12. Heat Equation

The heat equation for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ is the partial differential equation

$$\left(\partial_t - \frac{1}{2}\Delta\right) u = 0 \text{ with } u(0,x) = f(x), \tag{12.1}$$

where f is a given function on $\mathbb{R}^n$. By Fourier transforming Eq. (12.1) in the $x-$ variables only, one finds that (12.1) implies that

$$\left(\partial_t + \frac{1}{2}|\xi|^2\right) \hat{u}(t,\xi) = 0 \text{ with } \hat{u}(0,\xi) = \hat{f}(\xi). \tag{12.2}$$

and hence that $\hat{u}(t,\xi) = e^{-t|\xi|^2/2}\hat{f}(\xi)$. Inverting the Fourier transform then shows that

$$u(t,x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\hat{f}(\xi)\right)(x) = \left(\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right) \bigstar f\right)(x) =: e^{t\Delta/2}f(x).$$

From Example **??**,

$$\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right)(x) = p_t(x) = t^{-n/2}e^{-\frac{1}{2t}|x|^2}$$

and therefore,

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)\mathbf{d}y.$$

This suggests the following theorem.

Theorem 12.1. *Let*

$$p_t(x-y) := (2\pi t)^{-n/2}\, e^{-|x-y|^2/2t} \tag{12.3}$$

be the ***heat kernel*** *on* $\mathbb{R}^n$*. Then*

$$\left(\partial_t - \frac{1}{2}\Delta_x\right) p_t(x-y) = 0 \text{ and } \lim_{t\downarrow 0} p_t(x-y) = \delta_x(y), \tag{12.4}$$

where δ_x *is the* δ *– function at* x *in* $\mathbb{R}^n$*. More precisely, if* f *is a continuous bounded function on* $\mathbb{R}^n$*, then*

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)dy$$

is a solution to Eq. (12.1) where $u(0,x) := \lim_{t\downarrow 0} u(t,x)$*.*

Proof. Direct computations show that $\left(\partial_t - \frac{1}{2}\Delta_x\right) p_t(x-y) = 0$ and an application of Theorem **??** shows $\lim_{t\downarrow 0} p_t(x-y) = \delta_x(y)$ or equivalently that $\lim_{t\downarrow 0}\int_{\mathbb{R}^n} p_t(x-y)f(y)dy = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$. This shows that $\lim_{t\downarrow 0} u(t,x) = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$.

■

Proposition 12.2 (Properties of $e^{t\Delta/2}$)**.**

(1) *For* $f \in L^2(\mathbb{R}^n, dx)$*, the function*

$$\left(e^{t\Delta/2}f\right)(x) = (P_t f)(x) = \int_{\mathbb{R}^n} f(y)\frac{e^{-\frac{1}{2t}|x-y|^2}}{(2\pi t)^{n/2}}dy$$

is smooth in (t,x) *for* $t > 0$ *and* $x \in \mathbb{R}^n$ *and is in fact real analytic.*

(2) $e^{t\Delta/2}$ *acts as a contraction on* $L^p(\mathbb{R}^n, dx)$ *for all* $p \in [0,\infty]$ *and* $t > 0$*. Indeed,*

(3) *Moreover,* $p_t * f \to f$ *in* L^p *as* $t \to 0$*.*

Proof. Item 1. is fairly easy to check and is left the reader. One just notices that $p_t(x-y)$ analytically continues to $\operatorname{Re} t > 0$ and $x \in \mathbb{C}^n$ and then shows that it is permissible to differentiate under the integral.

Item 2.

$$|(p_t * f)(x)| \leq \int_{\mathbb{R}^n} |f(y)| p_t(x-y) dy$$

and hence with the aid of Jensen's inequality we have,

$$\|p_t * f\|_{L^p}^p \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)|^p p_t(x-y) dy dx = \|f\|_{L^p}^p$$

So P_t is a contraction $\forall t > 0$.

Item 3. It suffices to show, because of the contractive properties of $p_t *$, that $p_t * f \to f$ as $t \downarrow 0$ for $f \in C_c(\mathbb{R}^n)$. Notice that if f has support in the ball of radius R centered at zero, then

$$\begin{aligned} |(p_t * f)(x)| &\leq \int_{\mathbb{R}^n} |f(y)| P_t(x-y) dy \leq \|f\|_\infty \int_{|y| \leq R} P_t(x-y) dy \\ &= \|f\|_\infty C R^n e^{-\frac{1}{2t}(|x|-R)^2} \end{aligned}$$

and hence

$$\|p_t * f - f\|_{L^p}^p = \int_{|y| \leq R} |p_t * f - f|^p dy + \|f\|_\infty C R^n e^{-\frac{1}{2t}(|x|-R)^2}.$$

Therefore $p_t * f \to f$ in L^p as $t \downarrow 0 \quad \forall f \in C_c(\mathbb{R}^n)$. ■

Theorem 12.3 (Forced Heat Equation)**.** *Suppose $g \in C_b(\mathbb{R}^d)$ and $f \in C_b^{1,2}([0,\infty) \times \mathbb{R}^d)$ then*

$$u(t,x) := p_t * g(x) + \int_0^t p_{t-\tau} * f(\tau, x) d\tau$$

solves

$$\frac{\partial u}{\partial t} = \frac{1}{2} \triangle u + f \text{ with } u(0,\cdot) = g.$$

Proof. Because of Theorem 12.1, we may with out loss of generality assume $g = 0$ in which case

$$u(t,x) = \int_0^t p_t * f(t-\tau, x) d\tau.$$

Therefore

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= p_t * f(0,x) + \int_0^t p_\tau * \frac{\partial}{\partial t} f(t-\tau, x) d\tau \\ &= p_t * f_0(x) - \int_0^t p_\tau * \frac{\partial}{\partial \tau} f(t-\tau, x) d\tau \end{aligned}$$

and

$$\frac{\triangle}{2} u(t,x) = \int_0^t p_t * \frac{\triangle}{2} f(t-\tau, x) d\tau.$$

