5. Ordinary Differential Equations in a Banach Space

Let X be a Banach space, $U \subset_o X$, $J = (a, b) \ni 0$ and $Z \in C(J \times U, X)$ – Z is to be interpreted as a time dependent vector-field on $U \subset X$. In this section we will consider the ordinary differential equation (ODE for short)

$$\dot{y}(t) = Z(t, y(t)) \text{ with } y(0) = x \in U. \tag{5.1}$$

The reader should check that any solution $y \in C^1(J, U)$ to Eq. (5.1) gives a solution $y \in C(J, U)$ to the integral equation:

$$y(t) = x + \int_0^t Z(\tau, y(\tau))d\tau \tag{5.2}$$

and conversely if $y \in C(J, U)$ solves Eq. (5.2) then $y \in C^1(J, U)$ and y solves Eq. (5.1).

Remark 5.1. For notational simplicity we have assumed that the initial condition for the ODE in Eq. (5.1) is taken at $t = 0$. There is no loss in generality in doing this since if $\tilde{y}$ solves

$$\frac{d\tilde{y}}{dt}(t) = \tilde{Z}(t, \tilde{y}(t)) \text{ with } \tilde{y}(t_0) = x \in U$$

iff $y(t) := \tilde{y}(t + t_0)$ solves Eq. (5.1) with $Z(t, x) = \tilde{Z}(t + t_0, x)$.

5.1. Examples.

Let $X = \mathbb{R}$, $Z(x) = x^n$ with $n \in \mathbb{N}$ and consider the ordinary differential equation

$$\dot{y}(t) = Z(y(t)) = y^n(t) \text{ with } y(0) = x \in \mathbb{R}. \tag{5.3}$$

If y solves Eq. (5.3) with $x \neq 0$, then $y(t)$ is not zero for t near 0. Therefore up to the first time y possibly hits 0, we must have

$$t = \int_0^t \frac{\dot{y}(\tau)}{y(\tau)^n} d\tau = \int_0^{y(t)} u^{-n} du = \begin{cases} \frac{[y(t)]^{1-n} - x^{1-n}}{1-n} & \text{if} \quad n > 1 \\ \ln\left|\frac{y(t)}{x}\right| & \text{if} \quad n = 1 \end{cases}$$

and solving these equations for $y(t)$ implies

$$y(t) = y(t, x) = \begin{cases} \frac{x}{\sqrt[n-1]{1-(n-1)tx^{n-1}}} & \text{if} \quad n > 1 \\ e^t x & \text{if} \quad n = 1. \end{cases} \tag{5.4}$$

The reader should verify by direct calculation that $y(t, x)$ defined above does indeed solve Eq. (5.3). The above argument shows that these are the only possible solutions to the Equations in (5.3).

Notice that when $n = 1$, the solution exists for all time while for $n > 1$, we must require

$$1 - (n-1)tx^{n-1} > 0$$

or equivalently that

$$t < \frac{1}{(1-n)x^{n-1}} \text{ if } x^{n-1} > 0 \text{ and}$$
$$t > -\frac{1}{(1-n)|x|^{n-1}} \text{ if } x^{n-1} < 0.$$

Moreover for $n>1$, $y(t,x)$ blows up as t approaches the value for which $1-(n-1)tx^{n-1}=0$. The reader should also observe that, at least for s and t close to 0,

$$y(t,y(s,x))=y(t+s,x) \tag{5.5}$$

for each of the solutions above. Indeed, if $n=1$ Eq. (5.5) is equivalent to the well know identity, $e^te^s=e^{t+s}$ and for $n>1$,

$$\begin{aligned}
y(t,y(s,x)) &= \frac{y(s,x)}{\sqrt[n-1]{1-(n-1)ty(s,x)^{n-1}}}\\
&= \frac{\frac{x}{\sqrt[n-1]{1-(n-1)sx^{n-1}}}}{\sqrt[n-1]{1-(n-1)t\left[\frac{x}{\sqrt[n-1]{1-(n-1)sx^{n-1}}}\right]^{n-1}}}\\
&= \frac{\frac{x}{\sqrt[n-1]{1-(n-1)sx^{n-1}}}}{\sqrt[n-1]{1-(n-1)t\frac{x^{n-1}}{1-(n-1)sx^{n-1}}}}\\
&= \frac{x}{\sqrt[n-1]{1-(n-1)sx^{n-1}-(n-1)tx^{n-1}}}\\
&= \frac{x}{\sqrt[n-1]{1-(n-1)(s+t)x^{n-1}}}=y(t+s,x).
\end{aligned}$$

Now suppose $Z(x)=|x|^\alpha$ with $0<\alpha<1$ and we now consider the ordinary differential equation

$$\dot{y}(t)=Z(y(t))=|y(t)|^\alpha \text{ with } y(0)=x\in\mathbb{R}. \tag{5.6}$$

