6. Elliptic Ordinary Differential Operators

Let $\Omega \subset_o \mathbb{R}^n$ be a bounded connected open region. A function $u \in C^2(\Omega)$ is said to satisfy Laplace's equation if

$$\triangle u = 0 \text{ in } \Omega.$$

More generally if $f \in C(\Omega)$ is given we say u solves the **Poisson equation** if

$$-\triangle u = f \text{ in } \Omega.$$

In order to get a unique solution to either of these equations it is necessary to impose "boundary" conditions on u.

Example 6.1. For **Dirichlet boundary conditions** we impose $u = g$ on $\partial\Omega$ and for **Neumann boundary conditions** we impose $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$, where $g : \partial\Omega \to \mathbb{R}$ is a given function.

Lemma 6.2. *Suppose $f : \Omega \xrightarrow{C^0} \mathbb{R}$, $\partial\Omega$ is C^2 and $g : \partial\Omega \to \mathbb{R}$ is continuous. Then if there exists a solution to $-\triangle u = f$ with $u = g$ on $\partial\Omega$ such that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ then the solution is unique.*

Definition 6.3. Given an open set $\Omega \subset \mathbb{R}^n$ we say $u \in C^1(\overline{\Omega})$ if $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and ∇u extends to a continuous function on $\overline{\Omega}$.

Proof. If $\widetilde{u}$ is another solution then $v = \widetilde{u} - u$ solves $\triangle v = 0, v = 0$ on $\partial\Omega$. By the divergence theorem,

$$0 = \int_\Omega \triangle v \cdot v dm = -\int_\Omega |\nabla v|^2 dm + \int_{\partial\Omega} v \nabla v \cdot n d\sigma = -\int_\Omega |\nabla v|^2 dm,$$

where the boundary terms are zero since $v = 0$ on $\partial\Omega$. This identity implies $\int_\Omega |\nabla u|^2 dx = 0$ which then shows $\nabla v \equiv 0$ and since Ω is connected we learn v is constant on Ω. Because v is zero on $\partial\Omega$ we conclude $v \equiv 0$, that is $u = \tilde{u}$. ■

For the rest of this section we will now restrict to $n = 1$. However we will allow for more general operators than Δ in this case.

6.1. Symmetric Elliptic ODE.

Let $a \in C^1([0,1], (0,\infty))$ and

$$Lf = -(af')' = -af'' - a'f' \text{ for } f \in C^2([0,1]). \tag{6.1}$$

In the following theorem we will impose Dirichlet boundary conditions on L by restricting the domain of L to

$$D(L) := \{f \in C^2([0,1], \mathbb{R}) : f(0) = f(1) = 0\}.$$

Theorem 6.4. *The linear operator $L : D(L) \to C([0,1], \mathbb{R})$ is invertible and $L^{-1} : C([0,1], \mathbb{R}) \to D(L) \subset C^2([0,1], \mathbb{R})$ is a bounded operator.*

Proof.

(1) (Uniqueness) If $f, g \in D(L)$ then by integration by parts

$$(Lf, g) := \int_0^1 (Lf)(x) g(x) dx = \int_0^1 a(x) f'(x) g'(x)\, dx. \tag{6.2}$$

Therefore if $Lf = 0$ then

$$0 = (Lf, f) = \int_0^1 a(x) f'(x)^2\, dx$$

and hence $f' \equiv 0$ and since $f(0) = 0$, $f \equiv 0$. This shows L is injective.

(2) (Existence) Given $g \in C([0,1], \mathbb{R})$ we are looking for $f \in D(L)$ such that $Lf = g$, i.e. $(af')' = g$. Integrating this equation implies

$$-a(x)f'(x) = -C + \int_0^x g(y)dy.$$

Therefore

$$f'(z) = \frac{C}{a(z)} - \int 1_{y \le z} \frac{1}{a(z)} g(y)dy$$

which upon integration and using $f(0) = 0$ gives

$$f(x) = \int_0^x \frac{C}{a(z)}\, dz - \int 1_{y \le z \le x}\, \frac{1}{a(z)}\, g(y)\, dz\, dy.$$

If we let

$$\alpha(x) := \int_0^x \frac{1}{a(z)}\, dz \tag{6.3}$$

the last equation may be written as

$$f(x) = C\alpha(x) - \int_0^x (\alpha(x) - \alpha(y))g(y)\, dy. \tag{6.4}$$

It is a simple matter to work backwards to show the function f defined in Eq. (6.4) satisfies $Lf = g$ and $f(0) = 0$ for any constant C. So it only remains to choose C so that

$$0 = f(1) = C\,\alpha(1) - \int_0^1 (\alpha(1) - \alpha(y))g(y)dy.$$

Solving for C gives $C = \int_0^1 \left(1 - \frac{\alpha(y)}{\alpha(1)}\right) g(y)\, dy$ and the resulting function f may be written as

$$\begin{aligned} f(x) &= \int_0^1 \left[\left(1 - \frac{\alpha(y)}{\alpha(1)}\right)\alpha(x) - 1_{y \le x}(\alpha(x) - \alpha(y))\right] g(y)\, dy \\ &= \int_0^1 G(x,y)g(y)dy \end{aligned}$$

where

$$G(x,y) = \begin{cases} \alpha(x)\left(1 - \frac{\alpha(y)}{\alpha(1)}\right) & \text{if} \quad x \le y \\ \alpha(y)\left(1 - \frac{\alpha(x)}{\alpha(1)}\right) & \text{if} \quad y \le x. \end{cases} \tag{6.5}$$

For example when $a \equiv 1$,

$$G(x,y) = \begin{cases} x\,(1-y) & \text{if} \quad x \le y \\ y\,(1-x) & \text{if} \quad y \le x \end{cases}.$$

■

Definition 6.5. The function G defined in Eq. (6.5) is called the **Green's function** for the operator $L : D(L) \to C([0,1], \mathbb{R})$.

Remarks 6.6. The proof of Theorem 6.4 shows

$$\left(L^{-1}g\right)(x) := \int_0^1 G(x,y)g(y)dy \tag{6.6}$$

where G is defined in Eq. (6.5). The Green's function G has the following properties:

(1) Since L is invertible and G is a right inverse, G is also a left inverse, i.e. $GLf = f$ for all $f \in D(L)$.
(2) G is continuous.
(3) G is symmetric, $G(y,x) = G(x,y)$. (This reflects the symmetry in L, $(Lf,g) = (f,Lg)$ for all $f,g \in D(L)$, which follows from Eq. (6.2).)
(4) G may be written as

$$G(x,y) = \begin{cases} u(x)v(y) & \text{if} \quad x \le y \\ u(y)v(x) & \text{if} \quad y \le x. \end{cases}$$

where u and v are $L-$ harmonic functions (i.e. and $Lu = Lv = 0$) with $u(0) = 0$ and $v(1) = 0$. In particular $L_xG(x,y) = 0 = L_yG(x,y)$ for all $y \neq x$.
(5) The first order derivatives of the Green's function have a jump discontinuity on the diagonal. Explicitly,

$$G_y(x,x+) - G_y(x,x-) = -\frac{1}{a(x)}$$

which follows directly from

$$G_y(x,y) = \frac{1}{a(y)}\begin{cases} -\frac{\alpha(x)}{\alpha(1)} & \text{if} \quad x < y \\ \left(1 - \frac{\alpha(x)}{\alpha(1)}\right) & \text{if} \quad y < x. \end{cases} \tag{6.7}$$

By symmetry we also have

$$G_x(y+,y) - G_x(y-,y) = -\frac{1}{a(y)}.$$

(6) By Items 4. and 5. and Lemma 5.11 it follows that

$$L_yG(x,y) := L_yT_{G(x,y)} = \frac{d}{dy}\left(a(y)G_y(x,y)\right) = \delta(y-x)$$

and similarly that

$$L_xT_{G(x,y)} = L_xG(x,y) = \delta(x-y).$$

As a consequence of the above remarks we have the following representation theorem for function $f \in C^2([0,1])$.

Theorem 6.7 (Representation Theorem)**.** *For any* $f \in C^2([0,1])$,

$$f(x) = (GLf)(x) - G_y(x,y)a(y)f(y)\Big|_{y=0}^{y=1}. \tag{6.8}$$

Moreover if we are given $h : \partial[0,1] \to \mathbb{R}$ *and* $g \in C([0,1])$, *then the unique solution to*

$$Lf = g \text{ with } f = h \text{ on } \partial[0,1]$$

is

$$f(x) = (Gg)(x) - G_y(x,y)a(y)h(y)\Big|_{y=0}^{y=1}. \tag{6.9}$$

Proof. By repeated use of Lemma 5.11,

$$\begin{aligned}(GLf)(x) &= -\int_0^1 G(x,y)\frac{d}{dy}(a(y)f'(y))dy \\ &= \int_0^1 G_y(x,y)a(y)f'(y)dy \text{ (no boundary terms since } G(x,0)=G(x,1)=0) \\ &= G_y(x,y)a(y)f(y)\Big|_{y=0}^{y=1} + \int_0^1 L_y G(x,y)f(y)dy \\ &= G_y(x,y)a(y)f(y)\Big|_{y=0}^{y=1} + \int_0^1 \delta(x-y)f(y)dy \\ &= G_y(x,y)a(y)f(y)\Big|_{y=0}^{y=1} + f(x)\end{aligned}$$

which proves Eq. (6.8).