Hence we find, using integration by parts and approximate δ – function arguments, that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\frac{\triangle}{2}\right)u(t,x)&=p_t*f_0(x)+\int_0^t p_\tau*\left(-\frac{\partial}{\partial\tau}-\frac{1}{2}\triangle\right)f(t-\tau,x)d\tau\\&=p_t*f_0(x)+\lim_{\epsilon\downarrow0}\int_\epsilon^t p_\tau*\left(-\frac{\partial}{\partial\tau}-\frac{1}{2}\triangle\right)f(t-\tau,x)d\tau\\&=p_t*f_0(x)-\lim_{\epsilon\downarrow0}p_\tau*f(t-\tau,x)\big|_\epsilon^t\\&+\lim_{\epsilon\downarrow0}\int_\epsilon^t\left(\frac{\partial}{\partial\tau}-\frac{1}{2}\triangle\right)p_\tau*f(t-\tau,x)d\tau\\&=p_t*f_0(x)-p_t*f_0(x)+\lim_{\epsilon\downarrow0}\ p_\epsilon*f(t-\epsilon,x)=f(t,x).\end{aligned}$$

■

12.1. **Extensions of Theorem 12.1.**

Proposition 12.4. *Suppose $f:\mathbb{R}^n\to\mathbb{R}$ is a measurable function and there exists constants $c,C<\infty$ such that*

$$|f(x)|\le Ce^{\frac{c}{2}|x|^2}.$$

Then $u(t,x):=p_t*f(x)$ is smooth for $(t,x)\in(0,c^{-1})\times\mathbb{R}^n$ and for all $k\in\mathbb{N}$ and all multi-indices α,

$$D^\alpha\left(\frac{\partial}{\partial t}\right)^k u(t,x)=\left(D^\alpha\left(\frac{\partial}{\partial t}\right)^k p_t\right)*f(x). \tag{12.5}$$

In particular u satisfies the heat equation $u_t=\Delta u/2$ on $(0,c^{-1})\times\mathbb{R}^n$.

Proof. The reader may check that

$$D^\alpha\left(\frac{\partial}{\partial t}\right)^k p_t(x)=q(t^{-1},x)p_t(x)$$

where q is a polynomial in its variables. Let $x_0\in\mathbb{R}^n$ and $\epsilon>0$ be small, then for $x\in B(x_0,\epsilon)$ and any $\beta>0$,

$$\begin{aligned}|x-y|^2&=|x|^2-2|x||y|+|y|^2\ge|y|^2+|x|^2-\left(\beta^{-2}|x|^2+\beta^2|y|^2\right)\\&\ge\left(1-\beta^2\right)|y|^2-\left(\beta^{-2}-1\right)\left(|x_0|^2+\epsilon\right).\end{aligned}$$

Hence

$$\begin{aligned}g(y)&:=\sup\left\{\left|D^\alpha\left(\frac{\partial}{\partial t}\right)^k p_t(x-y)f(y)\right|:\epsilon\le t\le c-\epsilon\text{ and }x\in B(x_0,\epsilon)\right\}\\&\le\sup\left\{\left|q(t^{-1},x-y)\frac{e^{-\frac{1}{2t}|x-y|^2}}{(2\pi t)^{n/2}}Ce^{\frac{c}{2}|y|^2}\right|:\epsilon\le t\le c-\epsilon\text{ and }x\in B(x_0,\epsilon)\right\}\\&\le C(\beta,x_0,\epsilon)\sup\left\{\left|(2\pi t)^{-n/2}q(t^{-1},x-y)e^{\left[-\frac{1}{2t}\left(1-\beta^2\right)+\frac{c}{2}\right]|y|^2}\right|:\epsilon\le t\le c-\epsilon\text{ and }x\in B(x_0,\epsilon)\right\}.\end{aligned}$$

By choosing β close to 0, the reader should check using the above expression that for any $0<\delta<(1/t-c)/2$ there is a $\tilde{C}<\infty$ such that $g(y)\le\tilde{C}e^{-\delta|y|^2}$. In particular $g\in L^1(\mathbb{R}^n)$. Hence one is justified in differentiating past the integrals in p_t*f and this proves Eq. (12.5). ■

Lemma 12.5. *There exists a polynomial $q_n(x)$ such that for any $\beta > 0$ and $\delta > 0$,*

$$\int_{\mathbb{R}^n} 1_{|y|\geq\delta} e^{-\beta|y|^2} dy \leq \delta^n q_n\left(\frac{1}{\beta\delta^2}\right) e^{-\beta\delta^2}$$

Proof. Making the change of variables $y \to \delta y$ and then passing to polar coordinates shows

$$\int_{\mathbb{R}^n} 1_{|y|\geq\delta} e^{-\beta|y|^2} dy = \delta^n \int_{\mathbb{R}^n} 1_{|y|\geq 1} e^{-\beta\delta^2|y|^2} dy = \sigma\left(S^{n-1}\right)\delta^n \int_1^\infty e^{-\beta\delta^2 r^2} r^{n-1} dr.$$

Letting $\lambda = \beta\delta^2$ and $\phi_n(\lambda) := \int_{r=1}^\infty e^{-\lambda r^2} r^n dr$, integration by parts shows

$$\begin{aligned}\phi_n(\lambda) &= \int_{r=1}^\infty r^{n-1} d\left(\frac{e^{-\lambda r^2}}{-2\lambda}\right) = \frac{1}{2\lambda}e^{-\lambda} + \frac{1}{2}\int_{r=1}^\infty (n-1) r^{(n-2)} \frac{e^{-\lambda r^2}}{\lambda} dr \\ &= \frac{1}{2\lambda}e^{-\lambda} + \frac{n-1}{2\lambda}\phi_{n-2}(\lambda).\end{aligned}$$

Iterating this equation implies

$$\phi_n(\lambda) = \frac{1}{2\lambda}e^{-\lambda} + \frac{n-1}{2\lambda}\left(\frac{1}{2\lambda}e^{-\lambda} + \frac{n-3}{2\lambda}\phi_{n-4}(\lambda)\right)$$

and continuing in this way shows

$$\phi_n(\lambda) = e^{-\lambda} r_n(\lambda^{-1}) + \frac{(n-1)!!}{2^\delta \lambda^\delta}\phi_i(\lambda)$$

where δ is the integer part of $n/2$, $i = 0$ if n is even and $i = 1$ if n is odd and r_n is a polynomial. Since

$$\phi_0(\lambda) = \int_{r=1}^\infty e^{-\lambda r^2} dr \leq \phi_1(\lambda) = \int_{r=1}^\infty r e^{-\lambda r^2} dr = \frac{e^{-\lambda}}{2\lambda},$$

it follows that

$$\phi_n(\lambda) \leq e^{-\lambda} q_n(\lambda^{-1})$$

for some polynomial q_n. ■

Proposition 12.6. *Suppose $f \in C(\mathbb{R}^n, \mathbb{R})$ such that $|f(x)| \leq Ce^{\frac{c}{2}|x|^2}$ then $p_t * f \to f$ uniformly on compact subsets as $t \downarrow 0$. In particular in view of Proposition 12.4, $u(t,x) := p_t * f(x)$ is a solution to the heat equation with $u(0,x) = f(x)$.*

