma(S^{2n+1}) = \frac{2\pi^{n+1}}{n!}$  which shows that Eqs. (8.9) and (8.17) are in agreement. We may also write the formula in Eq. (8.17) as

$$\sigma(S^n) = \begin{cases} \frac{2(2\pi)^{n/2}}{(n-1)!!} & \text{for } n \text{ even} \\ \frac{(2\pi)^{\frac{n+1}{2}}}{(n-1)!!} & \text{for } n \text{ odd.} \end{cases}$$

8.3.  $n$  – dimensional manifolds with boundaries.

**Definition 8.18.** A set  $\Omega \subset \mathbb{R}^n$  is said to be a  $C^k$  – **manifold with boundary** if for each  $x_0 \in \partial\Omega := \Omega \setminus \Omega^\circ$  (here  $\Omega^\circ$  is the interior of  $\Omega$ ) there exists  $\epsilon > 0$  an open set  $0 \in D \subset \mathbb{R}^n$  and a  $C^k$ -diffeomorphism  $\psi : D \rightarrow B(x_0, \epsilon)$  such that  $\psi(D \cap \{y_n \geq 0\}) = B(x_0, \epsilon) \cap \Omega$ . See Figure 22 below. We call  $\partial\Omega$  the **manifold boundary** of  $\Omega$ .

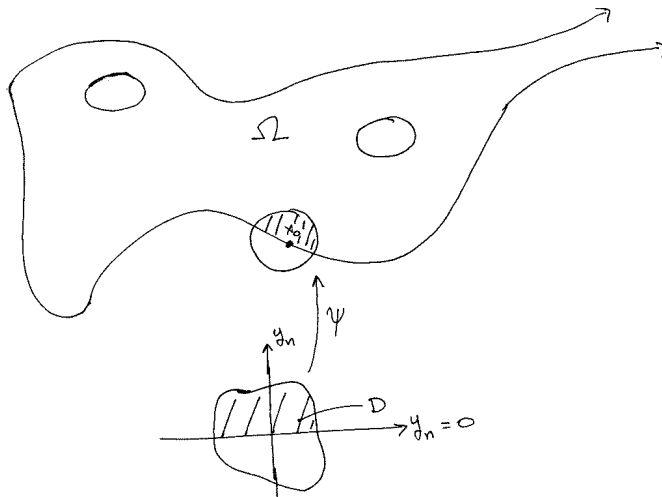


FIGURE 22. Flattening out a neighborhood of a boundary point.

- Remarks 8.19.* (1) In Definition 8.18 we have defined  $\partial\Omega = \Omega \setminus \Omega^\circ$  which is **not** the topological boundary of  $\Omega$ , defined by  $\text{bd}(\Omega) := \overline{\Omega} \setminus \Omega^\circ$ . Clearly we always have  $\partial\Omega \subset \text{bd}(\Omega)$  with equality iff  $\Omega$  is closed.
- (2) It is easily checked that if  $\Omega \subset \mathbb{R}^n$  is a  $C^k$  – manifold with boundary, then  $\partial\Omega$  is a  $C^k$  – hypersurface in  $\mathbb{R}^n$ .

The reader is left to verify the following examples.

**Example 8.20.** Let  $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ .

- (1)  $\overline{\mathbb{H}^n}$  is a  $C^\infty$  – manifold with boundary and
- $$\partial\overline{\mathbb{H}^n} = \text{bd}(\overline{\mathbb{H}^n}) = \mathbb{R}^{n-1} \times \{0\}.$$
- (2)  $\Omega = \overline{B(\xi, r)}$  is a  $C^\infty$  – manifold with boundary and  $\partial\Omega = \text{bd}(B(\xi, r))$ , as the reader should verify. See Exercise 8.2 for a general result containing this statement.
- (3) Let  $U$  be the open unit ball in  $\mathbb{R}^{n-1}$ , then  $\Omega = \mathbb{H}^n \cup (U \times \{0\})$  is a  $C^\infty$  – manifold with boundary and  $\partial\Omega = U \times \{0\}$  while  $\text{bd}(\Omega) = \mathbb{R}^{n-1} \times \{0\}$ .

- (4) Now let  $\Omega = \mathbb{H}^n \cup (\bar{U} \times \{0\})$ , then  $\Omega$  is **not** a  $C^1$  – manifold with boundary. The bad points are  $\text{bd}(U) \times \{0\}$ .
- (5) Suppose  $V$  is an open subset of  $\mathbb{R}^{n-1}$  and  $g : V \rightarrow \mathbb{R}$  is a  $C^k$  – function and set

$$\Omega := \{(y, z) \in V \times \mathbb{R} \subset \mathbb{R}^n : z \geq g(y)\},$$

then  $\Omega$  is a  $C^k$  – manifold with boundary and  $\partial\Omega = \Gamma(g)$  – the graph of  $g$ . Again the reader should check this statement.

- (6) Let

$$\Omega = [(0, 1) \times (0, 1)] \cup [(-1, 0) \times (-1, 0)] \cup [(-1, 1) \times \{0\}]$$

in which case

$$\Omega^o = [(0, 1) \times (0, 1)] \cup [(-1, 0) \times (-1, 0)]$$

and hence  $\partial\Omega = (-1, 1) \times \{0\}$  is a  $C^k$  – hypersurface in  $\mathbb{R}^2$ . Nevertheless  $\Omega$  is not a  $C^k$  – manifold with boundary as can be seen by looking at the point  $(0, 0) \in \partial\Omega$ .