Now suppose that f is defined as in Eq. (6.9). Observe from Eq. (6.7) that

$$\lim_{x\uparrow 1} a(1)G_y(x,1) = -1 \text{ and } \lim_{x\downarrow 0} a(0)G_y(x,0) = 1$$

and also notice that $G_y(x,1)$ and $G_y(x,0)$ are L_x – harmonic functions. Therefore by these remarks and Eq. (6.6), $f=h$ on $\partial[0,1]$ and

$$Lf(x) = g(x) - L_x G_y(x,y)a(y)h(y)\Big|_{y=0}^{y=1} = g(x)$$

as desired. ■

6.2. **General Regular 2nd order elliptic ODE.** Let $J=[r,s]$ be a closed bounded interval in $\mathbb{R}$.

Definition 6.8. A second order linear operator of the form

$$Lf = -af'' + bf' + cf \tag{6.10}$$

with $a\in C^2(J)$, $b\in C^1(J)$ and $c\in C^2(J)$ is said to be **elliptic** if $a>0$, (more generally if a is invertible if we are allowing for vector valued functions).

For this section L will denote an elliptic ordinary differential operator. We will now consider the Dirichlet boundary valued problem for $f\in C^2([r,s])$,

$$Lf = -af'' + bf' + cf = 0 \text{ with } f=0 \text{ on } \partial J. \tag{6.11}$$

Lemma 6.9. *Let $u,v\in C^2(J)$ be two L – harmonic functions, i.e. $Lu=0=Lv$ and let*

$$W := \det\begin{bmatrix} u & v \\ u' & v'\end{bmatrix} = uv' - vu'$$

be the Wronskian of u and v. Then W satisfies

$$W' = \frac{b}{a}W,\ \frac{d}{dx}\frac{1}{W} = -\frac{b}{a}\frac{1}{W} \text{ and}$$
$$W(x) = W(r)e^{\int_r^x \frac{b}{a}(t)dt}.$$

Proof. By direct computation

$$aW' = a(uv''-vu'') = u(bv'+cv) - v(bu'+cu) = bW.$$

■

Definition 6.10. Let $H^k(J)$ denote those $f \in C^{k-1}(J)$ such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in L^2(J)$. We also let $H_0^2(J) = \left\{ f \in H^2(J) : f|_{\partial J} = 0 \right\}$. We make $H^k(J)$ into a Hilbert space using the following inner product

$$(u,v)_{H^k} := \sum_{j=0}^{k} \left(D^j u, D^j v \right)_{L^2}.$$

Theorem 6.11. *As above, let* $D(L) = \left\{ f \in C^2(J) : f = 0 \text{ on } \partial J \right\}$. *If the* $\operatorname{Nul}(L) \cap D(L) = \{0\}$, *i.e. if the only solution* $f \in D(L)$ *to* $Lf = 0$ *is* $f = 0$, *then* $L : D(L) \to C(J)$ *is an invertible. Moreover there exists a continuous function* G *on* $J \times J$ *(called the Dirichlet Green's function for* L*) such that*

$$\left(L^{-1} g \right)(x) = \int_J G(x,y) g(y) dy \text{ for all } g \in C(J). \tag{6.12}$$

Moreover if $g \in L^2(J)$ *then* $Gg \in H_0^2(J)$ *and* $L(G\, g) = g$ *a.e. and more generally if* $g \in H^k(J)$ *then* $Gg \in H_0^{k+2}(J)$

Proof. To prove the surjectivity of $L : D(L) \to C(J)$, (i.e. existence of solutions $f \in D(L)$ to $Lf = g$ with $g \in C(J)$) we are going to construct the Green's function G.

(1) **Formal requirements on the Greens function.** Assuming Eq. (6.12) holds and working formally we should have

$$g(x) = L_x \int_J G(x,y) g(y) dy = \int_J L_x G(x,y) g(y) dy \tag{6.13}$$

for all $g \in C(J)$. Hence, again formally, this implies

$$L_x G(x,y) = \delta(y-x) \text{ with } G(r,y) = G(s,y) = 0. \tag{6.14}$$

This can be made more convincing by as follows. Let $\phi \in \mathcal{D} := \mathcal{D}(r,s)$, then multiplying

$$g(x) = L_x \int_J G(x,y) g(y) dy$$

by ϕ, integrating the result and then using integration by parts and Fubini's theorem gives

$$\begin{aligned} \int_J g(x)\phi(x)dx &= \int_J dx \phi(x) L_x \int_J dy G(x,y) g(y) \\ &= \int_J dx L_x \phi(x) \int_J dy G(x,y) g(y) \\ &= \int_J dy g(y) \int_J dx\ L_x \phi(x) G(x,y) \text{ for all } g \in C(J). \end{aligned}$$

From this we conclude

$$\int_J L_x \phi(x) G(x,y) dx = \phi(y),$$

i.e. $L_x T_{G(x,y)} = \delta(x-y)$.

(2) **Constructing** G. In order to construct a solution to Eq. (6.14), let u, v be two non-zero $L-$ harmonic functions chosen so that $u(r) = 0 = v(s)$ and $u'(r) = 1 = v'(s)$ and let W be the Wronskian of u and v. By Lemma 6.9, either W is never zero or is identically zero. If $W = 0$, then $(u(r), u'(r)) =$

$\lambda(v(r), v'(r))$ for some $\lambda \in \mathbb{R}$ and by uniqueness of solutions to ODE it would follow that $u \equiv \lambda v$. In this case $u(r) = 0$ and $u(s) = \lambda v(s) = 0$, and hence $u \in D(L)$ with $Lu = 0$. However by assumption, this implies $u = 0$ which is impossible since $u'(0) = 1$. Thus W is never 0.

By Eq. (6.14) we should require $L_x G(x, y) = 0$ for $x \neq y$ and $G(r, y) = G(s, y) = 0$ which implies that

$$G(x, y) = \begin{cases} u(x)\phi(y) & \text{if} \quad x < y \\ v(x)\psi(y) & \text{if} \quad x > y \end{cases}$$

for some functions ϕ and ψ. We now want to choose ϕ and ψ so that G is continuous and $L_x G(x, y) = \delta(x - y)$. Using

$$G_x(x, y) = \begin{cases} u'(x)\phi(y) & \text{if} \quad x < y \\ v'(x)\psi(y) & \text{if} \quad x > y \end{cases}$$

Lemma 6.9, we are led to require

$$\begin{aligned} 0 &= G(y+, y) - G(y-, y) = u(y)\phi(y) - v(y)\psi(y) \\ 1 &= -\left[a(x)G_x(x, y)\right]\big|_{x=y-}^{x=y+} = -a(y)\left[v'(y)\psi(y) - u'(y)\phi(y)\right]. \end{aligned}$$

Solving these equations for ϕ and ψ gives

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = -\frac{1}{aW} \begin{pmatrix} v \\ u \end{pmatrix}$$

and hence

$$G(x, y) = -\frac{1}{a(y)W(y)} \begin{cases} u(x)v(y) & \text{if} \quad x \leq y \\ v(x)u(y) & \text{if} \quad x \geq y. \end{cases} \tag{6.15}$$

(3) **With this G, Eq. (6.12) holds.** Given $g \in C(J)$, then f in Eq. (6.12) may be written as

$$\begin{aligned} f(x) &= \int_J G(x, y)g(y)dy \\ &= -v(x)\int_r^x \frac{u(y)}{a(y)W(y)}g(y)dy - u(x)\int_x^s \frac{v(y)}{a(y)W(y)}g(y)dy. \end{aligned} \tag{6.16}$$

Differentiating this equation twice gives

$$f'(x) = -v'(x)\int_r^x \frac{u(y)}{a(y)W(y)}g(y)dy - u'(x)\int_x^s \frac{v(y)}{a(y)W(y)}g(y)dy \tag{6.17}$$

and

$$\begin{aligned} f''(x) = &-v''(x)\int_r^x \frac{u(y)}{a(y)W(y)}g(y)dy - u''(x)\int_x^s \frac{v(y)}{a(y)W(y)}g(y)dy \\ &- v'(x)\frac{u(x)}{a(x)W(x)}g(x) + u'(x)\frac{v(x)}{a(x)W(x)}g(x). \end{aligned} \tag{6.18}$$

Using $Lv = 0 = Lu$, the definition of W and the last two equations we find

$$\begin{aligned} -a(x)f''(x) &= \left[b(x)v'(x) + c(x)v(x)\right]\int_r^x \frac{u(y)}{a(y)W(y)}g(y)dy \\ &\quad + \left[b(x)u'(x) + c(x)u(x)\right]\int_x^s \frac{v(y)}{a(y)W(y)}g(y)dy + g(x) \\ &= -b(x)f'(x) - c(x)f(x) + g(x), \end{aligned}$$

i.e. $Lf = g$.