Proof. Let $M > 0$ be fixed and assume $|x| \leq M$ throughout. By uniform continuity of f on compact set, given $\epsilon > 0$ there exists $\delta = \delta(t) > 0$ such that $|f(x) - f(y)| \leq \epsilon$ if $|x - y| \leq \delta$ and $|x| \leq M$. Therefore, choosing $a > c/2$ sufficiently small,

$$\begin{aligned}|p_t * f(x) - f(x)| &= \left|\int p_t(y)\left[f(x-y) - f(x)\right] dy\right| \leq \int p_t(y)\left|f(x-y) - f(x)\right| dy \\ &\leq \epsilon \int_{|y|\leq\delta} p_t(y) dy + C\left(2\pi t\right)^{-n/2} \int_{|y|\geq\delta} [e^{\frac{c}{2}|x-y|^2} + e^{\frac{c}{2}|x|^2}] e^{-\frac{1}{2t}|y|^2} dy \\ &\leq \epsilon + \tilde{C}\left(2\pi t\right)^{-n/2} \int_{|y|\geq\delta} e^{-\left(\frac{1}{2t} - a\right)|y|^2} dy.\end{aligned}$$

So by Lemma 12.5, it follows that

$$|p_t * f(x) - f(x)| \leq \epsilon + \tilde{C}\left(2\pi t\right)^{-n/2} \delta^n q_n\left(\frac{1}{\beta\left(\frac{1}{2t} - a\right)^2}\right) e^{-\left(\frac{1}{2t} - a\right)\delta^2}$$

and therefore

$$\limsup_{t\downarrow 0}\sup_{|x|\leq M}|p_t * f(x) - f(x)| \leq \epsilon \to 0 \text{ as } \epsilon \downarrow 0.$$

■

Lemma 12.7. *If $q(x)$ is a polynomial on $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} p_t(x-y)q(y)dy = \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\Delta^n}{2^n}q(x).$$

Proof. Since

$$f(t,x) := \int_{\mathbb{R}^n} p_t(x-y)q(y)dy = \int_{\mathbb{R}^n} p_t(y)\sum a_{\alpha\beta}x^\alpha y^\beta dy = \sum C_\alpha(t)x^\alpha,$$

$f(t,x)$ is a polynomial in x of degree no larger than that of q. Moreover $f(t,x)$ solves the heat equation and $f(t,x) \to q(x)$ as $t \downarrow 0$. Since $g(t,x) := \sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\Delta^n}{2^n}q(x)$ has the same properties of f and Δ is a bounded operator when acting on polynomials of a fixed degree we conclude $f(t,x) = g(t,x)$. ■

Example 12.8. Suppose $q(x) = x_1x_2 + x_3^4$, then

$$\begin{aligned} e^{t\Delta/2}q(x) &= x_1x_2 + x_3^4 + \frac{t}{2}\Delta\left(x_1x_2 + x_3^4\right) + \frac{t^2}{2!\cdot 4}\Delta^2\left(x_1x_2 + x_3^4\right) \\ &= x_1x_2 + x_3^4 + \frac{t}{2}12x_3^2 + \frac{t^2}{2!\cdot 4}4! \\ &= x_1x_2 + x_3^4 + 6tx_3^2 + 3t^2. \end{aligned}$$

Proposition 12.9. *Suppose $f \in C^\infty(\mathbb{R}^n)$ and there exists a constant $C < \infty$ such that*

$$\sum_{|\alpha|=2N+2}|D^\alpha f(x)| \leq Ce^{C|x|^2},$$

then

$$(p_t * f)(x) = \text{“}e^{t\Delta/2}f(x)\text{”} = \sum_{k=0}^{N}\frac{t^k}{k!}\Delta^k f(x) + O(t^{N+1}) \text{ as } t\downarrow 0$$

Proof. Fix $x \in \mathbb{R}^n$ and let

$$f_N(y) := \sum_{|\alpha|\leq 2N+1}\frac{1}{\alpha!}D^\alpha f(x)y^\alpha.$$

Then by Taylor's theorem with remainder

$$|f(x+y) - f_N(y)| \leq C|y|^{2N+2}\sup_{t\in[0,1]}e^{C|x+ty|^2} \leq C|y|^{2N+2}e^{2C\left[|x|^2+|y|^2\right]} \leq \tilde{C}|y|^{2N+2}e^{2C|y|^2}$$

and thus

$$\begin{aligned} \left|\int_{\mathbb{R}^n} p_t(y)f(x+y)dy - \int_{\mathbb{R}^n} p_t(y)f_N(y)dy\right| &\leq \tilde{C}\int_{\mathbb{R}^n} p_t(y)|y|^{2N+2}e^{2C|y|^2}dy \\ &= \tilde{C}t^{N+1}\int_{\mathbb{R}^n} p_1(y)|y|^{2N+2}e^{2t^2C|y|^2}dy \\ &= O(t^{N+1}). \end{aligned}$$

Since $f(x+y)$ and $f_N(y)$ agree to order $2N+1$ for y near zero, it follows that

$$\int_{\mathbb{R}^n} p_t(y) f_N(y) dy = \sum_{k=0}^{N} \frac{t^k}{k!} \Delta^k f_N(0) = \sum_{k=0}^{N} \frac{t^k}{k!} \Delta_y^k f(x+y)|_{y=0} = \sum_{k=0}^{N} \frac{t^k}{k!} \Delta^k f(x)$$

which completes the proof. ■

12.2. **Representation Theorem and Regularity.** In this section, suppose that Ω is a bounded domain such that $\bar{\Omega}$ is a C^2 – submanifold with C^2 boundary and for $T > 0$ let $\Omega_T := (0,T) \times \Omega$, and

$$\Gamma_T := ([0,T] \times \partial\Omega) \cup (\{0\} \times \Omega) \subset \mathrm{bd}(\Omega_T) = ([0,T] \times \partial\Omega) \cup (\{0,T\} \times \Omega)$$

as in Figure 36 below.