Working as above we find, if $x\neq 0$ that

$$t=\int_0^t \frac{\dot{y}(\tau)}{|y(t)|^\alpha}d\tau=\int_0^{y(t)}|u|^{-\alpha}\,du=\frac{[y(t)]^{1-\alpha}-x^{1-\alpha}}{1-\alpha},$$

where $u^{1-\alpha}:=|u|^{1-\alpha}\operatorname{sgn}(u)$. Since $\operatorname{sgn}(y(t))=\operatorname{sgn}(x)$ the previous equation implies

$$\begin{aligned}
\operatorname{sgn}(x)(1-\alpha)t &= \operatorname{sgn}(x)\left[\operatorname{sgn}(y(t))\,|y(t)|^{1-\alpha}-\operatorname{sgn}(x)\,|x|^{1-\alpha}\right]\\
&= |y(t)|^{1-\alpha}-|x|^{1-\alpha}
\end{aligned}$$

and therefore,

$$y(t,x)=\operatorname{sgn}(x)\left(|x|^{1-\alpha}+\operatorname{sgn}(x)(1-\alpha)t\right)^{\frac{1}{1-\alpha}} \tag{5.7}$$

is uniquely determined by this formula until the first time t where $|x|^{1-\alpha}+\operatorname{sgn}(x)(1-\alpha)t=0$. As before $y(t)=0$ is a solution to Eq. (5.6), however it is far from being the unique solution. For example letting $x\downarrow 0$ in Eq. (5.7) gives a function

$$y(t,0+)=((1-\alpha)t)^{\frac{1}{1-\alpha}}$$

which solves Eq. (5.6) for $t>0$. Moreover if we define

$$y(t):=\begin{cases}((1-\alpha)t)^{\frac{1}{1-\alpha}} & \text{if } t>0\\ 0 & \text{if } t\leq 0\end{cases},$$

(for example if $\alpha = 1/2$ then $y(t) = \frac{1}{4}t^2 1_{t\geq 0}$) then the reader may easily check y also solve Eq. (5.6). Furthermore, $y_a(t) := y(t-a)$ also solves Eq. (5.6) for all $a \geq 0$, see Figure 11 below.

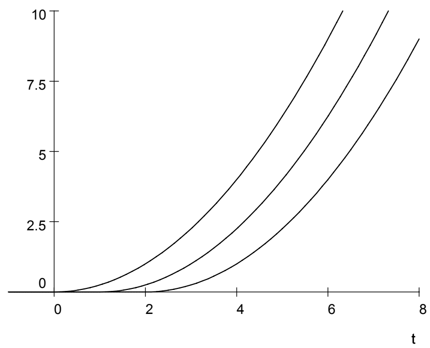

FIGURE 11. Three different solutions to the ODE $\dot{y}(t) = |y(t)|^{1/2}$ with $y(0) = 0$.

With these examples in mind, let us now go to the general theory starting with linear ODEs.

5.2. **Linear Ordinary Differential Equations.** Consider the linear differential equation

$$\dot{y}(t) = A(t)y(t) \text{ where } y(0) = x \in X. \tag{5.8}$$

Here $A \in C(J \to L(X))$ and $y \in C^1(J \to X)$. This equation may be written in its equivalent (as the reader should verify) integral form, namely we are looking for $y \in C(J, X)$ such that

$$y(t) = x + \int_0^t A(\tau)y(\tau)d\tau. \tag{5.9}$$

In what follows, we will abuse notation and use $\|\cdot\|$ to denote the operator norm on $L(X)$ associated to $\|\cdot\|$ on X we will also fix $J = (a, b) \ni 0$ and let $\|\phi\|_\infty := \max_{t\in J} \|\phi(t)\|$ for $\phi \in BC(J, X)$ or $BC(J, L(X))$.

Notation 5.2. For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, let

$$\Delta_n(t) = \begin{cases} \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n : 0 \leq \tau_1 \leq \dots \leq \tau_n \leq t\} & \text{if} \quad t \geq 0 \\ \{(\tau_1, \dots, \tau_n) \in \mathbb{R}^n : t \leq \tau_n \leq \dots \leq \tau_1 \leq 0\} & \text{if} \quad t \leq 0 \end{cases}$$

and also write $d\tau = d\tau_1 \dots d\tau_n$ and

$$\int_{\Delta_n(t)} f(\tau_1, \dots \tau_n)d\tau : = (-1)^{n\cdot 1_{t<0}} \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots \int_0^{\tau_2} d\tau_1 f(\tau_1, \dots \tau_n).$$

Lemma 5.3. *Suppose that $\psi \in C(\mathbb{R}, \mathbb{R})$, then*

$$(-1)^{n\cdot 1_{t<0}} \int_{\Delta_n(t)} \psi(\tau_1) \dots \psi(\tau_n)d\tau = \frac{1}{n!} \left(\int_0^t \psi(\tau)d\tau \right)^n. \tag{5.10}$$