Hence we have proved $L : D(L) \to C(J)$ is surjective and $L^{-1} : C(J) \to D(L)$ is given by Eq. (6.12).

Now suppose $g \in L^2(J)$, we will show that $f \in C^1(J)$ and Eq. (6.17) is still valid. The difficulty here is that it is clear that f is differentiable almost everywhere and Eq. (6.17) holds for almost every x. However this is not good enough, we need Eq. (6.17) to hold for all x. To remedy this, choose $g_n \in C(J)$ such that $g_n \to g$ in $L^2(J)$ and let $f_n := Gg_n$. Then by what we have just proved,

$$f_n'(x) = \int_J G_x(x, y) g_n(y) dy$$

Now by the Cauchy-Schwarz inequality,

$$\left| \int_J G_x(x, y) \left[g(y) - g_n(y) \right] dy \right|^2 \le \| g - g_n \|_{L^2(J)}^2 \int_J |G_x(x, y)|^2 \, dy \le C \| g - g_n \|_{L^2(J)}^2$$

where $C := \sup_{x \in J} \int_J |G_x(x, y)|^2 \, dy < \infty$. From this inequality it follows that $f_n'(x)$ converges uniformly to $\int_J G_x(x, y) g(y) dy$ as $n \to \infty$ and hence $f \in C^1(J)$ and

$$f'(x) = \int_J G_x(x, y) g(y) dy \text{ for all } x \in J,$$

i.e. Eq. (6.17) is valid for all $x \in J$. It now follows from Eq. (6.17) that $f \in H^2(J)$ and Eq. (6.18) holds for almost every x. Working as before we may conclude $Lf = g$ a.e. Finally if $g \in H^k(J)$ for $k \ge 1$, the reader may easily show $f \in H_0^{k+2}(J)$ by examining Eqs. (6.17) and (6.18). ■

Remark 6.12. When L is given as in Eq. (6.1), $b = -a'$ and by Lemma 6.9

$$W(x) = W(0) e^{-\int_0^x \frac{a'}{a}(t) dt} = W(0) e^{-\ln(a(x)/a(0))} = \frac{W(0) a(0)}{a(x)}.$$

So in this case

$$G(x, y) = -\frac{1}{W(0) a(0)} \begin{cases} u(x) v(y) & \text{if} \quad x \le y \\ v(x) u(y) & \text{if} \quad x \ge y \end{cases}$$

where we may take

$$u(x) = \alpha(x) := \int_0^x \frac{1}{a(z)} dz \text{ and } v(x) = \left(1 - \frac{\alpha(x)}{\alpha(1)} \right).$$

Finally for this choice of u and v we have

$$W(0) = u(0) v'(0) - u'(0) v(0) = -\frac{1}{a(0)}$$

giving

$$G(x, y) = \begin{cases} u(x) v(y) & \text{if} \quad x \le y \\ v(x) u(y) & \text{if} \quad x \ge y \end{cases}$$

which agrees with Eq. (6.5) above.

Lemma 6.13. *Let $L^* f := -(af)'' - (bf)' + cf$ be the **formal adjoint** of L. Then*

(6.19) $$(Lf, g) = (f, L^* g) \text{ for all } f, g \in D(L)$$

where $(f, g) := \int_J f(x) g(x) dx$. Moreover if $\mathrm{nul}(L) = \{0\}$ *then* $\mathrm{nul}(L^*) = \{0\}$ *and the Greens function for L^* is G^* defined by $G^*(x, y) = G(y, x)$, where G is the Green's function in Eq. (6.15). Consequently $L_y^* G(x, y) = \delta(x - y)$.*

Proof. First observe that G^* has been defined so that $(G^*g,f)=(g,Gf)$ for all $f\in L^2(J)$.Eq. (6.19) follows by two integration by parts after observing the boundary terms are zero because $f=g=0$ on ∂J. If $g\in \mathrm{nul}(L^*)$ and $f\in D(L)$, we find

$$0=(L^*g,f)=(g,Lf) \text{ for all } f\in D(L).$$

By Theorem 6.11, if $\mathrm{nul}(L)=\{0\}$ then $L:D(L)\to C(J)$ is invertible so the above equation implies $\mathrm{nul}(L^*)=\{0\}$. Another application of Theorem 6.11 then shows $L^*:D(L)\to C(J)$ is invertible and has a Green's function which we call $\tilde{G}(x,y)$. We will now complete the proof by showing $\tilde{G}=G^*$. To do this observe that

$$(f,g)=(L^*\tilde{G}f,g)=(\tilde{G}f,Lg)=(f,\tilde{G}^*Lg) \text{ for all } f,g\in D(L)$$

and this then implies $\tilde{G}^*L=Id_{D(L)}=GL$. Cancelling the L from this equation, show $\tilde{G}^*=G$ or equivalently that $\tilde{G}=G^*$. The remaining assertions of the Lemma follows from this observation.

Here is an **alternate proof** that $L_y^*G(x,y)=\delta(x-y)$, also see Using $GL=I_{D(L)}$, we learn for $u\in D(L)$ and $v\in C(J)$ that

$$(v,u)=(v,GLu)=(L^*G^*v,u)$$

which then implies $L^*G^*v=v$ for all $v\in C(J)$. This implies

$$f(x)=\int_J G(x,y)Lf(y)dy=\langle T_{G(x,\cdot)},Lf\rangle=\langle L^*T_{G(x,\cdot)},f\rangle \text{ for all } f\in D(L)$$

from which it follows that $L_y^*T_{G(x,y)}=\delta(x-y)$. ■

Definition 6.14. A **Green's function for** L is a function $G(x,y)$ as defined as in Eq. (6.15) where u and v are **any** two linearly independent $L-$ harmonic functions.[2]

The following theorem in is a generalization of Theorem 6.7.

Theorem 6.15 (Representation Theorem)**.** *Suppose and G is a Green's function for L then*

(1) $L_xT_{G(x,y)}=\delta(x-y)$ *and* $LG=I$ *on* $L^2(J)$. *(However Gg and G^*g may no longer satisfy the given Dirichlet boundary conditions.)*
(2) $L_y^*T_{G(x,y)}=\delta(x-y)$. *More precisely we have the following representation formula. For any $f\in H^2(J)$,*

$$f(x)=(GLf)(x)+\Big\{G(x,y)a(y)f'(y)-[a(y)G(x,y)]_y\,f(y)\Big\}\Big|_{y=r}^{y=s}. \tag{6.20}$$

(3) *Let us now assume* $\mathrm{nul}(L)=\{0\}$ *and G is the Dirichlet Green's function for L. The Eq. (6.20) specializes to*

$$f(x)=(GLf)(x)-[a(y)G(x,y)]_y\,f(y)\Big|_{y=r}^{y=s}.$$

Moreover if we are given $h:\partial J\to\mathbb{R}$ and $g\in L^2(J)$, then the unique solution $f\in H^2(J)$ to

$$Lf=g \text{ a.e. with } f=h \text{ on } \partial J$$

is

$$f(x)=(Gg)(x)+H(x) \tag{6.21}$$

[2]For example choose u,v so that $Lu=0=Lv$ and $u(\alpha)=v'(\alpha)=0$ and $u'(\alpha)=v(\alpha)=1$.

where, for $x \in J^0$,

$$H(x) := -\left[a(y)G(x,y)\right]_y h(y)\Big|_{y=r}^{y=s} \tag{6.22}$$

and $H(r) := H(r+)$ *and* $H(s) := H(s-)$.

Proof. 1. The first item follows from the proof of Theorem 6.11 with out any modification.

2. Using Lemma 6.9,

$$\begin{aligned}
L^*\left(\frac{u}{aW}\right) &= -(\frac{u}{W})'' - (\frac{bu}{aW})' + \frac{cu}{aW} \\
&= -(\frac{u'}{W} - \frac{b}{a}\frac{1}{W}u)' - (\frac{bu}{aW})' + \frac{cu}{aW} \\
&= -(\frac{u'}{W})' + \frac{cu}{aW} = -(\frac{u''}{W} - \frac{b}{a}\frac{1}{W}u) + \frac{cu}{aW} \\
&= \frac{1}{a}Lu = 0.
\end{aligned}$$

Similarly $L^*(\frac{v}{aW}) = 0$ and therefore $L_y^* G(x,y) = 0$ for $y \neq x$. Since

$$\begin{aligned}
G_y(x,y) &= -\left(\frac{d}{dy}\frac{1}{a(y)W(y)}\right)\begin{cases} u(x)v(y) & \text{if} \quad x \leq y \\ v(x)u(y) & \text{if} \quad x \geq y \end{cases} \\
&\quad - \frac{1}{a(y)W(y)}\begin{cases} u(x)v'(y) & \text{if} \quad x \leq y \\ v(x)u'(y) & \text{if} \quad x \geq y \end{cases}
\end{aligned} \tag{6.23}$$

we find

$$G_y(x,x+) - G_y(x,x-) = \frac{1}{a(x)W(x)}\left\{v(x)u'(x) - u(x)v'(x)\right\} = -\frac{1}{a(x)}.$$

Finally since

$$L_y^* = -a\frac{d^2}{dy^2} + \text{lower order terms}$$

we may conclude form Lemma 5.11 that $L_y^* G(x,y) = \delta(x-y)$.Using integration by parts for absolutely continuous functions and Lemma 6.13, for $f \in H^2(J)$,

$$\begin{aligned}
(GLf)(x) &= \int_J G(x,y)Lf(y)dy \\
&= \int_J G(x,y)\left(-a(y)\frac{d^2}{dy^2} + b(y)\frac{d}{dy} + c(y)\right)f(y)dy \\
&= \int_J \left[\frac{d}{dy}\left[a(y)G(x,y)\right]f'(y) + \left(-\frac{d}{dy}\left[b(y)G(x,y)\right]\ f + c(y)\right)f(y)\right]dy \\
&\quad - G(x,y)a(y)f'(y)|_{y=r}^{y=s} \\
&= -G(x,y)a(y)f'(y)|_{y=r}^{y=s} + \left[a(y)G(x,y)\right]_y f(y)|_{y=r}^{y=s} + \langle L_y^* G(x,y), f(y)\rangle \\
&= \left[a(y)G(x,y)\right]_y f(y)|_{y=r}^{y=s} - G(x,y)a(y)f'(y)|_{y=r}^{y=s} + f(x).
\end{aligned}$$

This proves Eq. (6.20).