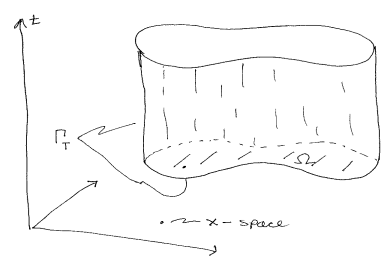

FIGURE 36. A cylindrical region Ω_T and the parabolic boundary Γ_T.

Theorem 12.10 (Representation Theorem)**.** *Suppose $u \in C^{2,1}(\bar{\Omega}_T)$ ($\bar{\Omega}_T = \bar{\Omega}_T = [0,T] \times \overline{\Omega}$) solves $u_t = \frac{1}{2} \triangle u + f$ on $\bar{\Omega}_T$. Then*

$$u(T,x) = \int_{\Omega} p_T(x-y) u(0,y) dy + \int_{[0,T]\times\Omega} p_{T-t}(x-y) f(t,y) dy dt$$

$$+ \frac{1}{2} \int_{[0,T]\times\partial\Omega} \left[\frac{\partial p_{T-t}}{\partial n_y}(x-y) u(t,y) - p_{T-t}(x-y) \frac{\partial u}{\partial n}(y) \right] d\sigma(y) dt \tag{12.6}$$

Proof. For $v \in C^{2,1}([0,T] \times \overline{\Omega})$, integration by parts shows

$$\begin{aligned} \int_{\Omega_T} f v dy dt &= \int_{\Omega_T} v(u_t - \frac{1}{2} \triangle v) dy dt \\ &= \int_{\Omega_T} (-v_t + \frac{1}{2} \nabla v \cdot \nabla u) dy dt + \int_{\Omega} v u \Big|_{t=0}^{t=T} dy + \frac{1}{2} \int_{[0,T]\times\partial\Omega} v \frac{\partial v}{\partial n} dt d\sigma \\ &= \int_{\Omega_T} (-v_t - \frac{1}{2} \triangle v) u dy dt + \int_{\Omega} v u \Big|_0^T dy + \frac{1}{2} \int_{[0,T]\times\partial\Omega} \left(\frac{\partial u}{\partial n} u - v \frac{\partial u}{\partial n} \right) d\sigma \ dt. \end{aligned}$$

Given $\epsilon > 0$, taking $v(t,y) := p_{T+\epsilon-t}(x-y)$ (note that $v_t + \frac{1}{2}\triangle v = 0$ and $v \in C^{2,1}([0,T]\times\Omega)$) implies

$$\int\limits_{[0,T]\times\Omega} f(t,y)p_{T+\epsilon-t}(x-y)dydt = 0 + \int_\Omega p_\epsilon(x-y)u(t,y)dy - \int\limits_\Omega p_{T+\epsilon}(x-y)u(t,y)dy$$
$$+\frac{1}{2}\int\limits_{[0,T]\times\partial\Omega}\left[\frac{\partial p_{T+\epsilon-t}(x-y)}{\partial n_y}u(t,y) - p_{T+\epsilon-t}(x-y)\frac{\partial u}{\partial n}(y)\right]d\sigma(y)dt$$

Let $\epsilon \downarrow 0$ above to complete the proof. ■

Corollary 12.11. *Suppose* $f := 0$ *so* $u_t(t,x) = \frac{1}{2}\triangle u(t,x)$. *Then* $u \in C^\infty((0,T)\times\Omega)$.

Proof. Extend $p_t(x)$ for $t \le 0$ by setting $p_t(x) := 0$ if $t \le 0$. It is not to hard to check that this extension is C^∞ on $\mathbb{R}\times\mathbb{R}^n\backslash\{0\}$. Using this notation we may write Eq. (12.6) as

$$u(t,x) = \int\limits_\Omega p_t(x-y)u(0,y)dy$$
$$+\frac{1}{2}\int\limits_{[0,\infty)\times\partial\Omega}\left[\frac{\partial p_{t-\tau}}{\partial n_y}(x-y)u(t,y) - p_{T-t}(x-y)\frac{\partial u}{\partial n}(y)\right]d\sigma(y)d\tau.$$

The result follows since now it permissible to differentiate under the integral to show $u \in C^\infty((0,T)\times\Omega)$. ■

Remark 12.12. Since $x \to p_t(x)$ is analytic one may show that $x \to u(t,x)$ is analytic for all $x \in \Omega$.

12.3. **Weak Max Principles.**

Notation 12.13. Let $a_{ij}, b_j \in C(\bar{\Omega}_T)$ satisfy $a_{ij} = a_{ji}$ and for $u \in C^2(\Omega)$ let

$$Lu(t,x) = \sum_{i,j=1}^n a_{ij}(t,x)u_{x_ix_j}(x) + \sum_{i=1}^n b_i(t,x)u_{x_i}(x). \tag{12.7}$$

We say L is **elliptic** if there exists $\theta > 0$ such that

$$\sum a_{ij}(t,x)\xi_i\xi_j \ge \theta|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } (t,x) \in \bar{\Omega}_T.$$

Assumption 3. In this section we assume L is elliptic. As an example $L = \frac{1}{2}\Delta$ is elliptic.

Lemma 12.14. *Let L be an elliptic operator as above and suppose $u \in C^2(\Omega)$ and $x_0 \in \Omega$ is a point where $u(x)$ has a local maximum. Then $Lu(t,x_0) \le 0$ for all $t \in [0,T]$.*

Proof. Fix $t \in [0,T]$ and set $B_{ij} = u_{x_ix_j}(x_0)$, $A_{ij} := a_{ij}(t,x_0)$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$ such that $Ae_i = \lambda_i e_i$. Notice that $\lambda_i \ge \theta > 0$ for all i. By the first derivative test, $u_{x_i}(x_0) = 0$ for all i and hence

$$\begin{aligned} Lu(t,x_0) &= \sum A_{ij}B_{ij} = \sum A_{ji}B_{ij} = \operatorname{tr}(AB) \\ &= \sum e_i \cdot ABe_i = \sum Ae_i \cdot Be_i = \sum_i \lambda_i e_i \cdot Be_i \\ &= \sum_i \lambda_i \partial_{e_i}^2 u(t,x_0) \le 0. \end{aligned}$$

The last inequality if a consequence of the second derivative test which asserts $\partial_v^2 u(t,x_0) \le 0$ for all $v \in \mathbb{R}^n$. ■

Theorem 12.15 (Elliptic weak maximum principle)**.** *Let Ω be a bounded domain and L be an elliptic operator as in Eq. (12.7). We now assume that a_{ij} and b_j are functions of x alone. For each $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ such that $Lu \ge 0$ on Ω (i.e. u is L – subharmonic) we have*

$$\max_{\bar{\Omega}} u \le \max_{\mathrm{bd}(\Omega)} u. \tag{12.8}$$

Proof. Let us first assume $Lu > 0$ on Ω. If u and had an interior local maximum at $x_0 \in \Omega$ then by Lemma 12.14, $Lu(x_0) \le 0$ which contradicts the assumption that $Lu(x_0) > 0$. So if $Lu > 0$ on Ω we conclude that Eq. (12.8) holds.