Proof. Let $\Psi(t) := \int_0^t \psi(\tau)d\tau$. The proof will go by induction on n. The case $n = 1$ is easily verified since

$$(-1)^{1\cdot 1_{t<0}} \int_{\Delta_1(t)} \psi(\tau_1)d\tau_1 = \int_0^t \psi(\tau)d\tau = \Psi(t).$$

Now assume the truth of Eq. (5.10) for $n-1$ for some $n \geq 2$, then

$$\begin{aligned}(-1)^{n\cdot 1_{t<0}} \int_{\Delta_n(t)} \psi(\tau_1)\dots\psi(\tau_n)d\tau &= \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1}\dots\int_0^{\tau_2} d\tau_1 \psi(\tau_1)\dots\psi(\tau_n) \\ &= \int_0^t d\tau_n \frac{\Psi^{n-1}(\tau_n)}{(n-1)!}\psi(\tau_n) = \int_0^t d\tau_n \frac{\Psi^{n-1}(\tau_n)}{(n-1)!}\dot{\Psi}(\tau_n) \\ &= \int_0^{\Psi(t)} \frac{u^{n-1}}{(n-1)!}du = \frac{\Psi^n(t)}{n!},\end{aligned}$$

wherein we made the change of variables, $u = \Psi(\tau_n)$, in the second to last equality. ■

Remark 5.4. Eq. (5.10) is equivalent to

$$\int_{\Delta_n(t)} \psi(\tau_1)\dots\psi(\tau_n)d\tau = \frac{1}{n!}\left(\int_{\Delta_1(t)} \psi(\tau)d\tau\right)^n$$

and another way to understand this equality is to view $\int_{\Delta_n(t)} \psi(\tau_1)\dots\psi(\tau_n)d\tau$ as a multiple integral (see Section 8 below) rather than an iterated integral. Indeed, taking $t > 0$ for simplicity and letting S_n be the permutation group on $\{1, 2, \dots, n\}$ we have

$$[0,t]^n = \cup_{\sigma\in S_n}\{(\tau_1,\dots,\tau_n) \in \mathbb{R}^n : 0 \leq \tau_{\sigma 1} \leq \dots \leq \tau_{\sigma n} \leq t\}$$

with the union being "essentially" disjoint. Therefore, making a change of variables and using the fact that $\psi(\tau_1)\dots\psi(\tau_n)$ is invariant under permutations, we find

$$\begin{aligned}\left(\int_0^t \psi(\tau)d\tau\right)^n &= \int_{[0,t]^n} \psi(\tau_1)\dots\psi(\tau_n)d\tau \\ &= \sum_{\sigma\in S_n} \int_{\{(\tau_1,\dots,\tau_n)\in\mathbb{R}^n : 0\leq\tau_{\sigma 1}\leq\dots\leq\tau_{\sigma n}\leq t\}} \psi(\tau_1)\dots\psi(\tau_n)d\tau \\ &= \sum_{\sigma\in S_n} \int_{\{(s_1,\dots,s_n)\in\mathbb{R}^n : 0\leq s_1\leq\dots\leq s_n\leq t\}} \psi(s_{\sigma^{-1}1})\dots\psi(s_{\sigma^{-1}n})d\mathbf{s} \\ &= \sum_{\sigma\in S_n} \int_{\{(s_1,\dots,s_n)\in\mathbb{R}^n : 0\leq s_1\leq\dots\leq s_n\leq t\}} \psi(s_1)\dots\psi(s_n)d\mathbf{s} \\ &= n!\int_{\Delta_n(t)} \psi(\tau_1)\dots\psi(\tau_n)d\tau.\end{aligned}$$

Theorem 5.5. *Let $\phi \in BC(J,X)$, then the integral equation*

$$y(t) = \phi(t) + \int_0^t A(\tau)y(\tau)d\tau \tag{5.11}$$

has a unique solution given by

$$y(t) = \phi(t) + \sum_{n=1}^{\infty} (-1)^{n\cdot 1_{t<0}} \int_{\Delta_n(t)} A(\tau_n)\dots A(\tau_1)\phi(\tau_1)d\tau \tag{5.12}$$

and this solution satisfies the bound

$$\|y\|_\infty \le \|\phi\|_\infty e^{\int_J \|A(\tau)\| d\tau}.$$

Proof. Define $\Lambda : BC(J,X) \to BC(J,X)$ by

$$(\Lambda y)(t) = \int_0^t A(\tau)y(\tau)d\tau.$$

Then y solves Eq. (5.9) iff $y = \phi + \Lambda y$ or equivalently iff $(I - \Lambda)y = \phi$.