3. Now suppose G is the Dirichlet Green's function for L. By Eq. (6.15),

$$[-a(y)G(x,y)]_y = \left(\frac{d}{dy}\frac{1}{W(y)}\right)\begin{cases} u(x)v(y) & \text{if} \quad x \le y \\ v(x)u(y) & \text{if} \quad x \ge y \end{cases}$$
$$+\frac{1}{W(y)}\begin{cases} u(x)v'(y) & \text{if} \quad x \le y \\ v(x)u'(y) & \text{if} \quad x \ge y \end{cases}$$

and hence the function H defined in Eq. (6.22) is more explicitly given by

$$H(x) = \frac{1}{W(s)}\left(u(x)v'(s)\right)h(s) - \frac{1}{W(r)}\left(v(x)u'(r)\right)h(r). \tag{6.24}$$

From this equation or the fact that $L_xG(x,r) = 0 = L_xG(x,s)$, H is is $L-$harmonic on J^0. Moreover, from Eq. (6.24),

$$H(r) = -\frac{1}{W(r)}\left(v(r)u'(r)\right)h(r) = \frac{1}{W(r)}\left(u(r)v'(r) - v(r)u'(r)\right)h(r) = h(r)$$

and

$$H(s) = \frac{1}{W(s)}\left(u(s)v'(s)\right)h(s) = \frac{1}{W(s)}\left(u(s)v'(s) - v(s)u'(s)\right)h(s) = h(s).$$

Therefore if f is defined by Eq. (6.21),

$$Lf = LGg - LH = g \text{ a.e. on } J^0$$

because $LG = I$ on $L^2(J)$ and

$$f|_{\partial J} = (Gg)|_{\partial J} + H|_{\partial J} = H|_{\partial J} = h$$

since $Gg \in H_0^2(J)$. ∎

Corollary 6.16 (Elliptic Regularity I)**.** *Suppose* $-\infty \le r_0 < s_0 \le \infty$, $J_0 := (r_0, s_0)$ *and* L *is as in Eq. (6.11) with the further assumption that* $a, b, c \in C^\infty(\mathbb{R})$. *If* $f \in C^2(J_0)$ *is a function such that* $g := Lf \in C^k(J_0)$ *for some* $k \ge 0$, *then* $f \in C^{k+2}(J_0)$.

Proof. Let $r < s$ be chosen so that $J := [r, s]$ is a bounded subinterval of J_0 and let G be a Green's function as in Definition 6.14. Since a, b, c are smooth, it follows from our general theory of ODE that $G(x,y) \in C^\infty(J \times J \setminus \Delta)$ where $\Delta = \{(x,x) : x \in J\}$ is the diagonal in $J \times J$. Now by Theorem 6.15,

$$f(x) = (Gg)(x) + \left\{G(x,y)a(y)f'(y) - [a(y)G(x,y)]_y f(y)\right\}\Big|_{y=r}^{y=s} \text{ for } x \in J^0.$$

Since

$$x \to \left\{G(x,y)a(y)f'(y) - [a(y)G(x,y)]_y f(y)\right\}\Big|_{y=r}^{y=s} \in C^\infty(J^0)$$

it suffices to show $Gg \in C^{k+2}(J^0)$. But this follows by examining the formula for $(Gg)''$ given on the right side of Eq. (6.18). ∎

In fact we have the following rather striking version of this result.

Theorem 6.17 (Hypoellipticity)**.** *Suppose* $-\infty \le r_0 < s_0 \le \infty$, $J_0 := (r_0, s_0)$ *and* L *is as in Eq. (6.11) with the further assumption that* $a, b, c \in C^\infty(\mathbb{R})$. *If* $u \in \mathcal{D}'(J_0)$ *is a generalized function such that* $v := Lu \in C^\infty(J_0)$, *then* $u \in C^\infty(J_0)$.

Proof. As in the proof of Corollary 6.16 let $r < s$ be chosen so that $J := [r, s]$ is a bounded subinterval of J_0 and let G be the Green's function constructed above.[3] Further suppose $\xi \in J^0$, $\theta \in C_c^\infty(J^0, [0,1])$ such that $\theta = 1$ in a neighborhood U of ξ and $\alpha \in C_c^\infty(V, [0,1])$ such that $\alpha = 1$ in a neighborhood V of ξ, see Figure 15.

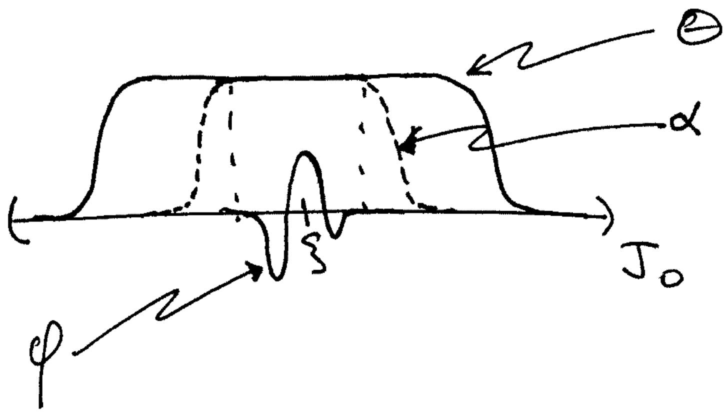

FIGURE 15. Constructing the cutoff functions, θ and α.

Finally suppose that $\phi \in C_c^\infty(V)$, then

$$\begin{aligned}\phi &= \theta\phi = \theta L^* G^* \phi = \theta L^* (M_\alpha + M_{1-\alpha}) G^* \phi \\ &= L^* M_\alpha G^* \phi + \theta L^* M_{1-\alpha} G^* \phi\end{aligned}$$

and hence

$$\begin{aligned}\langle u, \phi \rangle &= \langle u, L^* M_\alpha G^* \phi + \theta L^* M_{1-\alpha} G^* \phi \rangle \\ &= \langle Lu, M_\alpha G^* \phi \rangle + \langle u, \theta L^* M_{1-\alpha} G^* \phi \rangle.\end{aligned}$$

Now

$$\langle Lu, M_\alpha G^* \phi \rangle = \langle v, M_\alpha G^* \phi \rangle = \langle G M_\alpha v, \phi \rangle$$

and writing $u = D^n T_h$ for some continuous function h (which is always possible locally) we find

$$\begin{aligned}\langle u, \theta L^* M_{1-\alpha} G^* \phi \rangle &= (-1)^n \langle u, D^n M_\theta L^* M_{1-\alpha} G^* \phi \rangle \\ &= (-1)^n \int_{J \times J} h(x) D_x^n \left[\theta(x) L_x^* (1 - \alpha(x)) G(y, x)\right] \phi(y) dy dx \\ &= \int_J \psi(y) \phi(y) dy\end{aligned}$$

where

$$\psi(y) := \int_J h(x) D_x^n \left[\theta(x) L_x^* (1 - \alpha(x)) G(y, x)\right] dx$$

which is smooth for $y \in V$ because $1 - \alpha(x) = 0$ on V and so $(1 - \alpha(x)) G(y, x)$ is smooth for $(x, y) \in J \times V$. Putting this altogether shows

$$\langle u, \phi \rangle = \langle G M_\alpha v + \psi, \phi \rangle \text{ for all } \phi \in C_c^\infty(V).$$

[3] Actually we can simply define G^* to be a Green's function for L^*. It is not necessary to know $G^*(x, y) = G(y, x)$ where G is a Green's function for L.