- (7) If  $\Omega = S^{n-1} \subset \mathbb{R}^n$ , then  $\partial\Omega = \Omega$  is a  $C^\infty$  – hypersurface. However, as in the previous example  $\Omega$  is not an  $n$  – dimensional  $C^k$  – manifold with boundary despite the fact that  $\Omega$  is now closed. (**Warning:** there is a clash of notation here with that of the more general theory of manifolds where  $\partial S^{n-1} = \emptyset$  when viewing  $S^{n-1}$  as a manifold in its own right.)

**Lemma 8.21.** *Suppose  $\Omega \subset_o \mathbb{R}^n$  such that  $\text{bd}(\Omega)$  is a  $C^k$  – hypersurface, then  $\bar{\Omega}$  is  $C^k$  – manifold with boundary. (It is not necessarily true that  $\partial\bar{\Omega} = \text{bd}(\Omega)$ . For example, let  $\Omega := B(0, 1) \cup \{x \in \mathbb{R}^n : 1 < |x| < 2\}$ . In this case  $\bar{\Omega} = \bar{B}(0, 2)$  so  $\partial\bar{\Omega} = \{x \in \mathbb{R}^n : |x| = 2\}$  while  $\text{bd}(\Omega) = \{x \in \mathbb{R}^n : |x| = 2 \text{ or } |x| = 1\}$ .)*

**Proof. Claim:** Suppose  $U = (-1, 1)^n \subset_o \mathbb{R}^n$  and  $V \subset_o U$  such that  $\text{bd}(V) \cap U = \partial\mathbb{H}^n \cap U$ . Then  $V$  is either,  $U_+ := U \cap \mathbb{H}^n = U \cap \{x_n > 0\}$  or  $U_- := U \cap \{x_n < 0\}$  or  $U \setminus \partial\mathbb{H}^n = U_+ \cup U_-$ .

To prove the claim, first observe that  $V \subset U \setminus \partial\mathbb{H}^n$  and  $V$  is not empty, so either  $V \cap U_+$  or  $V \cap U_-$  is not empty. Suppose for example there exists  $\xi \in V \cap U_+$ . Let  $\sigma : [0, 1) \rightarrow U \cap \mathbb{H}^n$  be a continuous path such that  $\sigma(0) = \xi$  and

$$T = \sup \{t < 1 : \sigma([0, t]) \subset V\}.$$

If  $T \neq 1$ , then  $\eta := \sigma(T)$  is a point in  $U_+$  which is also in  $\text{bd}(V) = \bar{V} \setminus V$ . But this contradicts  $\text{bd}(V) \cap U = \partial\mathbb{H}^n \cap U$  and hence  $T = 1$ . Because  $U_+$  is path connected, we have shown  $U_+ \subset V$ . Similarly if  $V \cap U_- \neq \emptyset$ , then  $U_- \subset V$  as well and this completes the proof of the claim.

We are now ready to show  $\bar{\Omega}$  is a  $C^k$  – manifold with boundary. To this end, suppose

$$\xi \in \partial\bar{\Omega} = \text{bd}(\bar{\Omega}) = \bar{\Omega} \setminus \bar{\Omega}^o \subset \bar{\Omega} \setminus \Omega = \text{bd}(\Omega).$$

Since  $\text{bd}(\Omega)$  is a  $C^k$  – hypersurface, we may find an open neighborhood  $O$  of  $\xi$  such that there exists a  $C^k$  – diffeomorphism  $\psi : U \rightarrow O$  such that  $\psi(O \cap \text{bd}(\Omega)) = U \cap \mathbb{H}^n$ . Recall that

$$O \cap \text{bd}(\Omega) = O \cap \bar{\Omega} \cap \Omega^c = \overline{O \cap \Omega} \setminus (O \setminus \Omega) = \text{bd}_O(\Omega \cap O)$$

where  $\bar{A}^O$  and  $\text{bd}_O(A)$  denotes the closure and boundary of a set  $A \subset O$  in the relative topology on  $A$ . Since  $\psi$  is a  $C^k$  – diffeomorphism, it follows that  $V :=$

$\psi(O \cap \Omega)$  is an open set such that

$$\text{bd}(V) \cap U = \text{bd}_U(V) = \psi(\text{bd}_O(\Omega \cap O)) = \psi(O \cap \text{bd}(\Omega)) = U \cap \mathbb{H}^n.$$

Therefore by the claim, we learn either  $V = U_+$  of  $U_-$  or  $U_+ \cup U_-$ . However the latter case can not occur because in this case  $\xi$  would be in the interior of  $\bar{\Omega}$  and hence not in  $\text{bd}(\bar{\Omega})$ . This completes the proof, since by changing the sign on the  $n^{\text{th}}$  coordinate of  $\psi$  if necessary, we may arrange it so that  $\psi(\bar{\Omega} \cap O) = U_+$ . ■

**Exercise 8.2.** Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^k$  – function and assume

$$\{F < 0\} := \{x \in \mathbb{R}^n : F(x) < 0\} \neq \emptyset$$

and  $F'(\xi) : \mathbb{R}^n \rightarrow \mathbb{R}$  is surjective (or equivalently  $\nabla F(\xi) \neq 0$ ) for all

$$\xi \in \{F = 0\} := \{x \in \mathbb{R}^n : F(x) = 0\}.$$

Then  $\Omega := \{F \leq 0\}$  is a  $C^k$  – manifold with boundary and  $\partial\Omega = \{F = 0\}$ .