An induction argument shows

$$\begin{aligned}(\Lambda^n \phi)(t) &= \int_0^t d\tau_n A(\tau_n)(\Lambda^{n-1}\phi)(\tau_n) \\ &= \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} A(\tau_n)A(\tau_{n-1})(\Lambda^{n-2}\phi)(\tau_{n-1}) \\ &\vdots \\ &= \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_n) \dots A(\tau_1)\phi(\tau_1) \\ &= (-1)^{n \cdot 1_{t<0}} \int_{\Delta_n(t)} A(\tau_n) \dots A(\tau_1)\phi(\tau_1) d\tau.\end{aligned}$$

Taking norms of this equation and using the triangle inequality along with Lemma 5.3 gives,

$$\begin{aligned}\|(\Lambda^n \phi)(t)\| &\le \|\phi\|_\infty \cdot \int_{\Delta_n(t)} \|A(\tau_n)\| \dots \|A(\tau_1)\| d\tau \\ &\le \|\phi\|_\infty \cdot \frac{1}{n!} \left(\int_{\Delta_1(t)} \|A(\tau)\| d\tau \right)^n \\ &\le \|\phi\|_\infty \cdot \frac{1}{n!} \left(\int_J \|A(\tau)\| d\tau \right)^n.\end{aligned}$$

Therefore,

$$\|\Lambda^n\|_{op} \le \frac{1}{n!} \left(\int_J \|A(\tau)\| d\tau \right)^n \tag{5.13}$$

and

$$\sum_{n=0}^\infty \|\Lambda^n\|_{op} \le e^{\int_J \|A(\tau)\| d\tau} < \infty$$

where $\|\cdot\|_{op}$ denotes the operator norm on $L(BC(J,X))$. An application of Proposition 3.69 now shows $(I-\Lambda)^{-1} = \sum_{n=0}^\infty \Lambda^n$ exists and

$$\left\|(I - \Lambda)^{-1}\right\|_{op} \le e^{\int_J \|A(\tau)\| d\tau}.$$

It is now only a matter of working through the notation to see that these assertions prove the theorem. ■

Corollary 5.6. *Suppose that $A \in L(X)$ is independent of time, then the solution to*

$$\dot{y}(t) = Ay(t) \text{ with } y(0) = x$$

is given by $y(t) = e^{tA}x$ *where*

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n. \tag{5.14}$$

Proof. This is a simple consequence of Eq. 5.12 and Lemma 5.3 with $\psi = 1$. ∎

We also have the following converse to this corollary whose proof is outlined in Exercise 5.11 below.

Theorem 5.7. *Suppose that* $T_t \in L(X)$ *for* $t \geq 0$ *satisfies*

1. *(Semi-group property.)* $T_0 = Id_X$ *and* $T_t T_s = T_{t+s}$ *for all* $s, t \geq 0$.
2. *(Norm Continuity)* $t \to T_t$ *is continuous at* 0, *i.e.* $\|T_t - I\|_{L(X)} \to 0$ *as* $t \downarrow 0$.

Then there exists $A \in L(X)$ *such that* $T_t = e^{tA}$ *where* e^{tA} *is defined in Eq. (5.14).*

5.3. **Uniqueness Theorem and Continuous Dependence on Initial Data.**

Lemma 5.8. *Gronwall's Lemma.* *Suppose that* f, ϵ, *and* k *are non-negative functions of a real variable* t *such that*

$$f(t) \leq \epsilon(t) + \left| \int_0^t k(\tau) f(\tau) d\tau \right|. \tag{5.15}$$

Then

$$f(t) \leq \epsilon(t) + \left| \int_0^t k(\tau)\epsilon(\tau) e^{\left|\int_\tau^t k(s)ds\right|} d\tau \right|, \tag{5.16}$$

and in particular if ϵ *and* k *are constants we find that*

$$f(t) \leq \epsilon e^{k|t|}. \tag{5.17}$$

Proof. I will only prove the case $t \geq 0$. The case $t \leq 0$ can be derived by applying the $t \geq 0$ to $\tilde{f}(t) = f(-t)$, $\tilde{k}(t) = k(-t)$ and $\tilde{\epsilon}(t) = \epsilon(-t)$.