That is to say $u = GM_\alpha v + \psi$ on V which proves the theorem since $GM_\alpha v + \psi \in C^\infty(V)$. ■

Example 6.18. Let $L = \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}$ be the wave operator on $\mathbb{R}^2$ which is not elliptic. Given $f \in C^2(\mathbb{R})$ we have already seen that $Lf(y-x) = 0 \in C^\infty(\mathbb{R}^2)$. Clearly since f was arbitrary, it does not follow that $F(x,y) := f(y-x) \in C^\infty(\mathbb{R}^2)$. Moreover, if f is merely continuous and $F(x,y) := f(y-x)$, then $LT_F = 0$ with $F \notin C^2(\mathbb{R}^2)$. To check $LT_F = 0$ we first observe

$$\begin{aligned} -\langle(\partial_x + \partial_y)\, T_F, \phi\rangle = \langle T_F, (\partial_x + \partial_y)\,\phi\rangle &= \int_{\mathbb{R}^2} f(y-x)\,(\partial_x + \partial_y)\,\phi(x,y)dxdy \\ &= \int_{\mathbb{R}^2} f(y)\,[\phi_x(x, y+x) + \phi_y(x, y+x)]\, dxdy \\ &= \int_{\mathbb{R}^2} f(y)\frac{\partial}{\partial x}\,[\phi(x, y+x)]\, dxdy = 0. \end{aligned}$$

Therefore $LT_F = (\partial_x - \partial_y)\,(\partial_x + \partial_y)\, T_F = 0$ as well.

Corollary 6.19. *Suppose a, b, c are smooth and $u \in \mathcal{D}'(J^0)$ is an eigenvector for L, i.e. $Lu = \lambda u$ for some $\lambda \in \mathbb{C}$. Then $u \in C^\infty(J)$.*

Proof. Since $L - \lambda$ is an elliptic ordinary differential operator and $(L-\lambda)\, u = 0 \in C^\infty(J^0)$, it follows by Theorem 6.17 that $u \in C^\infty(J^0)$. ■

6.3. **Elementary Sobolev Inequalities.**

Notation 6.20. Let $\overline{\int_J} f dm := \frac{1}{|J|}\int_J f dm$ denote the average of f over $J = [r,s]$.

Proposition 6.21. *For $f \in H^1(J)$,*

$$\begin{aligned} |f(x)| &\le \left|\overline{\int_J} f dm\right| + \|f'\|_{L^1(J)} \\ &\le \left|\overline{\int_J} f dm\right| + \sqrt{|J|}\left(\int_J |f'(y)|^2\, dy\right)^{1/2} \le C\,(|J|)\,\|f\|_{H^1(J)}\,. \end{aligned}$$

where $C\,(|J|) = \max\left(\frac{1}{\sqrt{|J|}}, \sqrt{|J|}\right)$.

Proof. By the fundamental theorem of calculus for absolutely continuous functions

$$f(x) = f(a) + \int_a^x f'(y)dy$$

for any $a, x \in J$. Integrating this equation on a and then dividing by $|J| := s - r$ implies

$$f(x) = \overline{\int_J} f dm + \overline{\int_J} da \int_a^x f'(y)dy$$

and hence

$$\begin{aligned}|f(x)| &\le \left|\overline{\int_J} f dm\right| + \overline{\int_J} da \left|\int_a^x |f'(y)|\, dy\right| \\ &\le \left|\overline{\int_J} f dm\right| + \int_J |f'(y)|\, dy \\ &\le \left|\overline{\int_J} f dm\right| + \sqrt{|J|}\left(\int_J |f'(y)|^2\, dy\right)^{1/2} \\ &\le \frac{1}{\sqrt{|J|}}\left(\int_J |f|^2\, dm\right)^{1/2} + \sqrt{|J|}\left(\int_J |f'(y)|^2\, dy\right)^{1/2}.\end{aligned}$$

■

Notation 6.22. For the remainder of this section, suppose $Lf = -\frac{1}{\rho}D\left(\rho a f'\right) + cf.$ is an elliptic ordinary differential operator on $J = [r, s]$, $\rho \in C^2(J, (0, \infty))$ is a **positive weight** and

$$(f, g)_\rho := \int_J f(x)g(x)\rho(x)dx.$$

We will also take $D(L) = H_0^2(J)$, so that we are imposing Dirichlet boundary conditions on L. Finally let

$$\mathcal{E}(f, g) := \int_J \left[af'g' + cfg\right]\rho dm \text{ for } f, g \in H^1(J).$$

Lemma 6.23. *For $f, g \in D(L)$,*

$$(Lf, g)_\rho = \mathcal{E}(f, g) = (f, Lg)_\rho. \tag{6.25}$$

Moreover

$$\mathcal{E}(f, f) \ge a_0 \left\|f'\right\|_2^2 + c_0 \left\|f\right\|_2^2 \text{ for all } f \in H^1(J)$$

where $c_0 := \min_J c$ and $a_0 = \min_J a$. If $\lambda_0 \in \mathbb{R}$ with $\lambda_0 + c_0 > 0$ then

$$\|f\|_{H^1(J)}^2 \le K\left[\mathcal{E}(f, f) + \lambda_0 \|f\|_2^2\right] \tag{6.26}$$

where $K = \left[\min(a_0, c_0 + \lambda_0)\right]^{-1}$.

Proof. Eq. (6.25) is a simple consequence of integration by parts. By elementary estimates

$$\mathcal{E}(f, f) \ge a_0 \left\|f'\right\|_2^2 + c_0 \left\|f\right\|_2^2$$

and

$$\mathcal{E}(f, f) + \lambda_0 \|f\|_2^2 \ge a_0 \left\|f'\right\|_2^2 + (c_0 + \lambda_0)\|f\|_2^2 \ge \min(a_0, c_0 + \lambda_0)\|f\|_{H^1(J)}^2$$

which proves Eq. (6.26). ■

Corollary 6.24. *Suppose $\lambda_0 + c_0 > 0$ then* $\mathrm{Nul}(L + \lambda_0) \cap D(L) = 0$ *and hence*

$$(L + \lambda_0) : H_0^2(J) \to L^2(J)$$

is invertible and the ***resolvent*** $(L + \lambda_0)^{-1}$ *has a continuous integral kernel $G(x, y)$, i.e.*

$$(L + \lambda_0)^{-1} u(x) = \int_J G(x, y)u(y)dy.$$

Moreover if we define $D(L^k)$ inductively by

$$D(L^k) := \left\{u \in D(L^{k-1}) : L^{k-1}u \in D(L)\right\}$$

we have $D(L^k) = H_0^{2k}(J)$.

Proof. By Lemma 6.23, for all $u \in D(L)$,

$$\|u\|_{H^1(J)}^2 \le K\left((Lu, u) + \lambda_0 \|u\|_2^2\right) = K\left(((L+\lambda_0)u, u)\right)$$

so that if $(L+\lambda_0)u = 0$, then $\|u\|_{H^1(J)}^2 = 0$ and hence $u = 0$. The remaining assertions except for $D(L^k) = H_0^k(J)$ now follow directly from Theorem 6.11 applied with L replaced by $L+\lambda_0$. Finally if $u \in D(L)$ then $(L+\lambda_0)u = Lu + \lambda_0 u \in L^2(J)$ and therefore

$$u = (L+\lambda_0)^{-1}(Lu + \lambda_0 u) \in H_0^2(J).$$

Now suppose we have shown, $D(L^k) = H_0^{2k}(J)$ and $u \in D\left(L^{k+1}\right)$, then

$$(L+\lambda_0)u = Lu + \lambda_0 u \in D(L^k) + D(L^{k+1}) \subset D(L^k) = H_0^{2k}(J)$$

and so by Theorem 6.11, $u \in (L+\lambda_0)^{-1} H_0^{2k}(J) \subset H_0^{2k+2}(J)$. ■

Corollary 6.25. *There exists an orthonormal basis* $\{\phi_n\}_{n=0}^\infty$ *for* $L^2(J, \rho dm)$ *of eigenfunctions of* L *with eigenvalues* $\lambda_n \in \mathbb{R}$ *such that* $-c_0 \le \lambda_0 < \lambda_1 < \lambda_2 < \dots$.

Proof. Let $\lambda_0 > -c_0$ and let $G := (L+\lambda_0)^{-1} : L^2(J) \to H_0^2(J) = D(L) \subset L^2(J)$. From the theory of compact operators to be developed later, G is a compact symmetric positive definite operator on $L^2(J)$ and hence there exists an orthonormal basis $\{\phi_n\}_{n=0}^\infty$ for $L^2(J, \rho dm)$ of eigenfunctions of G with eigenvalues $\mu_n > 0$ such that $\mu_0 \ge \mu_1 \ge \mu_2 \ge \dots \to 0$.[4] Since

$$\mu_n \phi_n = G\phi_n = (L+\lambda_0)^{-1}\phi_n,$$

it follows that $\mu_n (L+\lambda_0)\phi_n = \phi_n$ for all n and therefore $L\phi_n = \lambda_n \phi_n$ with $\lambda_n = (\mu_n^- 1 - \lambda_0) \uparrow \infty$. Finally since L is a second order ordinary differential equation there can be at most one linearly independent eigenvector for a given eigenvalue λ_n and hence $\lambda_n < \lambda_n + 1$ for all n. ■

Example 6.26. Let $J = [0, \pi]$, $\rho = 1$ and $L = -D^2$ on $H_0^2(J)$. Then $L\phi = \lambda\phi$ implies $\phi'' + \lambda\phi = 0$. Since L is positive, we need only consider the case where $\lambda \ge 0$ in which case $\phi(x) = a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)$. The boundary conditions for f imply $a = 0$ and $0 = \sin\left(\sqrt{\lambda}\pi\right)$, i.e. $\sqrt{\lambda} \in \mathbb{N}_+$. Therefore in this example

$$\phi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx) \text{ with } \lambda_k = k^2.$$

The collection of functions $\{\phi_k\}_{k=1}^\infty$ is an orthonormal basis for $L^2(J)$.