**Hint:** For  $\xi \in \{F = 0\}$ , let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  be a linear transformation such that  $A|_{\text{Nul}(F'(\xi))} : \text{Nul}(F'(\xi)) \rightarrow \mathbb{R}^{n-1}$  is invertible and  $A|_{\text{Nul}(F'(\xi))^\perp} \equiv 0$  and then define

$$\phi(x) := (A(x - \xi), -F(x)) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n.$$

Now use the inverse function theorem to construct  $\psi$ .

**Definition 8.22** (Outward pointing unit normal vector). Let  $\Omega$  be a  $C^1$  – manifold with boundary, the **outward pointing unit normal** to  $\partial\Omega$  is the unique function  $n : \partial\Omega \rightarrow \mathbb{R}^n$  satisfying the following requirements.

- (1) (Unit length.)  $|n(x)| = 1$  for all  $x \in \partial\Omega$ .
- (2) (Orthogonality to  $\partial\Omega$ .) If  $x_0 \in \partial\Omega$  and  $\psi : D \rightarrow B(x_0, \epsilon)$  is as in the Definition 8.18, then  $n(x_0) \perp \psi'(0)(\partial\mathbb{H}^n)$ , i.e.  $n(x_0)$  is perpendicular of  $\partial\Omega$ .
- (3) (Outward Pointing.) If  $\phi := \psi^{-1}$ , then  $\phi'(0)n(x_0) \cdot e_n < 0$  or equivalently put  $\psi'(0)e_n \cdot n(x_0) < 0$ , see Figure 23 below.

#### 8.4. Divergence Theorem.

**Theorem 8.23** (Divergence Theorem). Let  $\Omega \subset \mathbb{R}^n$  be a manifold with  $C^2$  – boundary and  $n : \partial\Omega \rightarrow \mathbb{R}^n$  be the unit outward pointing normal to  $\Omega$ . If  $Z \in C_c(\Omega, \mathbb{R}^n) \cap C^1(\Omega^\circ, \mathbb{R}^n)$  and

$$(8.18) \quad \int_{\Omega} |\nabla \cdot Z| dm < \infty$$

then

$$(8.19) \quad \int_{\partial\Omega} Z(x) \cdot n(x) d\sigma(x) = \int_{\Omega} \nabla \cdot Z(x) dx.$$

The proof of Theorem 8.23 will be given after stating a few corollaries and then a number preliminary results.

**Example 8.24.** Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{on } [0, 1], \\ 0 & \text{if } x = 0 \end{cases}$$



then  $f \in C([0, 1]) \cap C^\infty((0, 1))$  and  $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \sin\left(\frac{1}{x}\right)$  for  $x > 0$ . Since

$$\int_0^1 \frac{1}{x} \left| \sin\left(\frac{1}{x}\right) \right| dx = \int_1^\infty u |\sin(u)| \frac{1}{u^2} du = \int_1^\infty \frac{|\sin(u)|}{u} du = \infty,$$

$\int_0^1 |f'(x)| dx = \infty$  and the integrability assumption,  $\int_\Omega |\nabla \cdot Z| dx < \infty$ , in Theorem 8.23 is necessary.

**Corollary 8.25.** *Let  $\Omega \subset \mathbb{R}^n$  be a **closed** manifold with  $C^2$  - boundary and  $n : \partial\Omega \rightarrow \mathbb{R}^n$  be the outward pointing unit normal to  $\Omega$ . If  $Z \in C(\Omega, \mathbb{R}^n) \cap C^1(\Omega^\circ, \mathbb{R}^n)$  and*

$$(8.20) \quad \int_\Omega \{|Z| + |\nabla \cdot Z|\} dm + \int_{\partial\Omega} |Z \cdot n| d\sigma < \infty$$

then Eq. (8.19) is valid, i.e.

$$\int_{\partial\Omega} Z(x) \cdot n(x) d\sigma(x) = \int_\Omega \nabla \cdot Z(x) dx.$$

**Proof.** Let  $\psi \in C_c^\infty(\mathbb{R}^n, [0, 1])$  such that  $\psi = 1$  in a neighborhood of 0 and set  $\psi_k(x) := \psi(x/k)$  and  $Z_k := \psi_k Z$ . We have  $\text{supp}(Z_k) \subset \text{supp}(\psi_k) \cap \Omega$  - which is a compact set since  $\Omega$  is closed. Since  $\nabla \psi_k(x) = \frac{1}{k} (\nabla \psi)(x/k)$  is bounded,

$$\int_\Omega |\nabla \cdot Z_k| dm = \int_\Omega |\nabla \psi_k \cdot Z + \psi_k \nabla \cdot Z| dm \leq C \int_\Omega |Z| dm + \int_\Omega |\nabla \cdot Z| dm < \infty.$$

Hence Theorem 8.23 implies

$$(8.21) \quad \int_\Omega \nabla \cdot Z_k dm = \int_{\partial\Omega} Z_k \cdot n d\sigma.$$