Set $F(t) = \int_0^t k(\tau) f(\tau) d\tau$. Then by (5.15),

$$\dot{F} = kf \leq k\epsilon + kF.$$

Hence,

$$\frac{d}{dt}(e^{-\int_0^t k(s)ds} F) = e^{-\int_0^t k(s)ds}(\dot{F} - kF) \leq k\epsilon e^{-\int_0^t k(s)ds}.$$

Integrating this last inequality from 0 to t and then solving for F yields:

$$F(t) \leq e^{\int_0^t k(s)ds} \cdot \int_0^t d\tau k(\tau)\epsilon(\tau) e^{-\int_0^\tau k(s)ds} = \int_0^t d\tau k(\tau)\epsilon(\tau) e^{\int_\tau^t k(s)ds}.$$

But by the definition of F we have that

$$f \leq \epsilon + F,$$

and hence the last two displayed equations imply (5.16). Equation (5.17) follows from (5.16) by a simple integration. ∎

Corollary 5.9 (Continuous Dependence on Initial Data)**.** *Let* $U \subset_o X$, $0 \in (a, b)$ *and* $Z : (a, b) \times U \to X$ *be a continuous function which is* K*–Lipschitz function on* U, *i.e.* $\|Z(t,x)-Z(t,x')\| \leq K\|x-x'\|$ *for all* x *and* x' *in* U. *Suppose* $y_1, y_2 : (a,b) \to U$ *solve*

$$\frac{dy_i(t)}{dt} = Z(t, y_i(t)) \quad \text{with } y_i(0) = x_i \quad \text{for } i = 1,2. \tag{5.18}$$

Then

$$\|y_2(t) - y_1(t)\| \leq \|x_2 - x_1\| e^{K|t|} \text{ for } t \in (a,b) \tag{5.19}$$

and in particular, there is at most one solution to Eq. (5.1) under the above Lipschitz assumption on Z.

Proof. Let $f(t) \equiv \|y_2(t) - y_1(t)\|$. Then by the fundamental theorem of calculus,

$$\begin{aligned} f(t) &= \|y_2(0) - y_1(0) + \int_0^t (\dot{y}_2(\tau) - \dot{y}_1(\tau))\, d\tau\| \\ &\leq f(0) + \left| \int_0^t \|Z(\tau, y_2(\tau)) - Z(\tau, y_1(\tau))\| \, d\tau \right| \\ &= \|x_2 - x_1\| + K \left| \int_0^t f(\tau)\, d\tau \right|. \end{aligned}$$

Therefore by Gronwall's inequality we have,

$$\|y_2(t) - y_1(t)\| = f(t) \leq \|x_2 - x_1\| e^{K|t|}.$$

■

5.4. Local Existence (Non-Linear ODE).

Theorem 5.10 (Local Existence)**.** *Let* $T > 0$, $J = (-T, T)$, $x_0 \in X$, $r > 0$ *and*

$$C(x_0, r) := \{x \in X : \|x - x_0\| \leq r\}$$

be the closed r – *ball centered at* $x_0 \in X$. *Assume*

$$M = \sup \{\|Z(t,x)\| : (t,x) \in J \times C(x_0, r)\} < \infty \tag{5.20}$$

and there exists $K < \infty$ *such that*

$$\|Z(t,x) - Z(t,y)\| \leq K \|x - y\| \text{ for all } x, y \in C(x_0, r) \text{ and } t \in J. \tag{5.21}$$

Let $T_0 < \min\{r/M, T\}$ *and* $J_0 := (-T_0, T_0)$, *then for each* $x \in B(x_0, r - MT_0)$ *there exists a unique solution* $y(t) = y(t,x)$ *to Eq. (5.2) in* $C(J_0, C(x_0, r))$. *Moreover* $y(t,x)$ *is jointly continuous in* (t,x), $y(t,x)$ *is differentiable in* t, $\dot{y}(t,x)$ *is jointly continuous for all* $(t,x) \in J_0 \times B(x_0, r - MT_0)$ *and satisfies Eq. (5.1).*

Proof. The uniqueness assertion has already been proved in Corollary 5.9. To prove existence, let $C_r := C(x_0, r)$, $Y := C(J_0, C(x_0, r))$ and

$$S_x(y)(t) := x + \int_0^t Z(\tau, y(\tau)) d\tau. \tag{5.22}$$

With this notation, Eq. (5.2) becomes $y = S_x(y)$, i.e. we are looking for a fixed point of S_x. If $y \in Y$, then

$$\begin{aligned} \|S_x(y)(t) - x_0\| &\leq \|x - x_0\| + \left| \int_0^t \|Z(\tau, y(\tau))\| \, d\tau \right| \leq \|x - x_0\| + M|t| \\ &\leq \|x - x_0\| + MT_0 \leq r - MT_0 + MT_0 = r, \end{aligned}$$

showing $S_x(Y) \subset Y$ for all $x \in B(x_0, r - MT_0)$. Moreover if $y, z \in Y$,

$$\begin{aligned} \|S_x(y)(t) - S_x(z)(t)\| &= \left\| \int_0^t \left[Z(\tau, y(\tau)) - Z(\tau, z(\tau))\right] d\tau \right\| \\ &\le \left| \int_0^t \|Z(\tau, y(\tau)) - Z(\tau, z(\tau))\| \, d\tau \right| \\ &\le K \left| \int_0^t \|y(\tau) - z(\tau)\| \, d\tau \right|. \end{aligned} \tag{5.23}$$