Theorem 6.27. *Let* $J = [r, s]$ *and* $\rho, a \in C^2(J, (0,\infty))$, $c \in C^2(J)$ *and* L *be defined by*

$$Lf = -\frac{1}{\rho} D(\rho a f') + cf.$$

and for $\lambda \in \mathbb{R}$ *let*

$$E^\lambda := \left\{\phi \in H_0^2(J) : L\phi = \alpha\phi \text{ for some } \alpha < \lambda\right\}.$$

[4]In fact G is "Hilbert Schmidt" which then implies

$$\sum_{n=0}^\infty \mu_n^2 < \infty.$$

Then there are constants $d_1, d_2 > 0$ *such that*

$$\dim(E^{\lambda}) \leq d_1\lambda + d_2. \tag{6.27}$$

Proof. For $\lambda \in \mathbb{R}$ let $E_\lambda := \left\{\phi \in H_0^2(J) : L\phi = \lambda\phi\right\}$. By Corollary 6.24, $E_\lambda = \{0\}$ if $\lambda < c_0$ and since $(Lf, g)_\rho = (f, Lg)_\rho$ for all $f, g \in H_0^2(J)$ it follows that $E_\lambda \perp E_\beta$ for all $\lambda \neq \beta$. Indeed, if $f \in E_\lambda$ and $g \in E_\beta$, then

$$(\beta - \lambda)(f, g)_\rho = (f, Lg)_\rho - (Lf, g)_\rho = 0.$$

Thus it follow that any finite dimensional subspace $W \subset E^{\lambda}$ has an orthonormal basis (relative to $(\cdot, \cdot)_\rho$ – inner product) of eigenvectors $\{\phi_k\}_{k=1}^n \subset E^{\lambda}$ of L, say $L\phi_k = \lambda_k\phi_k$. Let $u = \sum_{k=1}^n u_k\phi_k$ where $u_k \in \mathbb{R}$. By Proposition 6.21 and Lemma 6.23,

$$\|u\|_u^2 \leq C\|u\|_{H^1(J)}^2 \leq C\left((L + \lambda_0)u, u\right)_\rho = C\left(\sum_{k=1}^n u_k(\lambda_k + \lambda_0)\phi_k, u\right)_\rho$$

(where C is a constant varying from place to place but independent of u) and hence for any $x \in J$,

$$\left|\sum_{k=1}^n u_k\phi_k(x)\right|^2 \leq \|u\|_u^2 \leq C(\lambda + \lambda_0)\sum_{k=1}^n |u_k|^2.$$

Now choose $u_k = \phi_k(x)$ in this equation to find

$$\left|\sum_{k=1}^n |\phi_k(x)|^2\right|^2 \leq C(\lambda + \lambda_0)\sum_{k=1}^n |\phi_k(x)|^2$$

or equivalently that

$$\sum_{k=1}^n |\phi_k(x)|^2 \leq C(\lambda + \lambda_0).$$

Multiplying this equation by ρ and then integrating shows

$$\dim(W) = n = \sum_{k=1}^n (\phi_k, \phi_k)_\rho \leq C(\lambda + \lambda_0)\int_J \rho\, dm = C'(\lambda + \lambda_0).$$

Since $W \subset E^{\lambda}$ is arbitrary, it follows that

$$\dim(E^{\lambda}) \leq C'(\lambda + \lambda_0).$$

■

Remarks 6.28. Notice that for all $\lambda \in \mathbb{R}$, $\dim(E_\lambda) \leq 1$ because if $u, v \in E_\lambda$ then by uniqueness of solutions to ODE, $u = [u'(r)/v'(r)]\, v$. Let $\{\phi_k\}_{k=1}^\infty \subset H_0^2(J) \cap C^\infty(J)$ be the eigenvectors of L ordered so that the corresponding eigenvalues are increasing. With this ordering we have $k = \dim(E^{\lambda_k}) \leq d_1\lambda_k + d_2$ and therefore,

$$\lambda_k \geq d_1^{-1}(k - d_2). \tag{6.28}$$

The estimates in Eqs. (6.27) and (6.28) are not particularly good as Example 6.26 illustrates.

6.4. **Application to Heat and Wave Equations.**

Lemma 6.29. *L is a closed operator, i.e. if $s_n \in D(L)$ and $s_n \to s$ and $Ls_n \to g$ in L^2, then $s \in D(L)$ and $Ls = g$. In particular if $f_k \in D(L)$ and $\sum_{k=1}^\infty f_k$ and $\sum_{k=1}^\infty Lf_k$ exists in L^2, then $\sum_{k=1}^\infty f_k \in D(L)$ and*

$$L\sum_{k=1}^\infty f_k = \sum_{k=1}^\infty Lf_k.$$

Proof. Let $\lambda_0 + c_0 > 0$ and $G = (L+\lambda_0)^{-1}$. Then by assumption $(L+\lambda_0)s_n \to g + \lambda_0 s$ and so

$$s \leftarrow s_n = G(L+\lambda_0)s_n \to G(g+\lambda_0 s) \text{ as } n\to\infty$$

showing $s = Gg \in D(L+\lambda_0) = D(L)$ and

$$(L+\lambda_0)s = (L+\lambda_0)G(g+\lambda_0 s) = g + \lambda_0 s$$

and hence $Ls = g$ as desired. The assertions about the sums follow by applying the sequence results to $s_n = \sum_{k=1}^n f_k$. ■

Theorem 6.30. *Given $f \in L^2$, let*

$$u(t) = e^{-tL}f = \sum_{n=0}^\infty (f,\phi_n)e^{-t\lambda_n}\phi_n. \tag{6.29}$$

Then for $t>0$, $u(t,x)$ is smooth in (t,x) and solves the heat equation

$$u_t(t,x) = -Lu(t,x),\ u(t,x) = 0 \text{ for } x\in\partial J \tag{6.30}$$

$$\text{and } f = L^2 - \lim_{t\downarrow 0} u(t) \tag{6.31}$$

Moreover, $u(t,x) = \int_J p_t(x,y)f(y)\rho(y)dy$ where

$$p_t(x,y) := \sum_{n=0}^\infty e^{-t\lambda_n}\phi_n(x)\phi_n(y) \tag{6.32}$$

*is a smooth function in $t>0$ and $x,y\in J$. The function p_t is called the **Diurichlet Heat Kernel** for L.*

Proof. (Sketch.) For any $t>0$ and $k\in\mathbb{N}$, $\sup_n \left(e^{-t\lambda_n}\lambda_n^k\right) < \infty$ and so by Lemma 6.29, for $t>0$, $u(t) \in D\left(L^k\right) = H_0^{2k}(J)$[5] (Corollary 6.24) and

$$L^k u(t) = \sum_{n=0}^\infty (f,\phi_n)e^{-t\lambda_n}\lambda_n^k\phi_n.$$

Also we have $L^k u^{(m)}(t)$ exists in L^2 for all $k,m\in\mathbb{N}$ and

$$L^k u^{(m)}(t) = (-1)^m\sum_{n=0}^\infty (f,\phi_n)e^{-t\lambda_n}\lambda_n^{k+m}\phi_n.$$

By Sobolev inequalities and elliptic estimates such as Proposition 6.21 and Lemma 6.23, one concludes that $u \in C^\infty((0,\infty), H_0^k(J))$ for all k and then that $u \in$

[5]Basically, if $L^k u = g \in L^2(J)$ then $u = G^k g \in H_0^{2k}(J)$.

$C^\infty((0,\infty)\times J,\mathbb{R})$. Eq. (6.30) is now relatively easy to prove and Eq. (6.31) follows from the following computation

$$\|f-u(t)\|_2^2=\sum_{n=1}^\infty |(f,\phi_n)|^2\left|1-e^{-t\lambda_n}\right|^2$$

which goes to 0 as $t\downarrow 0$ by the D.C.T. for sums.