By the D.C.T.,

$$\int_\Omega \nabla \cdot Z_k dm = \int_\Omega \left[ \frac{1}{k} (\nabla \psi)(x/k) \cdot Z(x) + \psi(x/k) \nabla \cdot Z(x) \right] dx \rightarrow \int_\Omega \nabla \cdot Z dm$$

and

$$\int_{\partial\Omega} Z_k \cdot n d\sigma = \int_{\partial\Omega} \psi_k Z \cdot n d\sigma \rightarrow \int_{\partial\Omega} Z \cdot n d\sigma,$$

which completes the proof by passing the limit in Eq. (8.21). ■

**Corollary 8.26** (Integration by parts I). *Let  $\Omega \subset \mathbb{R}^n$  be a closed manifold with  $C^2$  - boundary,  $n : \partial\Omega \rightarrow \mathbb{R}^n$  be the outward pointing normal to  $\Omega$ ,  $Z \in C(\Omega, \mathbb{R}^n) \cap C^1(\Omega^\circ, \mathbb{R}^n)$  and  $f \in C(\Omega, \mathbb{R}) \cap C^1(\Omega^\circ, \mathbb{R})$  such that*

$$\int_\Omega \{|f| \{ |Z| + |\nabla \cdot Z| \} + |\nabla f| |Z| \} dm + \int_{\partial\Omega} |f| |Z \cdot n| d\sigma < \infty$$

then

$$\int_\Omega f(x) \nabla \cdot Z(x) dx = - \int_\Omega \nabla f(x) \cdot Z(x) dx + \int_{\partial\Omega} Z(x) \cdot n(x) d\sigma(x).$$

**Proof.** Apply Corollary 8.25 with  $Z$  replaced by  $fZ$ . ■

**Corollary 8.27** (Integration by parts II). *Let  $\Omega \subset \mathbb{R}^n$  be a closed manifold with  $C^2$ -boundary,  $n : \partial\Omega \rightarrow \mathbb{R}^n$  be the outward pointing normal to  $\Omega$  and  $f, g \in C(\Omega, \mathbb{R}) \cap C^1(\Omega^\circ, \mathbb{R})$  such that*

$$\int_{\Omega} \{|f||g| + |\partial_i f||g| + |f||\partial_i g|\} dm + \int_{\partial\Omega} |fgn_i| d\sigma < \infty$$

then

$$\int_{\Omega} f(x)\partial_i g(x) dm = - \int_{\Omega} \partial_i f(x) \cdot g(x) dm + \int_{\partial\Omega} f(x)g(x)n_i(x)d\sigma(x).$$

**Proof.** Apply Corollary 8.26 with  $Z$  chosen so that  $Z_j = 0$  if  $j \neq i$  and  $Z_i = g$ , (i.e.  $Z = (0, \dots, g, 0, \dots, 0)$ ). ■

**Proposition 8.28.** *Let  $\Omega$  be as in Corollary 8.25 and suppose  $u, v \in C^2(\Omega^\circ) \cap C^1(\Omega)$  such that  $u, v, \nabla u, \nabla v, \Delta u, \Delta v \in L^2(\Omega)$  and  $u, v, \frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \in L^2(\partial\Omega, d\sigma)$  then*

$$(8.22) \quad \int_{\Omega} \Delta u \cdot v dm = - \int_{\Omega} \nabla u \cdot \nabla v dm + \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\sigma$$

and

$$(8.23) \quad \int_{\Omega} (\Delta uv - \Delta v u) dm = \int_{\partial\Omega} \left( v \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} u \right) d\sigma.$$

**Proof.** Eq. (8.22) follows by applying Corollary 8.26 with  $f = v$  and  $Z = \nabla u$ . Similarly applying Corollary 8.26 with  $f = u$  and  $Z = \nabla v$  implies

$$\int_{\Omega} \Delta v \cdot u dm = - \int_{\Omega} \nabla v \cdot \nabla u dm + \int_{\partial\Omega} u \frac{\partial v}{\partial n} d\sigma$$

and subtracting this equation from Eq. (8.22) implies Eq. (8.23). ■

### 8.5. The proof of Theorem 8.23.

**Lemma 8.29.** *Suppose  $\Omega \subset \mathbb{R}^n$  and  $Z \in C^1(\Omega, \mathbb{R}^n)$  and  $f \in C_c^1(\Omega, \mathbb{R})$ , then*

$$\int_{\Omega} f \nabla \cdot Z dx = - \int_{\Omega} \nabla f \cdot Z dx.$$

**Proof.** Let  $W := fZ$  on  $\Omega$  and  $W = 0$  on  $\Omega^c$ , then  $W \in C_c(\mathbb{R}^n, \mathbb{R}^n)$ . By Fubini's theorem and the fundamental theorem of calculus,

$$\int_{\Omega} \nabla \cdot (fZ) dx = \int_{\mathbb{R}^n} (\nabla \cdot W) dx = \sum_{i=1}^n \int_{\mathbb{R}^n} \frac{\partial W^i}{\partial x^i} dx_1 \dots dx_n = 0.$$

This completes the proof because  $\nabla \cdot (fZ) = \nabla f \cdot Z + f \nabla \cdot Z$ . ■

**Corollary 8.30.** *If  $\Omega \subset \mathbb{R}^n$ ,  $Z \in C^1(\Omega, \mathbb{R}^n)$  and  $g \in C(\Omega, \mathbb{R})$  then  $g = \nabla \cdot Z$  iff*

$$(8.24) \quad \int_{\Omega} gf dx = - \int_{\Omega} Z \cdot \nabla f dx \text{ for all } f \in C_c^1(\Omega).$$

**Proof.** By Lemma 8.29, Eq. (8.24) holds iff

$$\int_{\Omega} gf \, dx = \int_{\Omega} \nabla \cdot Z f \, dx \text{ for all } f \in C_c^1(\Omega)$$

which happens iff  $g = \nabla \cdot Z$ . ■

**Proposition 8.31** (Behavior of  $\nabla$  under coordinate transformations). *Let  $\psi : W \rightarrow \Omega$  is a  $C^2$ -diffeomorphism where  $W$  and  $\Omega$  are open subsets of  $\mathbb{R}^n$ . Given  $f \in C^1(\Omega, \mathbb{R})$  and  $Z \in C^1(\Omega, \mathbb{R}^n)$  let  $f^\psi = f \circ \psi \in C^1(W, \mathbb{R})$  and  $Z^\psi \in C^1(W, \mathbb{R}^n)$  be defined by  $Z^\psi(y) = \psi'(y)^{-1}Z(\psi(y))$ . Then*

- (1)  $\nabla f^\psi = \nabla(f \circ \psi) = (\psi')^{\text{tr}}(\nabla f) \circ \psi$  and
- (2)  $\nabla \cdot [\det \psi' Z^\psi] = (\nabla \cdot Z) \circ \psi \cdot \det \psi'$ . (Notice that we use  $\psi$  is  $C^2$  at this point.)