Now suppose that $Lu \ge 0$ on Ω. Let $\phi(x) := e^{\lambda x_1}$ with $\lambda > 0$, then

$$L\phi(x) = \left(\lambda^2 a_{11}(x) + b_1(x)\lambda\right) e^{\lambda x_1} \ge \lambda\left(\lambda\theta + b_1(x)\right) e^{\lambda x_1}.$$

By continuity of $b(x)$ we may choose λ sufficiently large so that $\lambda\theta + b_1(x) > 0$ on $\bar{\Omega}$ in which case $L\phi > 0$ on Ω. The results in the first paragraph may now be applied to $u_\epsilon(x) := u(x) + \epsilon\phi(x)$ (for any $\epsilon > 0$) to learn

$$u(x) + \epsilon\phi(x) = u_\epsilon(x) \le \max_{\mathrm{bd}(\Omega)} u_\epsilon \le \max_{\mathrm{bd}(\Omega)} u + \epsilon \max_{\mathrm{bd}(\Omega)} \phi \text{ for all } x \in \bar{\Omega}.$$

Letting $\epsilon \downarrow 0$ in this expression then implies

$$u(x) \le \max_{\mathrm{bd}(\Omega)} u \text{ for all } x \in \bar{\Omega}$$

which is equivalent to Eq. (12.8). ■

Theorem 12.16 (Parabolic weak maximum principle)**.** *Assume $u \in C^{1,2}(\overline{\Omega}_T \backslash \Gamma_T) \cap C(\overline{\Omega}_T)$.*

1. *If $u_t - Lu \le 0$ in Ω_T then*

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u. \tag{12.9}$$

2. *If $u_t - Lu \ge 0$ in Ω_T then $\min_{\overline{\Omega}_T} u = \min_{\Gamma_T} u$.*

Proof. Item 1. follows from Item 2. by replacing $u \to -u$, so it suffices to prove item 1. We begin by assuming $u_t - Lu < 0$ on $\bar{\Omega}_T$ and suppose for the sake of contradiction that there exists a point $(t_0, x_0) \in \overline{\Omega}_T \backslash \Gamma_T$ such that $u(t_0,x_0) = \max_{\overline{\Omega}_T} u$.

1. If $(t_0,x_0) \in \Omega_T$ (i.e. $0 < t_0 < T$) then by the first derivative test $\frac{\partial u}{\partial t}(t_0,x_0) = 0$ and by Lemma 12.14 $Lu(t_0,x_0) \le 0$. Therefore,

$$\left(u_t - Lu\right)(t_0,x_0) = -Lu(t_0,x_0) \ge 0$$

which contradicts the assumption that $u_t - Lu < 0$ in Ω_T.

(2) If $(t_0, x_0) \in \overline{\Omega}_T \backslash \Gamma_T$ with $t_0 = T$, then by the first derivative test, $\frac{\partial u}{\partial t}(T, x_0) \geq 0$ and by Lemma 12.14 $Lu(t_0, x_0) \leq 0$. So again

$$(u_t - Lu)(t_0, x_0) \geq 0$$

which contradicts the assumption that $u_t - Lu < 0$ in Ω_T.

Thus we have proved Eq. (12.9) holds if $u_t - Lu < 0$ on $\bar{\Omega}_T$. Finally if $u_t - Lu \leq 0$ on $\bar{\Omega}_T$ and $\epsilon > 0$, the function $u^\epsilon(t, x) := u(t, x) - \epsilon t$ satisfies $u_t^\epsilon - Lu^\epsilon \leq -\epsilon < 0$. Therefore by what we have just proved

$$u(t, x) - \epsilon t \leq \max_{\bar{\Omega}_T} u^\epsilon = \max_{\Gamma_T} u^\epsilon \leq \max_{\Gamma_T} u \text{ for all } (t, x) \in \bar{\Omega}_T.$$

Letting $\epsilon \downarrow 0$ in the last equation shows that Eq. (12.9) holds. ■

Corollary 12.17. *There is at most one solution $u \in C^{1,2}(\overline{\Omega}_T \backslash \Gamma_T) \cap C(\overline{\Omega}_T)$ to the partial differential equation*

$$\frac{\partial u}{\partial t} = Lu \text{ with } u = f \text{ on } \Gamma_T.$$

Proof. If there were another solution v, then $w := u - v$ would solve $\frac{\partial w}{\partial t} = Lw$ with $w = 0$ on Γ_T. So by the maximum principle in Theorem 12.16, $w = 0$ on $\bar{\Omega}_T$. ■

We now restrict back to $L = \frac{1}{2}\triangle$ and we wish to see what can be said when $\Omega = \mathbb{R}^n$ – an unbounded set.