Let $y_0(t, x) = x$ and $y_n(\cdot, x) \in Y$ defined inductively by

$$y_n(\cdot, x) := S_x(y_{n-1}(\cdot, x)) = x + \int_0^t Z(\tau, y_{n-1}(\tau, x)) d\tau. \tag{5.24}$$

Using the estimate in Eq. (5.23) repeatedly we find

$$\begin{aligned} \|y_{n+1}(t) - y_n(t)\| &\le K \left| \int_0^t \|y_n(\tau) - y_{n-1}(\tau)\| \, d\tau \right| \\ &\le K^2 \left| \int_0^t dt_1 \left| \int_0^{t_1} dt_2 \, \|y_{n-1}(t_2) - y_{n-2}(t_2)\| \right| \right| \\ &\dots \\ &\le K^n \left| \int_0^t dt_1 \left| \int_0^{t_1} dt_2 \dots \left| \int_0^{t_{n-1}} dt_n \, \|y_1(t_n) - y_0(t_n)\| \right| \dots \right| \right| \\ &\le K^n \|y_1(\cdot, x) - y_0(\cdot, x)\|_\infty \int_{\Delta_n(t)} d\tau \\ &= \frac{K^n |t|^n}{n!} \|y_1(\cdot, x) - y_0(\cdot, x)\|_\infty \le 2r \frac{K^n |t|^n}{n!} \end{aligned} \tag{5.25}$$

wherein we have also made use of Lemma 5.3. Combining this estimate with

$$\|y_1(t, x) - y_0(t, x)\| = \left\| \int_0^t Z(\tau, x) d\tau \right\| \le \left| \int_0^t \|Z(\tau, x)\| \, d\tau \right| \le M_0,$$

where

$$M_0 = T_0 \max \left\{ \int_0^{T_0} \|Z(\tau, x)\| \, d\tau, \int_{-T_0}^0 \|Z(\tau, x)\| \, d\tau \right\} \le MT_0,$$

shows

$$\|y_{n+1}(t, x) - y_n(t, x)\| \le M_0 \frac{K^n |t|^n}{n!} \le M_0 \frac{K^n T_0^n}{n!}$$

and this implies

$$\sum_{n=0}^\infty \sup \left\{ \|y_{n+1}(\cdot, x) - y_n(\cdot, x)\|_{\infty, J_0} : t \in J_0 \right\} \le \sum_{n=0}^\infty M_0 \frac{K^n T_0^n}{n!} = M_0 e^{KT_0} < \infty$$

where

$$\|y_{n+1}(\cdot, x) - y_n(\cdot, x)\|_{\infty, J_0} := \sup \left\{ \|y_{n+1}(t, x) - y_n(t, x)\| : t \in J_0 \right\}.$$

So $y(t, x) := \lim_{n \to \infty} y_n(t, x)$ exists uniformly for $t \in J$ and using Eq. (5.21) we also have

$$\sup \left\{ \|Z(t, y(t)) - Z(t, y_{n-1}(t))\| : t \in J_0 \right\} \le K \|y(\cdot, x) - y_{n-1}(\cdot, x)\|_{\infty, J_0} \to 0 \text{ as } n \to \infty.$$

Now passing to the limit in Eq. (5.24) shows y solves Eq. (5.2). From this equation it follows that $y(t,x)$ is differentiable in t and y satisfies Eq. (5.1).

The continuity of $y(t,x)$ follows from Corollary 5.9 and mean value inequality (Corollary 4.10):

$$\begin{aligned}\|y(t,x)-y(t',x')\| &\le \|y(t,x)-y(t,x')\| + \|y(t,x')-y(t',x')\| \\ &= \|y(t,x)-y(t,x')\| + \left\|\int_{t'}^{t} Z(\tau,y(\tau,x'))d\tau\right\| \\ &\le \|y(t,x)-y(t,x')\| + \left|\int_{t'}^{t} \|Z(\tau,y(\tau,x'))\|\, d\tau\right| \\ &\le \|x-x'\|e^{KT} + \left|\int_{t'}^{t} \|Z(\tau,y(\tau,x'))\|\, d\tau\right| \qquad (5.26) \\ &\le \|x-x'\|e^{KT} + M\,|t-t'|\,.\end{aligned}$$

The continuity of $\dot{y}(t,x)$ is now a consequence Eq. (5.1) and the continuity of y and Z. ■

Corollary 5.11. *Let $J=(a,b)\ni 0$ and suppose $Z\in C(J\times X,X)$ satisfies*

$$\|Z(t,x)-Z(t,y)\| \le K\,\|x-y\| \text{ for all } x,y\in X \text{ and } t\in J. \qquad (5.27)$$

Then for all $x\in X$, there is a unique solution $y(t,x)$ (for $t\in J$) to Eq. (5.1). Moreover $y(t,x)$ and $\dot{y}(t,x)$ are jointly continuous in (t,x).