**Proof.** 1. Let  $v \in \mathbb{R}^n$ , then by definition of the gradient and using the chain rule,

$$\nabla(f \circ \psi) \cdot v = \partial_v(f \circ \psi) = \nabla f(\psi) \cdot \psi'v = (\psi')^{\text{tr}} \nabla f(\psi) \cdot v.$$

2. Let  $f \in C_c^1(\Omega)$ . By the change of variables formula,

$$\begin{aligned} \int_{\Omega} f \nabla \cdot Z \, dm &= \int_W f \circ \psi (\nabla \cdot Z) \circ \psi |\det \psi'| \, dm \\ (8.25) \qquad &= \int_W f^\psi (\nabla \cdot Z) \circ \psi |\det \psi'| \, dm. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{\Omega} f \nabla \cdot Z \, dm &= - \int_{\Omega} \nabla f \cdot Z \, dm = - \int_W \nabla f(\psi) \cdot Z(\psi) |\det \psi'| \, dm \\ &= - \int_W [(\psi')^{\text{tr}}]^{-1} \nabla f^\psi \cdot Z(\psi) |\det \psi'| \, dm \\ &= - \int_W \nabla f^\psi \cdot (\psi')^{-1} Z(\psi) |\det \psi'| \, dm \\ &= - \int_W (\nabla f^\psi \cdot Z^\psi) |\det \psi'| \, dm \\ (8.26) \qquad &= \int_W f^\psi \nabla \cdot (|\det \psi'| Z^\psi) \, dm. \end{aligned}$$

Since Eqs. (8.25) and (8.26) hold for all  $f \in C_c^1(\Omega)$  we may conclude

$$\nabla \cdot (|\det \psi'| Z^\psi) = (\nabla \cdot Z) \circ \psi |\det \psi'|$$

and by linearity this proves item 2. ■

**Lemma 8.32.** *Eq. (8.19 of the divergence Theorem 8.23 holds when  $\Omega = \bar{\mathbb{H}}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$  and  $Z \in C_c(\bar{\mathbb{H}}^n, \mathbb{R}^n) \cap C^1(\mathbb{H}^n, \mathbb{R}^n)$  satisfies*

$$\int_{\mathbb{H}^n} |\nabla \cdot Z| \, dx < \infty$$

**Proof.** In this case  $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$  and  $n(x) = -e_n$  for  $x \in \partial\Omega$  is the outward pointing normal to  $\Omega$ . By Fubini's theorem and the fundamental theorem of calculus,

$$\sum_{i=1}^{n-1} \int_{x_n > \delta} \frac{\partial Z^i}{\partial x^i} dx = 0$$

and

$$\int_{x_n > \delta} \frac{\partial Z_n}{\partial x_n} dx = - \int_{\mathbb{R}^{n-1}} Z_n(y, \delta) dy.$$

Therefore

$$\begin{aligned} \int_{\mathbb{H}^n} \nabla \cdot Z dx &\stackrel{\text{D.C.T.}}{=} \lim_{\delta \downarrow 0} \int_{x_n > \delta} \nabla \cdot Z dx = - \lim_{\delta \downarrow 0} \int_{\mathbb{R}^{n-1}} Z_n(y, \delta) dy \\ &= - \int_{\mathbb{R}^{n-1}} Z_n(y, 0) dy = \int_{\partial\mathbb{H}^n} Z(x) \cdot n(x) d\sigma(x). \end{aligned}$$

■

*Remark 8.33.* The same argument used in the proof of Lemma 8.32 shows Theorem 8.23 holds when

$$\Omega = \bar{\mathbb{R}}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}.$$

Notice that  $\bar{\mathbb{R}}_+^n$  has a corners and edges, etc. and so  $\partial\Omega$  is not smooth in this case.

8.5.1. *The Proof of the Divergence Theorem 8.23. Proof.* First suppose that  $\text{supp}(Z)$  is a compact subset of  $B(x_0, \epsilon) \cap \Omega$  for some  $x_0 \in \partial\Omega$  and  $\epsilon > 0$  is sufficiently small so that there exists  $V \subset_o \mathbb{R}^n$  and  $C^2$ -diffeomorphism  $\psi : V \rightarrow B(x_0, \epsilon)$  (see Figure 23) such that  $\psi(V \cap \{y_n > 0\}) = B(x_0, \epsilon) \cap \Omega^\circ$  and

$$\psi(V \cap \{y_n = 0\}) = B(x_0, \epsilon) \cap \partial\Omega.$$

Because  $n$  is the outward pointing normal,  $n(\psi(y)) \cdot \psi'(y)e_n < 0$  on  $y_n = 0$ . Since

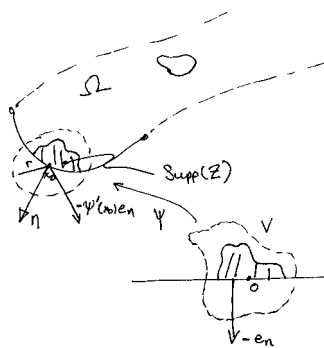


FIGURE 23. Reducing the divergence theorem for general  $\Omega$  to  $\Omega = \mathbb{H}^n$ .