Theorem 12.18. *Suppose $u \in C([0, T] \times \mathbb{R}^n) \cap C^{2,1}((0, T) \times \mathbb{R}^n)$,*

$$u_t - \frac{1}{2}\triangle u \leq 0 \text{ on } [0, T] \times \mathbb{R}^n$$

and there exists constants $A, a < \infty$ such that

$$u(t, x) \leq Ae^{a|x|^2} \text{ for } (t, x) \in (0, T) \times \mathbb{R}^n.$$

Then

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} u(t, x) \leq K := \sup_{x \in \mathbb{R}^n} u(0, x).$$

Proof. Recall that

$$p_t(x) = \left(\frac{1}{t}\right)^{n/2} e^{-\frac{1}{2t}|x|^2} = \left(\frac{1}{t}\right)^{n/2} e^{-\frac{1}{2t}x \cdot x}$$

solves the heat equation

$$\partial_t p_t(x) = \frac{1}{2}\triangle p_t(x). \tag{12.10}$$

Since both sides of Eq. (12.10) are analytic as functions in x, so[7]

$$\frac{\partial p_t}{\partial t}(ix) = \frac{1}{2}(\triangle p_t)(ix) = -\frac{1}{2}\triangle_x p_t(ix)$$

and therefore for all $\tau > 0$ and $t < \tau$

$$\frac{\partial p_{\tau - t}}{\partial t}(ix) = -\dot{p}_{\tau - t}(ix) = \frac{1}{2}\triangle_x p_{\tau - t}(ix).$$

[7]Similarly since both sides of Eq. (12.10) are analytic functions in t, it follows that

$$\frac{\partial}{\partial t} p_{-t}(x) = -\dot{p}_t(x) = -\frac{1}{2}\triangle p_{-t}.$$

That is to say the function

$$\rho(t,x) := p_{\tau-t}(ix) = \left(\frac{1}{\tau-t}\right)^{n/2} e^{\frac{1}{2(\tau-t)}|x|^2} \text{ for } 0 \le t < \tau$$

solves the heat equation. (This can be checked directly as well.)

Let $\epsilon, \tau > 0$ (to be chosen later) and set

$$v(t,x) = u(t,x) - \epsilon\rho(t,x) \text{ for } 0 \le t \le \tau/2.$$

Since $\rho(t,x)$ is increasing in t,

$$v(t,x) \le Ae^{a|x|^2} - \epsilon\left(\frac{1}{\tau}\right)^{n/2} e^{\frac{1}{2\tau}|x|^2} \text{ for } 0 \le t \le \tau/2.$$

Hence if we require $\frac{1}{2\tau} > a$ or $\tau < \frac{1}{2a}$ it will follows that

$$\lim_{|x|\to\infty}\left[\sup_{0\le t\le\tau/2} v(t,x)\right] = -\infty.$$

Therefore we may choose M sufficiently large so that

$$v(t,x) \le K := \sup_z u(0,z) \text{ for all } |x| \ge M \text{ and } 0 \le t \le \tau/2.$$

Since

$$\left(\partial_t - \frac{\triangle}{2}\right)v = \left(\partial_t - \frac{\triangle}{2}\right)u \le 0$$

we may apply the maximum principle with $\Omega = B(0,M)$ and $T = \tau/2$ to conclude for $(t,x) \in \Omega_T$ that

$$u(t,x) - \epsilon\rho(t,x) = v(t,x) \le \sup_{z\in\Omega} v(0,z) \le K \text{ if } 0 \le t \le \tau/2.$$

We may now let $\epsilon \downarrow 0$ in this equation to conclude that

$$u(t,x) \le K \text{ if } 0 \le t \le \tau/2. \tag{12.11}$$

By applying Eq. (12.11) to $u(t+\tau/2, x)$ we may also conclude

$$u(t,x) \le K \text{ if } 0 \le t \le \tau.$$

Repeating this argument then enables us to show $u(t,x) \le K$ for all $0 \le t \le T$. ■

Corollary 12.19. *The heat equation*

$$u_t - \frac{1}{2}\triangle u = 0 \text{ on } [0,T] \times \mathbb{R}^n \text{ with } u(0,\cdot) = f(\cdot) \in C(\mathbb{R}^n)$$

has at most one solution in the class of functions $u \in C([0,T]\times\mathbb{R}^n) \cap C^{2,1}((0,T)\times\mathbb{R}^n)$ *which satisfy*

$$u(t,x) \le Ae^{a|x|^2} \text{ for } (t,x) \in (0,T) \times \mathbb{R}^n$$

for some constants A *and* a.

Theorem 12.20 (Max Principle a la Hamilton)**.** *Suppose* $u \in C^{1,2}\left([0,T]\times\mathbb{R}^d\right)$ *satisfies*

1. $u(t,x) \le Ae^{a|x|^2}$ *for some* A, a *(for all* $t \le T$*)*
2. $u(0,x) \le 0$ *for all* x
3. $\frac{\partial u}{\partial t} \le \triangle u$ *i.e.* $(\partial_t - \triangle)u \le 0$.

Then $u(t,x) \le 0$ *for all* $(t,x) \in [0,T]\times\mathbb{R}^d$.

Proof. Special Case. Assume $\frac{\partial u}{\partial t} < \triangle u$ on $[0,T] \times \mathbb{R}^d$, $u(0,x) < 0$ for all $x \in \mathbb{R}^d$ and there exists $M > 0$ such that $u(t,x) < 0$ if $|x| \geq M$ and $t \in [0,T]$.For the sake of contradiction suppose there is some point $(t,x) \in [0,T] \times \mathbb{R}^d$ such that $u(t,x) > 0$.

By the intermediate value theorem there exists $\tau \in [0,t]$ such that $u(\tau,x) = 0$. In particular the set $\{u = 0\}$ is a non-empty closed compact subset of $(0,T] \times B(0,M)$. Let

$$\pi : (0,T] \times B(0,M) \to (0,T]$$

be projection onto the first factor, since $\{u \neq 0\}$ is a compact subset of $(0,T] \times B(0,M)$ if follows that

$$t_0 := \min\{t \in \pi(\{u = 0\})\} > 0.$$

Choose a point $x_0 \in B(0,M)$ such that $(t_0,x_0) \in \{u = 0\}$, i.e. $u(t_0,x_0) = 0$, see Figure 37 below. Since $u(t,x) < 0$ for all $0 \leq t < t_0$ and $x \in \mathbb{R}^d$, $u(t_0,x) \leq 0$

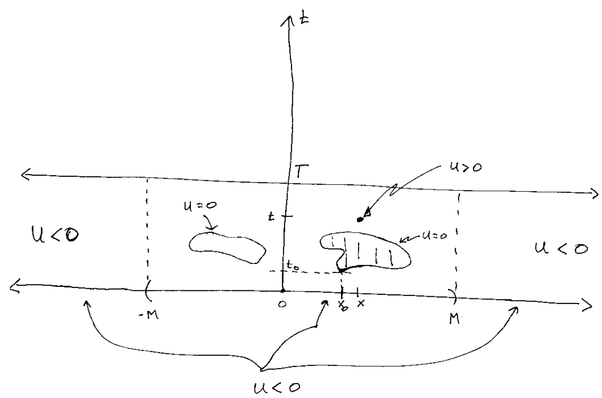

FIGURE 37. Finding a point (t_0,x_0) such that t_0 is as small as possible and $u(t_0,x_0) = 0$.

for all $x \in \mathbb{R}^d$ with $u(t_0,x_0) = 0$. This information along with the first and second derivative tests allows us to conclude

$$\nabla u(t_0,x_0) = 0,\ \triangle u(t_0,x_0) \leq 0 \text{ and } \frac{\partial u}{\partial t}(t_0,x_0) \geq 0.$$

This then implies that

$$0 \leq \frac{\partial u}{\partial t}(t_0,x_0) < \triangle u(t_0,x_0) \leq 0$$

which is absurd. Hence we conclude that $u \leq 0$ on $[0,T] \times \mathbb{R}^d$.