Finally from Eq. (6.29)

$$u(t,x)=\sum_{n=0}^\infty\int_J f(y)\phi(y)\rho(y)dy e^{-t\lambda_n}\phi_n(x)=\int_J\sum_{n=0}^\infty e^{-t\lambda_n}\phi_n(x)\phi(y)f(y)\rho(y)dy$$

where the interchange of the sum and the integral is permissible since

$$\int_J\sum_{n=0}^\infty e^{-t\lambda_n}\left|\phi_n(x)\phi(y)f(y)\right|\rho(y)dy\le C\int_J\sum_{n=0}^\infty e^{-t\lambda_n}\left(\lambda_0+\lambda_n\right)^2|f(y)|\rho(y)dy<\infty$$

since $\sum_{n=0}^\infty e^{-t\lambda_n}\left(\lambda_0+\lambda_n\right)^2<\infty$ because λ_n grows linearly in n. Moreover one similarly shows

$$\left(\frac{\partial}{\partial t}\right)^j\partial_x^{2k-1}\partial_y^{2l-1}p_t(x,y)=\sum_{n=0}^\infty\left(-\lambda_n\right)^j e^{-t\lambda_n}\partial_x^{2k-1}\phi_n(x)\partial_y^{2l-1}\phi(y)$$

where the above operations are permissible since

$$\left\|\phi_n^{(2k-1)}\right\|_u\le C\left\|\phi_n\right\|_{H_0^{2k}(J)}\le C\left\|\left(L+\lambda_0\right)^k\phi_n\right\|_2=C\left(\lambda_n+\lambda_0\right)^k$$

and therefore

$$\sum_{n=0}^\infty\left|\left(-\lambda_n\right)^j e^{-t\lambda_n}\partial_x^{2k-1}\phi_n(x)\partial_y^{2l-1}\phi(y)\right|\le C\sum_{n=0}^\infty|\lambda_n|^j\left(\lambda_n+\lambda_0\right)^{k+l}e^{-t\lambda_n}<\infty.$$

Again we use λ_n grows linearly with n. From this one may conclude that $p_t(x,y)$ is smooth for $t>0$ and $x,y\in J$. (We will do this in more detail when we work out the higher dimensional analogue.) ■

Remark 6.31 (Wave Equation). Suppose $f\in D(L^k)$, then

$$|(f,\phi_n)|=\left|\frac{1}{\lambda_n^k}(f,L^k\phi_n)\right|=\left|\frac{1}{\lambda_n^k}(L^kf,\phi_n)\right|\le\frac{1}{|\lambda_n^k|}$$

and therefore

$$\cos\left(t\sqrt{L}\right)f:=\sum_{n=0}^\infty\cos\left(t\sqrt{\lambda_n}\right)(f,\phi_n)\phi_n$$

will be convergent in L^2 but moreover

$$L^k\cos\left(t\sqrt{L}\right)f:=\sum_{n=0}^\infty\cos\left(t\sqrt{\lambda_n}\right)(f,\phi_n)\lambda_n^k\phi_n=\sum_{n=0}^\infty\cos\left(t\sqrt{\lambda_n}\right)(L^kf,\phi_n)\phi_n$$

will also be convergent. Therefore if we let

$$u(t)=\cos\left(t\sqrt{L}\right)f+\frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}g$$

where $f,g\in D(L^k)$ for all k. Then we will get a solution to the wave equation

$$u_{tt}(t,x)+Lu(t,x)=0\text{ with }u(0)=f\text{ and }\dot{u}(0)=g.$$

More on all of this later.

6.5. **Extensions to Other Boundary Conditions.** In this section, we will assume $\rho \in C^2(J,(0,\infty))$,

$$Lu = -\rho^{-1}(\rho a u')' + bu' + cu \tag{6.33}$$

is an elliptic ODE on $L^2(J)$ with smooth coefficients and

$$(u,v) = (u,v)_\rho = \int_J u(x)v(x)\rho(x)dx. \tag{6.34}$$

Theorem 6.32. *For $v \in H^2(J)$ let*

$$L^*v = -\rho^{-1}(\rho a v')' - bv' + \left[c - \rho^{-1}(\rho b)'\right] v. \tag{6.35}$$

Then for $u, v \in H^2(J)$,

$$(Lu, v) = (u, L^*v) + \mathcal{B}(u,v)|_{\partial J} \tag{6.36}$$

where

$$\mathcal{B}(u,v) = \rho a \left\{(u', u)\cdot(-v, v' + \frac{b}{a}v)\right\}. \tag{6.37}$$

Proof. This is an exercise in integration by parts,

$$\begin{aligned}
(Lu,v) &= \int_J \Big(-(\rho a u')' + \rho b u' + \rho c u\Big) v dm \\
&= \int_J \left(\rho a u' v' - (\rho b v)' u + \rho c u\right) dm + [\rho b u v - \rho a u' v]\,|_{\partial J} \\
&= \int_J \Big(-u\,(\rho a v')' - (\rho b v)' u + \rho c v u\Big) dm + [\rho b u v + \rho a u v' - \rho a u' v]\,|_{\partial J} \\
&= \int_J \Big(-u\rho^{-1}(\rho a v')' - \rho^{-1}(\rho b v)' u + c v u\Big) \rho dm + \left[\rho a\left(\frac{b}{a} u v + u v' - v u'\right)\right]|_{\partial J} \\
&= (u, L^*v) + \left[\rho a (u', u)\cdot(-v, v' + \frac{b}{a}v)\right]|_{\partial J}.
\end{aligned}$$

■

Notation 6.33. Given $(\alpha,\beta) : \partial J \to \mathbb{R}^2 \setminus \{0\}$ and $u, v \in H^2(J)$ let

$$Bu = \alpha u' + \beta u = (\alpha,\beta)\cdot(u',u) \text{ on } \partial J$$

and

$$B^*v = \alpha v' + \left(\beta + \frac{b}{a}\alpha\right) v = \alpha v' + \tilde{\beta} v \text{ on } \partial J$$

where $\tilde{\beta} := \left(\beta + \frac{b}{a}\alpha\right)$.

Remarks 6.34. The function $(\alpha,\tilde{\beta}) : \partial J \to \mathbb{R}^2$ also takes values in $\mathbb{R}^2 \setminus \{0\}$ because $(\alpha,\tilde{\beta}) = 0$ iff $(\alpha,\beta) = 0$. Furthermore if $\alpha = 0$ then $\tilde{\beta} = \beta$.

Proposition 6.35. *Let B and B^* be as defined in Notation 6.33 and define*

$$D(L) = \left\{u \in H^2(J) : Bu = 0 \text{ on } \partial J\right\}.$$
$$D(L^*) = \left\{u \in H^2(J) : B^*u = 0 \text{ on } \partial J\right\},$$

Then $v \in H^2(J)$ satisfies

$$(Lu, v) = (u, L^*v) \text{ for all } u \in D(L) \tag{6.38}$$

iff $v \in D(L^*)$.

Proof. We have to check that $\mathcal{B}(u,v)$ appearing in Eq. (6.36) is 0. (Actually we must check that $\mathcal{B}(u,v)|_{\partial J} = 0$ which we might arrange by using something like "periodic boundary conditions." I am not considering this type of condition at the moment. Since u may be chosen to be zero near r or s we must require $\mathcal{B}(u,v) = 0$ on ∂J.) Now $\mathcal{B}(u,v) = 0$ iff

$$(u', u) \cdot \left(-v, v' + \frac{b}{a} v\right) = 0 \tag{6.39}$$

which happens iff (u', u) is parallel to $\left(v' + \frac{b}{a} v, v\right)$. The boundary condition $Bu = 0$ may be rewritten as saying $(u', u) \cdot (\alpha, \beta) = 0$ or equivalently that (u', u) is parallel to $(-\beta, \alpha)$ on ∂J. Therefore the condition in Eq. (6.39) is equivalent to $(-\beta, \alpha)$ is parallel to $\left(v' + \frac{b}{a} v, v\right)$ or equivalently that

$$0 = (\alpha, \beta) \cdot \left(v' + \frac{b}{a} v, v\right) = B^* v.$$

■

Corollary 6.36. *The formulas for* L *and* L^* *agree iff* $b = 0$ *in which case*

$$Lu = -\rho^{-1} D\left(a \rho u'\right) + cu,$$

$B = B^*$, $D(L) = D(L^*)$ *and*

$$(Lu, v) = (u, Lv) \text{ for all } u, v \in D(L). \tag{6.40}$$

(In fact L *is a "self-adjoint operator," as we will see later by showing* $(L + \lambda_0)^{-1}$ *exists for* λ_0 *sufficiently large. Eq. (6.40) then may be used to deduce* $(L + \lambda_0)^{-1}$ *is a bounded self-adjoint operator with a symmetric Green's functions* G*.)*

6.5.1. *Dirichlet Forms Associated to* $(L, D(L))$. For the rest of this section let $a, b_1, b_2, c_0, \rho \in C^2(J)$, with $a > 0$ and $\rho > 0$ on J and for $u, v \in H^1(J)$, let

$$\mathcal{E}(u,v) := \int_J \left(a u' v' + b_1 u v' + b_2 u' v + c_0 u v\right) \rho dm \text{ and} \tag{6.41}$$

$$\|u\|_{H^1(J)} := \left(\|u'\|^2 + \|u\|^2\right)^{1/2}$$

where $\|u\|^2 = (u,u)_\rho$ as defined in Eq. (6.34).