$V$  is connected and  $\det \psi'(y)$  is never zero on  $V$ ,  $\varsigma := \text{sgn}(\det \psi'(y)) \in \{\pm 1\}$  is

constant independent of  $y \in V$ . For  $y \in \partial\bar{\mathbb{H}}^n$ ,

$$\begin{aligned}
(Z \cdot n)(\psi(y)) & |\det[\psi'(y)e_1 | \dots | \psi'(y)e_{n-1} | n(\psi(y))]| \\
&= -\zeta(Z \cdot n)(\psi(y)) \det[\psi'(y)e_1 | \dots | \psi'(y)e_{n-1} | n(\psi(y))] \\
&= -\zeta \det[\psi'(y)e_1 | \dots | \psi'(y)e_{n-1} | Z(\psi(y))] \\
&= -\zeta \det[\psi'(y)e_1 | \dots | \psi'(y)e_{n-1} | \psi'(y)Z^\psi(y)] \\
&= -\zeta \det \psi'(y) \cdot \det[e_1 | \dots | e_{n-1} | Z^\psi(y)] \\
&= -|\det \psi'(y)| Z^\psi(y) \cdot e_n,
\end{aligned}$$

wherein the second equality we used the linearity properties of the determinant and the identity

$$Z(\psi(y)) = Z \cdot n(\psi(y)) + \sum_{i=1}^{n-1} \alpha_i \psi'(y)e_i \text{ for some } \alpha_i.$$

Starting with the definition of the surface integral we find

$$\begin{aligned}
\int_{\partial\Omega} Z \cdot n \, d\sigma &= \int_{\partial\bar{\mathbb{H}}^n} (Z \cdot n)(\psi(y)) |\det[\psi'(y)e_1 | \dots | \psi'(y)e_{n-1} | n(\psi(y))]| \, dy \\
&= \int_{\partial\bar{\mathbb{H}}^n} \det \psi'(y) Z^\psi(y) \cdot (-e_n) \, dy \\
&= \int_{\mathbb{H}^n} \nabla \cdot [\det \psi' Z^\psi] \, dm \text{ (by Lemma 8.32)} \\
&= \int_{\mathbb{H}^n} [(\nabla \cdot Z) \circ \psi] \det \psi' \, dm \text{ (by Proposition 8.31)} \\
&= \int_{\Omega} (\nabla \cdot Z) \, dm \text{ (by the Change of variables theorem).}
\end{aligned}$$

2) We now prove the general case where  $Z \in C_c(\Omega, \mathbb{R}^n) \cap C^1(\Omega^o, \mathbb{R}^n)$  and  $\int_{\Omega} |\nabla \cdot Z| \, dm < \infty$ . Using Theorem 7.23, we may choose  $\phi_i \in C_c^\infty(\mathbb{R}^n)$  such that

- (1)  $\sum_{i=1}^N \phi_i \leq 1$  with equality in a neighborhood of  $K = \text{Supp}(Z)$ .
- (2) For all  $i$  either  $\text{supp}(\phi_i) \subset \Omega$  or  $\text{supp}(\phi_i) \subset B(x_0, \epsilon)$  where  $x_0 \in \partial\Omega$  and  $\epsilon > 0$  are as in the previous paragraph.

Then by special cases proved in the previous paragraph and in Lemma 8.29,

$$\begin{aligned}
\int_{\Omega} \nabla \cdot Z \, dx &= \int_{\Omega} \nabla \cdot \left( \sum_i \phi_i Z \right) \, dx = \sum_i \int_{\Omega} \nabla \cdot (\phi_i Z) \, dx = \sum_i \int_{\partial\Omega} (\phi_i Z) \cdot n \, d\sigma \\
&= \int_{\partial\Omega} \sum_i \phi_i Z \cdot n \, d\sigma = \int_{\partial\Omega} Z \cdot n \, d\sigma.
\end{aligned}$$

■

**8.6. Application to Holomorphic functions.** Let  $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$  be a compact manifold with  $C^2$  – boundary.

**Definition 8.34.** Let  $\Omega \subset \mathbb{C} \cong \mathbb{R}^2$  be a compact manifold with  $C^2$  – boundary and  $f \in C(\partial\Omega, \mathbb{C})$ . The contour integral,  $\int_{\partial\Omega} f(z)dz$ , of  $f$  along  $\partial\Omega$  is defined by

$$\int_{\partial\Omega} f(z)dz := i \int_{\partial\Omega} fnd\sigma$$

where  $\mathbf{n} := (\text{Re } n, \text{Im } n)$  is the outward pointing normal, see Figure 24.

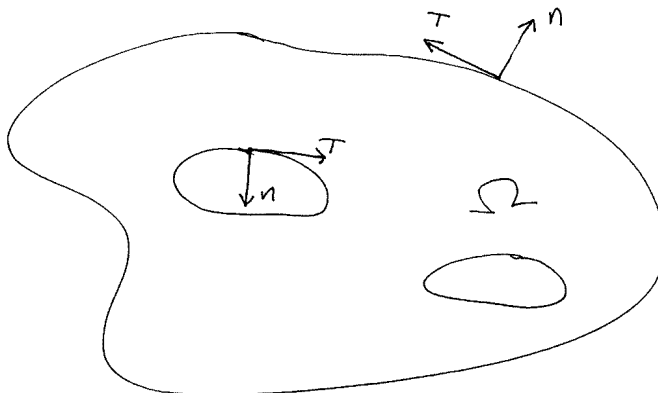


FIGURE 24. The induced direction for countour integrals along boundaries of regions.