General Case: Let $p_t(x) = \frac{1}{t^{d/2}} e^{-\frac{1}{4t}|x|^2}$ be the fundamental solution to the heat equation

$$\partial_t p_t = \triangle p_t.$$

Let $\tau > 0$ to be determined later. As in the proof of Theorem 12.18, the function

$$\rho(t,x) := p_{\tau - t}(ix) = \left(\frac{1}{\tau - t}\right)^{d/2} e^{\frac{1}{4(\tau - t)}|x|^2} \text{ for } 0 \leq t < \tau$$

is still a solution to the heat equation. Given $\epsilon > 0$, define, for $t \le \tau/2$,

$$u_\epsilon(t,x) = u(t,x) - \epsilon - \epsilon t - \epsilon\rho(t,x).$$

Then

$$\begin{aligned}(\partial_t - \triangle)u_\epsilon &= (\partial_t - \triangle)u - \epsilon \le -\epsilon < 0,\\ u_\epsilon(0,x) &= u(0,x) - \epsilon \le 0 - \epsilon \le -\epsilon < 0\end{aligned}$$

and for $t \le \tau/2$

$$u_\epsilon(t,x) \le Ae^{a|x|^2} - \epsilon - \epsilon\frac{1}{\tau^{d/2}}e^{\frac{1}{4\tau}|x|^2}.$$

Hence if we choose τ such that $\frac{1}{4\tau} > a$, we will have $u_\epsilon(t,x) < 0$ for $|x|$ sufficiently large. Hence by the special case already proved, $u_\epsilon(t,x) \le 0$ for all $0 \le t \le \frac{\tau}{2}$ and $\epsilon > 0$. Letting $\epsilon \downarrow 0$ implies that $u(t,x) \le 0$ for all $0 \le t \le \tau/2$. As in the proof of Theorem 12.18 we may step our way up by applying the previous argument to $u(t+\tau/2,x)$ and then to $u(t+\tau,x)$, etc. to learn $u(t,x) \le 0$ for all $0 \le t \le T$. ■

12.4. Non-Uniqueness of solutions to the Heat Equation.

Theorem 12.21 (See Fritz John §7)**.** *For any $\alpha > 1$, let*

$$g(t) := \begin{cases} e^{-t^{-\alpha}} & t > 0\\ 0 & t \le 0\end{cases} \tag{12.12}$$

and define

$$u(t,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)x^{2k}}{(2k)!}.$$

Then $u \in C^\infty(\mathbb{R}^2)$ and

$$u_t = u_{xx} \text{ and } u(0,x) := 0. \tag{12.13}$$

In particular, the heat equation does not have unique solutions.

Proof. We are going to look for a solution to Eq. (12.13) of the form

$$u(t,x) = \sum_{n=0}^{\infty} g_n(t)x^n$$

in which case we have (formally) that

$$\begin{aligned}0 = u_t - u_{xx} &= \sum_{n=0}^{\infty}(\dot g_n(t)x^n - g_n(t)n(n-1)x^{n-2})\\ &= \sum_{n=0}^{\infty}\left[\dot g_n(t) - (n+2)(n+1)g_{n+2}(t)\right]x^n.\end{aligned}$$

This implies

$$g_{n+2} = \frac{\dot g_n}{(n+2)(n+1)}. \tag{12.14}$$

To simplify the final answer, we will now assume $u_x(0,x) = 0$, i.e. $g_1 \equiv 0$ in which case Eq. (12.14) implies $g_n \equiv 0$ for all n odd. We also have with $g := g_0$,

$$g_2 = \frac{\dot g_0}{2\cdot 1} = \frac{\dot g}{2!},\ g_4 = \frac{\dot g_2 0}{4\cdot 3} = \frac{\ddot g}{4!},\ g_6 = \frac{g^{(3)}}{6!}\ \dots\ g_{2k} = \frac{g^{(k)}}{(2k)!}$$

and hence

$$u(t,x)=\sum_{k=0}^{\infty}\frac{g^{(k)}(t)x^{2k}}{(2k)!}. \tag{12.15}$$

The function $u(t,x)$ will solve $u_t=u_{xx}$ for $(t,x)\in\mathbb{R}^2$ with $u(0,x)=0$ provided the convergence in the sum is adequate to justify the above computations.

Now let $g(t)$ be given by Eq. (12.12) and extend g to $\mathbb{C}\backslash(-\infty,0]$ via $g(z)=e^{-z^{-\alpha}}$ where

$$z^{-\alpha}=e^{-\alpha\log(z)}=e^{-\alpha(\ln r+i\theta)}\text{ for }z=re^{i\theta}\text{ with }-\pi<\theta<\pi.$$

In order to estimate $g^{(k)}(t)$ we will use of the Cauchy estimates on the contour $|z-t|=\gamma t$ where γ is going to be chosen sufficiently close to 0. Now

$$\operatorname{Re}(z^{-\alpha})=e^{-\alpha\ln r}\cos(\alpha\theta)=|z|^{-\alpha}\cos(\alpha\theta)$$

and hence

$$|g(z)|=e^{-\operatorname{Re}(z^{-\alpha})}=e^{-|z|^{-\alpha}\cos(\alpha\theta)}.$$

From Figure 38, we see

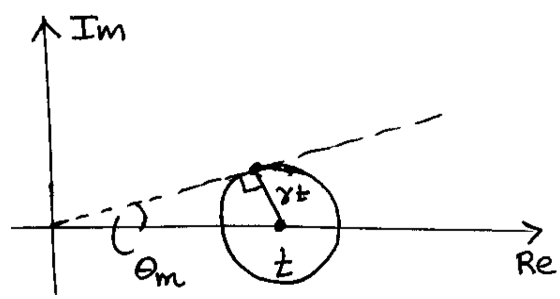

FIGURE 38. Here is a picture of the maximum argument θ_m that a point z on $\partial B(t,\gamma t)$ may attain. Notice that $\sin\theta_m=\gamma t/t=\gamma$ is independent of t and $\theta_m\to0$ as $\gamma\to0$.