Proof. Let $J_0=(a_0,b_0)\ni 0$ be a precompact subinterval of J and $Y:=BC\left(J_0,X\right)$. By compactness, $M:=\sup_{t\in\bar{J}_0}\|Z(t,0)\|<\infty$ which combined with Eq. (5.27) implies

$$\sup_{t\in\bar{J}_0}\|Z(t,x)\| \le M+K\,\|x\| \text{ for all } x\in X.$$

Using this estimate and Lemma 4.4 one easily shows $S_x(Y)\subset Y$ for all $x\in X$. The proof of Theorem 5.10 now goes through without any further change. ■

5.5. Global Properties.

Definition 5.12 (Local Lipschitz Functions)**.** Let $U\subset_o X$, J be an open interval and $Z\in C(J\times U,X)$. The function Z is said to be locally Lipschitz in x if for all $x\in U$ and all compact intervals $I\subset J$ there exists $K=K(x,I)<\infty$ and $\epsilon=\epsilon(x,I)>0$ such that $B(x,\epsilon(x,I))\subset U$ and

(5.28)
$$\|Z(t,x_1)-Z(t,x_0)\| \le K(x,I)\|x_1-x_0\| \text{ for all } x_0,x_1\in B(x,\epsilon(x,I)) \text{ and } t\in I.$$

For the rest of this section, we will assume J is an open interval containing 0, U is an open subset of X and $Z\in C(J\times U,X)$ is a locally Lipschitz function.

Lemma 5.13. *Let $Z\in C(J\times U,X)$ be a locally Lipschitz function in X and E be a compact subset of U and I be a compact subset of J. Then there exists $\epsilon>0$ such that $Z(t,x)$ is bounded for $(t,x)\in I\times E_\epsilon$ and and $Z(t,x)$ is K – Lipschitz on E_ϵ for all $t\in I$, where*

$$E_\epsilon := \{x\in U : \operatorname{dist}(x,E)<\epsilon\}\,.$$

Proof. Let $\epsilon(x,I)$ and $K(x,I)$ be as in Definition 5.12 above. Since E is compact, there exists a finite subset $\Lambda \subset E$ such that $E \subset V := \cup_{x\in\Lambda} B(x,\epsilon(x,I)/2)$. If $y \in V$, there exists $x \in \Lambda$ such that $\|y-x\| < \epsilon(x,I)/2$ and therefore

$$\|Z(t,y)\| \le \|Z(t,x)\| + K(x,I)\,\|y-x\| \le \|Z(t,x)\| + K(x,I)\epsilon(x,I)/2$$
$$\le \sup_{x\in\Lambda,t\in I}\{\|Z(t,x)\| + K(x,I)\epsilon(x,I)/2\} =: M < \infty.$$

This shows Z is bounded on $I \times V$.

Let

$$\epsilon := d(E,V^c) \le \frac{1}{2}\min_{x\in\Lambda}\epsilon(x,I)$$

and notice that $\epsilon > 0$ since E is compact, V^c is closed and $E \cap V^c = \emptyset$. If $y,z \in E_\epsilon$ and $\|y-z\| < \epsilon$, then as before there exists $x \in \Lambda$ such that $\|y-x\| < \epsilon(x,I)/2$. Therefore

$$\|z-x\| \le \|z-y\| + \|y-x\| < \epsilon + \epsilon(x,I)/2 \le \epsilon(x,I)$$

and since $y,z \in B(x,\epsilon(x,I))$, it follows that

$$\|Z(t,y)-Z(t,z)\| \le K(x,I)\|y-z\| \le K_0\|y-z\|$$

where $K_0 := \max_{x\in\Lambda} K(x,I) < \infty$. On the other hand if $y,z \in E_\epsilon$ and $\|y-z\| \ge \epsilon$, then

$$\|Z(t,y)-Z(t,z)\| \le 2M \le \frac{2M}{\epsilon}\,\|y-z\|\,.$$

Thus if we let $K := \max\{2M/\epsilon, K_0\}$, we have shown

$$\|Z(t,y)-Z(t,z)\| \le K\|y-z\| \text{ for all } y,z \in E_\epsilon \text{ and } t \in I.$$

■

Proposition 5.14 (Maximal Solutions)**.** *Let $Z \in C(J\times U, X)$ be a locally Lipschitz function in x and let $x \in U$ be fixed. Then there is an interval $J_x = (a(x), b(x))$ with $a \in [-\infty, 0)$ and $b \in (0,\infty]$ and a C^1–function $y : J \to U$ with the following properties:*

1. *y solves ODE in Eq. (5.1).*
2. *If $\tilde{y} : \tilde{J} = (\tilde{a},\tilde{b}) \to U$ is another solution of Eq. (5.1) (we assume that $0 \in \tilde{J}$) then $\tilde{J} \subset J$ and $\tilde{y} = y|_{\tilde{J}}$.*

The function $y : J \to U$ is called the maximal solution to Eq. (5.1).