In order to carry out the integral in Definition 8.34 more effectively, suppose that  $z = \gamma(t)$  with  $a \leq t \leq b$  is a parametrization of a part of the boundary of  $\Omega$  and  $\gamma$  is chosen so that  $T := \dot{\gamma}(t)/|\dot{\gamma}(t)| = in(\gamma(t))$ . That is to say  $T$  is gotten from  $n$  by a  $90^\circ$  rotation in the counterclockwise direction. Combining this with  $d\sigma = |\dot{\gamma}(t)| dt$  we see that

$$i n d\sigma = T |\dot{\gamma}(t)| dt = \dot{\gamma}(t)dt =: dz$$

so that

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\dot{\gamma}(t)dt.$$

**Proposition 8.35.** Let  $f \in C^1(\bar{\Omega}, \mathbb{C})$  and  $\bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$ , then

$$(8.27) \quad \int_{\partial\Omega} f(z)dz = 2i \int_{\Omega} \bar{\partial}f dm.$$

Now suppose  $w \in \Omega$ , then

$$(8.28) \quad f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \int_{\Omega} \frac{\bar{\partial}f(z)}{z-w} dm(z).$$

**Proof.** By the divergence theorem,

$$\begin{aligned} \int_{\Omega} \bar{\partial}f dm &= \frac{1}{2} \int_{\Omega} (\partial_x + i\partial_y) f dm = \frac{1}{2} \int_{\partial\Omega} f(n_1 + in_2) d\sigma \\ &= \frac{1}{2} \int_{\partial\Omega} fnd\sigma = -\frac{i}{2} \int_{\partial\Omega} f(z)dz. \end{aligned}$$

Given  $\epsilon > 0$  small, let  $\Omega_\epsilon := \Omega \setminus B(w, \epsilon)$ . Eq. (8.27) with  $\Omega = \Omega_\epsilon$  and  $f$  being replaced by  $\frac{f(z)}{z-w}$  implies

$$(8.29) \quad \int_{\partial\Omega_\epsilon} \frac{f(z)}{z-w} dz = 2i \int_{\Omega_\epsilon} \frac{\bar{\partial}f}{z-w} dm$$

wherein we have used the product rule and the fact that  $\bar{\partial}(z-w)^{-1} = 0$  to conclude

$$\bar{\partial} \left[ \frac{f(z)}{z-w} \right] = \frac{\bar{\partial}f(z)}{z-w}.$$

Noting that  $\partial\Omega_\epsilon = \partial\Omega \cup \partial B(w, \epsilon)$  and  $\partial B(w, \epsilon)$  may be parametrized by  $z = w + \epsilon e^{-i\theta}$  with  $0 \leq \theta \leq 2\pi$ , we have

$$\begin{aligned} \int_{\partial\Omega_\epsilon} \frac{f(z)}{z-w} dz &= \int_{\partial\Omega} \frac{f(z)}{z-w} dz + \int_0^{2\pi} \frac{f(w + \epsilon e^{-i\theta})}{\epsilon e^{-i\theta}} (-i\epsilon) e^{-i\theta} d\theta \\ &= \int_{\partial\Omega} \frac{f(z)}{z-w} dz - i \int_0^{2\pi} f(w + \epsilon e^{-i\theta}) d\theta \end{aligned}$$

and hence

$$(8.30) \quad \int_{\partial\Omega} \frac{f(z)}{z-w} dz - i \int_0^{2\pi} f(w + \epsilon e^{-i\theta}) d\theta = 2i \int_{\Omega_\epsilon} \frac{\bar{\partial}f(z)}{z-w} dm(z)$$

Since

$$\lim_{\epsilon \downarrow 0} \int_0^{2\pi} f(w + \epsilon e^{-i\theta}) d\theta = 2\pi f(w)$$

and

$$\lim_{\epsilon \downarrow 0} \int_{\Omega_\epsilon} \frac{\bar{\partial}f}{z-w} dm = \int_{\Omega} \frac{\bar{\partial}f(z)}{z-w} dm(z).$$

we may pass to the limit in Eq. (8.30) to find

$$\int_{\partial\Omega} \frac{f(z)}{z-w} dz - 2\pi i f(w) = 2i \int_{\Omega} \frac{\bar{\partial}f(z)}{z-w} dm(z)$$

which is equivalent to Eq. (8.28). ■

**Exercise 8.3.** Let  $\Omega$  be as above and assume  $f \in C^1(\bar{\Omega}, \mathbb{C})$  satisfies  $g := \bar{\partial}f \in C^\infty(\Omega, \mathbb{C})$ . Show  $f \in C^\infty(\Omega, \mathbb{C})$ . Hint, let  $w_0 \in \Omega$  and  $\epsilon > 0$  be small and choose  $\phi \in C_c^\infty(B(z_0, \epsilon))$  such that  $\phi = 1$  in a neighborhood of  $w_0$  and let  $\psi = 1 - \phi$ . Then by Eq. (8.28),

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \int_{\Omega} \frac{g(z)}{z-w} \phi(z) dm(z) - \frac{1}{\pi} \int_{\Omega} \frac{g(z)}{z-w} \psi(z) dm(z).$$

Now show each of the three terms above are smooth in  $w$  for  $w$  near  $w_0$ . To handle the middle term notice that

$$\int_{\Omega} \frac{g(z)}{z-w} \phi(z) dm(z) = \int_{\mathbb{C}} \frac{g(z+w)}{z} \phi(z+w) dm(z)$$

for  $w$  near  $w_0$ .