$$\beta(\gamma):=\min\left\{\cos(\alpha\theta):-\pi<\theta<\pi\text{ and }|re^{i\theta}-t|=\gamma t\right\}$$

is independent of t and $\beta(\gamma)\to1$ as $\gamma\to0$. Therefore for $|z-t|=\gamma t$ we have

$$|g(z)|\le e^{-|z|^{-\alpha}\beta(\gamma)}\le e^{-([\gamma+1]t)^{-\alpha}\beta(\gamma)}=e^{-\frac{\beta(\gamma)}{1+\gamma}t^{-\alpha}}\le e^{-\frac{1}{2}t^{-\alpha}}$$

provided γ is chosen so small that $\frac{\beta(\gamma)}{1+\gamma}\ge\frac{1}{2}$.

By for $w\in B(t,t\gamma)$, the Cauchy integral formula and its derivative give

$$g(w)=\frac{1}{2\pi i}\oint_{|z-t|=\gamma t}\frac{g(z)}{z-w}\,dz\text{ and}$$
$$g^{(k)}(w)=\frac{k!}{2\pi i}\oint_{|z-t|=\gamma t}\frac{g(z)}{(z-w)^{k+1}}\,dz$$

and in particular

(12.16)

$$\left|g^{(k)}(t)\right|\le\frac{k!}{2\pi}\oint_{|z-t|=\gamma t}\frac{|g(z)|}{|z-w|^{k+1}}\,|dz|\le\frac{k!}{2\pi}e^{-\frac{1}{2}t^{-\alpha}}\frac{2\pi\gamma t}{|\gamma t|^{k+1}}=\frac{k!}{|\gamma t|^{k}}e^{-\frac{1}{2}t^{-\alpha}}.$$

We now use this to estimate the sum in Eq. (12.15) as

$$|u(t,x)| \le \sum_{k=0}^{\infty} \left| \frac{g^{(k)}(t)x^{2k}}{(2k)!} \right| \le e^{-\frac{1}{2}t^{-\alpha}} \sum_{k=0}^{\infty} \frac{k!}{(\gamma t)^k} \frac{|x|^{2k}}{(2k)!}$$
$$\le e^{-\frac{1}{2}t^{-\alpha}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{\gamma t} \right)^k = \exp\left(\frac{x^2}{\gamma t} - \frac{1}{2}t^{-\alpha} \right) < \infty.$$

Therefore $\lim_{t \downarrow 0} u(t,x) = 0$ uniformly for x in compact subsets of $\mathbb{R}$. Similarly one may use the estimate in Eq. (12.16) to show u is smooth and

$$u_{xx} = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)(2k)(2k-1)x^{2k-2}}{(2k)!} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)x^{2(k-1)}}{(2(k-1))!}$$
$$= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)x^{2k}}{(2k)!} = u_t.$$

■

12.5. **The Heat Equation on the Circle and $\mathbb{R}$.** In this subsection, let $S_L := \{Lz : z \in S\}$ – be the circle of radius L. As usual we will identify functions on S_L with $2\pi L$ – periodic functions on $\mathbb{R}$. Given two $2\pi L$ periodic functions f, g, let

$$(f,g)_L := \frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(x)\bar{g}(x)dx$$

and denote $H_L := L^2_{2\pi L}$ to be the $2\pi L$ – periodic functions f on $\mathbb{R}$ such that $(f,f)_L < \infty$. By Fourier's theorem we know that the functions $\chi_k^L(x) := e^{ikx/L}$ with $k \in \mathbb{Z}$ form an orthonormal basis for H_L and this basis satisfies

$$\frac{d^2}{dx^2}\chi_k^L = -\left(\frac{k}{L}\right)^2 \chi_k^L.$$

Therefore the solution to the heat equation on S_L,

$$u_t = \frac{1}{2}u_{xx} \text{ with } u(0,\cdot) = f \in H_L$$

is given by

$$u(t,x) = \sum_{k \in \mathbb{Z}} (f, \chi_k^L) e^{-\frac{1}{2}\left(\frac{k}{L}\right)^2 t} e^{ikx/L}$$
$$= \sum_{k \in \mathbb{Z}} \left(\frac{1}{2\pi L} \int_{-\pi L}^{\pi L} f(y) e^{-iky/L} dy \right) e^{-\frac{1}{2}\left(\frac{k}{L}\right)^2 t} e^{ikx/L}$$
$$= \int_{-\pi L}^{\pi L} p_t^L(x-y) f(y) dy$$

where

$$p_t^L(x) = \frac{1}{2\pi L} \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}\left(\frac{k}{L}\right)^2 t} e^{ikx/L}.$$

If f is L periodic then it is nL – periodic for all $n \in \mathbb{N}$, so we also would learn

$$u(t,x) = \int_{-\pi nL}^{\pi nL} p_t^{nL}(x-y) f(y) dy \text{ for all } n \in \mathbb{N}.$$

this suggest that we might pass to the limit as $n \to \infty$ in this equation to learn

$$u(t,x) = \int_{\mathbb{R}} p_t(x-y) f(y) dy$$

where

$$p_t(x) := \lim_{n\to\infty} p_t^{nL}(x) = \lim_{L\to\infty} \frac{1}{2\pi L} \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}\left(\frac{k}{L}\right)^2 t} e^{i\left(\frac{k}{L}\right)x}$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{1}{2}\xi^2 t} e^{i\xi x} d\xi = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

From this we conclude

$$u(t,x) = \int_{\mathbb{R}} p_t(x-y) f(y) dy = \int_{-\pi L}^{\pi L} \sum_{n\in\mathbb{Z}} p_t(x-y+2\pi n L) f(y) dy$$

and we arrive at the identity

$$\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+2\pi n L)^2}{2t}} = \sum_{n\in\mathbb{Z}} p_t(x+2\pi n L) = \frac{1}{2\pi L} \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}\left(\frac{k}{L}\right)^2 t} e^{ikx/L}$$

which is a special case of Poisson's summation formula.