Lemma 6.37 (A Coercive inequality for $\mathcal{E}$)**.** *There is a constant* $K < \infty$ *such that*

$$|\mathcal{E}(u,v)| \le K \|u\|_{H^1(J)} \|v\|_{H^1(J)} \text{ for } u, v \in H^1(J). \tag{6.42}$$

Let $a_0 = \min_J a$, $\bar{c} = \min_J c_0$ *and* $B := \max_J |b_1 + b_2|$, *then for* $u \in H^1(J)$,

$$\mathcal{E}(u,u) \ge \frac{a_0}{2} \|u'\|^2 + \left(\bar{c} - \frac{B^2}{2a_0}\right) \|u\|^2. \tag{6.43}$$

Proof. Let $A = \max_J a$, $B_i = \max_J |b_i|$ and $C_0 := \max_J |c_0|$, then

$$\begin{aligned} |\mathcal{E}(u,v)| &\le \int_J \left(a |u'| |v'| + |b_1| |u| |v'| + |b_2| |u'| |v| + |c_0| |u| |v|\right) \rho dm \\ &\le A \|u'\| \|v'\| + B_1 \|u\| \|v'\| + B_2 \|u'\| \|v\| + C_0 \|u\| \|v\| \\ &\le K \left(\|u'\|^2 + \|u\|^2\right)^{1/2} \left(\|v'\|^2 + \|v\|^2\right)^{1/2}. \end{aligned}$$

Let $a_0 = \min_J a$, $\bar{c} = \min_J c$ and $B := \max_J |b_1 + b_2|$, then for any $\delta > 0$,

$$\begin{aligned}\mathcal{E}(u,u) &= \int_J \left(a\left|u'\right|^2 + (b_1+b_2)\,uu' + c_0\left|u\right|^2\right)\rho dm \\ &\geq a_0\left\|u'\right\|^2 + \bar{c}\left\|u\right\|^2 - B\int_J |u|\left|u'\right|\rho dm \\ &\geq a_0\left\|u'\right\|^2 + \bar{c}\left\|u\right\|^2 - \frac{B}{2}\left(\delta\left\|u'\right\|^2 + \delta^{-1}\left\|u\right\|^2\right) \\ &= \left(a_0 - \frac{B\delta}{2}\right)\left\|u'\right\|^2 + \left(\bar{c} - \frac{B}{2}\delta^{-1}\right)\left\|u\right\|^2.\end{aligned}$$

Taking $\delta = a_0/B$ in this equation proves Eq. (6.43). ■

Theorem 6.38. *Let*

$$\begin{aligned} b &= (b_2 - b_1)\,,\ \ c := c_0 - \rho^{-1}\left(\rho b_1\right)', \\ Lu &= -\rho^{-1}\left(a\rho u'\right)' + bu' + cu \ \ \textit{and} \\ Bu &= \left.\left(\rho a u' + \rho b_1 u\right)\right|_{\partial J}. \end{aligned} \tag{6.44}$$

Then for $u \in H^2(J)$ *and* $v \in H^1(J)$

$$\mathcal{E}(u,v) = (Lu,v) + \left[(Bu)\,v\right]_{\partial J}$$

and for $u \in H^1(J)$ *and* $v \in H^2(J)$,

$$\mathcal{E}(u,v) = (u, L^*v) + \left[(B^*v)\,u\right]_{\partial J}.$$

Here (as in Eq. (6.35)

$$L^*v = -\rho^{-1}\left(a\rho u'\right)' - \rho^{-1}\left[\rho b u\right]' + cu$$

and (as in Notation 6.33)

$$B^*v = \rho a v' + \left(\rho b_1 + \frac{b}{a}\rho a\right)v = \rho a v' + \rho b_2 v.$$

Proof. Let $u \in H^2(J)$ and $v \in H^1(J)$ and integrating Eq. (6.41) by parts to find

$$\begin{aligned}\mathcal{E}(u,v) &= \int_J \left(-\rho^{-1}\left(a\rho u'\right)' v - \rho^{-1}\left(\rho b_1 u\right)' v + b_2 u' v + c_0 uv\right)\rho dm + \left[\rho a u' v + \rho b_1 uv\right]_{\partial J} \\ &= (Lu,v) + \left[Bu\cdot v\right]_{\partial J}\end{aligned} \tag{6.45}$$

where

$$\begin{aligned} Lu &= -\rho^{-1}\left(a\rho u'\right)' - \rho^{-1}\left(\rho b_1 u\right)' + b_2 u' + c_0 u \\ &= -\rho^{-1}\left(a\rho u'\right)' + (b_2 - b_1)\,u' + \left[c_0 - \rho^{-1}\left(\rho b_1\right)'\right]u \\ &= -\rho^{-1}\left(a\rho u'\right)' + bu' + cu \end{aligned}$$

and

$$Bu = \rho a u' + \rho b_1 u.$$

Similarly

$$\begin{aligned}\mathcal{E}(u,v) &= \int_J \left(-u\rho^{-1}\left(a\rho v'\right)' + b_1 uv' - u\rho^{-1}\left(\rho b_2 v\right)' + c_0 uv\right)\rho dm + \left[\left(\rho a u v' + \rho b_2 uv\right)\right]_{\partial J} \\ &= (u, L^{\dagger} v) + \left[B^{\dagger} v\cdot u\right]_{\partial J}\end{aligned}$$

where

$$\begin{aligned} L^\dagger v &= -\rho^{-1}\left(a\rho v'\right)' + b_1 v' - \rho^{-1}\left(\rho b_2 v\right)' + c_0 v \\ &= -\rho^{-1}\left(a\rho v'\right)' + \left(b_1 - b_2\right) v' + \left[c_0 - \rho^{-1}\left(\rho b_2\right)'\right] v \\ &= -\rho^{-1}\left(a\rho v'\right)' - bv' + \left[c + \rho^{-1}\left(\rho(b_1 - b_2)\right)^{-1}\right] v \\ &= -\rho^{-1}\left(a\rho v'\right)' - bv' + \left[c - \rho^{-1}\left(\rho b\right)'\right] v = L^* v. \end{aligned}$$

and

$$B^\dagger v = (\rho a v' + \rho b_2 v) = B^* v.$$

■

Remark 6.39. As a consequence of Theorem 6.38, the mapping

$$(a, b_1, b_2, c_0) \to \left[(u, v) \to \mathcal{E}(u, v) := \int_J \left(au'v' + b_1 uv' + b_2 u'v + c_0 uv\right) \rho dm\right]$$

is highly **non-**injective. In fact $\mathcal{E}$ depends only on a, $b = b_2 - b_1$ and $c := c_0 - \rho^{-1}\left(\rho b_1\right)'$ on J and b_1 on ∂J.

Corollary 6.40. *As above let* $(\alpha, \beta) : \partial J \to \mathbb{R}^2 \setminus \{0\}$ *and let*

$$\begin{aligned} D(L) &= \left\{u \in H^2(J) : Bu = \alpha u' + \beta u = 0 \text{ on } \partial J\right\} \text{ and} \\ Lu &= -\rho^{-1}\left(a\rho u'\right)' + bu' + cu. \end{aligned}$$

Given $\lambda_0 > 0$ *sufficiently large,* $(L + \lambda_0) : D(L) \to L^2(J)$ *is invertible and there is a continuous Green's function* $G(x, y)$ *such that*

$$(L + \lambda_0)^{-1} f(x) = \int_J G(x, y) f(y) dy.$$

Proof. Let us normalize α so that $\alpha = a$ whenever $\alpha \neq 0$. The boundary term in Eq. (6.45) will be zero whenever

$$au' + b_1 u = 0 \text{ when } v \neq 0 \text{ on } \partial J.$$

This suggests that we define a subspace χ of $H^1(J)$ by

$$\chi := \left\{u \in H^1(J) : u = 0 \text{ on } \partial J \text{ where } \alpha = 0 \text{ on } \partial J\right\}.$$

Hence χ is either $H^1_0(J)$, $H^1(J)$, $\left\{u \in H^1(J) : u(r) = 0\right\}$ or $\left\{u \in H^1(J) : u(s) = 0\right\}$. Now choose a function $b_1 \in C^2(J)$ such that $b_1 = \beta$ on ∂J, then set $b_2 := b + b_1$ and $c_0 = c + \rho^{-1}\left(\rho b_1\right)'$, then

$$D(L) = \chi \cap \left\{u \in H^2(J) : Bu = au' + b_1 u = 0 \text{ on } \partial J\right\}$$

and

$$(Lu, v) = \mathcal{E}(u, v) \text{ for all } u \in D(L) \text{ and } v \in \chi.$$

Using this observation, it follows from Eq. (6.43) of Lemma 6.37, for λ_0 sufficiently large and any $u \in D(L)$, that

$$\begin{aligned} \left(\left(L + \lambda_0\right) u, u\right) &= \mathcal{E}(u, u) + \lambda_0 (u, u) \\ &\geq \frac{a_0}{2}\left\|u'\right\|^2 + \left(\bar{c} - \frac{B^2}{2a_0} + \lambda_0\right)\|u\|^2 \geq \frac{a_0}{2}\|u\|^2_{H^1(J)}. \end{aligned}$$

As usual this equation shows $\mathrm{Nul}(L + \lambda_0) = \{0\}$. The remaining assertions are now proved as in the proof of Corollary 6.24.