**Definition 8.36.** A function  $f \in C^1(\Omega, \mathbb{C})$  is said to be holomorphic if  $\bar{\partial}f = 0$ .

By Proposition 8.35, if  $f \in C^1(\bar{\Omega}, \mathbb{C})$  and  $\bar{\partial}f = 0$  on  $\Omega$ , then **Cauchy's integral formula** holds for  $w \in \Omega$ , namely

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz$$

and  $f \in C^\infty(\Omega, \mathbb{C})$ . For more details on Holomorphic functions, see the complex variable appendix.

8.6.1. *Appendix: More Proofs of Proposition 8.31.*

**Exercise 8.4.**  $\det'(A)B = \det(A) \operatorname{tr}(A^{-1}B)$ .

8.4.

$$\left. \frac{d}{dt} \right|_0 \det(A + tB) = \det(A) \left. \frac{d}{dt} \right|_0 \det(A + tA^{-1}B) = \det(A) \operatorname{tr}(A^{-1}B).$$

■

**Proof. 2nd Proof of Proposition 8.31** by direct computation. Letting  $A = \psi'$ ,

$$\begin{aligned} \frac{1}{\det A} \nabla \cdot (\det A Z^\psi) &= \frac{1}{\det A} \{Z^\psi \cdot \nabla \det A + \det A \nabla \cdot Z^\psi\} \\ (8.31) \qquad \qquad \qquad &= \operatorname{tr}[A^{-1} \partial_{Z^\psi} A] + \nabla \cdot Z^\psi \end{aligned}$$

and

$$\begin{aligned} \nabla \cdot Z^\psi &= \nabla \cdot (A^{-1} Z \circ \psi) = \partial_i (A^{-1} Z \circ \psi) \cdot e_i \\ &= e_i \cdot (-A^{-1} \partial_i A A^{-1} Z \circ \psi) + e_i \cdot A^{-1} (Z' \circ \psi) A e_i \\ &= -e_i \cdot (A^{-1} \psi'' \langle e_i, A^{-1} Z \circ \psi \rangle) + \operatorname{tr}(A^{-1} (Z' \circ \psi) A) \\ &= -e_i \cdot (A^{-1} \psi'' \langle e_i, A^{-1} Z \circ \psi \rangle) + \operatorname{tr}(Z' \circ \psi) \\ &= -\operatorname{tr}(A^{-1} \psi'' \langle Z^\psi, - \rangle) + (\nabla \cdot Z) \circ \psi \\ (8.32) \qquad \qquad \qquad &= -\operatorname{tr}[A^{-1} \partial_{Z^\psi} A] + (\nabla \cdot Z) \circ \psi. \end{aligned}$$

Combining Eqs. (8.31) and (8.32) gives the desired result:

$$\nabla \cdot (\det \psi' Z^\psi) = \det \psi' (\nabla \cdot Z) \circ \psi.$$

■

**Lemma 8.37** (Flow interpretation of the divergence). *Let  $Z \in C^1(\Omega, \mathbb{R}^n)$ . Then*

$$\nabla \cdot Z = \left. \frac{d}{dt} \right|_0 \det(e^{tZ})'$$

and

$$\int_{\Omega} \nabla \cdot (f Z) dm = \left. \frac{d}{dt} \right|_0 \int_{e^{tZ}(\Omega)} f dm.$$

**Proof.** By Exercise 8.4 and the change of variables formula,

$$\left. \frac{d}{dt} \right|_0 \det(e^{tZ})' = \operatorname{tr} \left( \left. \frac{d}{dt} \right|_0 (e^{tZ})' \right) = \operatorname{tr}(Z') = \nabla \cdot Z$$



and

$$\begin{aligned}
 \frac{d}{dt}\Big|_0 \int_{e^{tZ}(\Omega)} f(x) dx &= \frac{d}{dt}\Big|_0 \int_{\Omega} f(e^{tZ}(y)) \det(e^{tZ})'(y) dy \\
 &= \int_{\Omega} \{\nabla f(y) \cdot Z(y) + f(y) \nabla \cdot Z(y)\} dy \\
 &= \int_{\Omega} \nabla \cdot (f Z) dm.
 \end{aligned}$$

■

**Proof. 3rd Proof of Proposition 8.31.** Using Lemma 8.37 with  $f = \det \psi'$  and  $Z = Z^\psi$  and the change of variables formula,

$$\begin{aligned}
 \int_{\Omega} \nabla \cdot (\det \psi' Z^\psi) dm &= \frac{d}{dt}\Big|_0 \int_{e^{tZ}(\Omega)} \det \psi' dm = \frac{d}{dt}\Big|_0 m(\psi \circ e^{tZ^\psi}(\Omega)) \\
 &= \frac{d}{dt}\Big|_0 m(\psi \circ \psi^{-1} \circ e^{tZ} \circ \psi(\Omega)) = \frac{d}{dt}\Big|_0 m(e^{tZ}(\psi(\Omega))) \\
 &= \frac{d}{dt}\Big|_0 \int_{e^{tZ}(\psi(\Omega))} 1 dm = \int_{\psi(\Omega)} \nabla \cdot Z dm \\
 &= \int_{\Omega} (\nabla \cdot Z) \circ \psi \det \psi' dm.
 \end{aligned}$$

Since this is true for all regions  $\Omega$ , it follows that  $\nabla \cdot (\det \psi' Z^\psi) = \det \psi' (\nabla \cdot Z^\psi)$ .

■