Proof. Suppose that $y_i : J_i = (a_i,b_i) \to U$, $i = 1,2$, are two solutions to Eq. (5.1). We will start by showing the $y_1 = y_2$ on $J_1 \cap J_2$. To do this[9] let $J_0 = (a_0,b_0)$ be chosen so that $0 \in J_0 \subset J_1 \cap J_2$, and let $E := y_1(J_0) \cup y_2(J_0)$ – a compact subset of X. Choose $\epsilon > 0$ as in Lemma 5.13 so that Z is Lipschitz on E_ϵ. Then $y_1|_{J_0}, y_2|_{J_0} : J_0 \to E_\epsilon$ both solve Eq. (5.1) and therefore are equal by Corollary 5.9.

[9]Here is an alternate proof of the uniqueness. Let

$$T \equiv \sup\{t \in [0,\min\{b_1,b_2\}) : y_1 = y_2 \text{ on } [0,t]\}.$$

(T is the first positive time after which y_1 and y_2 disagree.

Suppose, for sake of contradiction, that $T < \min\{b_1,b_2\}$. Notice that $y_1(T) = y_2(T) =: x'$. Applying the local uniqueness theorem to $y_1(\cdot - T)$ and $y_2(\cdot - T)$ thought as function from $(-\delta,\delta) \to B(x',\epsilon(x'))$ for some δ sufficiently small, we learn that $y_1(\cdot - T) = y_2(\cdot - T)$ on $(-\delta,\delta)$. But this shows that $y_1 = y_2$ on $[0, T+\delta)$ which contradicts the definition of T. Hence we must have the $T = \min\{b_1,b_2\}$, i.e. $y_1 = y_2$ on $J_1 \cap J_2 \cap [0,\infty)$. A similar argument shows that $y_1 = y_2$ on $J_1 \cap J_2 \cap (-\infty,0]$ as well.

Since $J_0 = (a_0, b_0)$ was chosen arbitrarily so that $[a, b] \subset J_1 \cap J_2$, we may conclude that $y_1 = y_2$ on $J_1 \cap J_2$.

Let $(y_\alpha, J_\alpha = (a_\alpha, b_\alpha))_{\alpha \in A}$ denote the possible solutions to (5.1) such that $0 \in J_\alpha$. Define $J_x = \cup J_\alpha$ and set $y = y_\alpha$ on J_α. We have just checked that y is well defined and the reader may easily check that this function $y : J_x \to U$ satisfies all the conclusions of the theorem. ■

Notation 5.15. For each $x \in U$, let $J_x = (a(x), b(x))$ be the maximal interval on which Eq. (5.1) may be solved, see Proposition 5.14. Set $\mathcal{D}(Z) \equiv \cup_{x \in U}(J_x \times \{x\}) \subset J \times U$ and let $\phi : \mathcal{D}(Z) \to U$ be defined by $\phi(t, x) = y(t)$ where y is the maximal solution to Eq. (5.1). (So for each $x \in U$, $\phi(\cdot, x)$ is the maximal solution to Eq. (5.1).)

Proposition 5.16. *Let $Z \in C(J \times U, X)$ be a locally Lipschitz function in x and $y : J_x = (a(x), b(x)) \to U$ be the maximal solution to Eq. (5.1). If $b(x) < b$, then either $\limsup_{t \uparrow b(x)} \|Z(t, y(t))\| = \infty$ or $y(b(x)-) \equiv \lim_{t \uparrow b(x)} y(t)$ exists and $y(b(x)-) \notin U$. Similarly, if $a > a(x)$, then either $\limsup_{t \downarrow a(x)} \|y(t)\| = \infty$ or $y(a(x)+) \equiv \lim_{t \downarrow a} y(t)$ exists and $y(a(x)+) \notin U$.*

Proof. Suppose that $b < b(x)$ and $M \equiv \limsup_{t \uparrow b(x)} \|Z(t, y(t))\| < \infty$. Then there is a $b_0 \in (0, b(x))$ such that $\|Z(t, y(t))\| \le 2M$ for all $t \in (b_0, b(x))$. Thus, by the usual fundamental theorem of calculus argument,

$$\|y(t) - y(t')\| \le \left| \int_t^{t'} \|Z(t, y(\tau))\| \, d\tau \right| \le 2M|t - t'|$$

for all $t, t' \in (b_0, b(x))$. From this it is easy to conclude that $y(b(x)-) = \lim_{t \uparrow b(x)} y(t)$ exists. Now if $y(b(x)-) \in U$, by the local existence Theorem 5.10, there exists $\delta > 0$ and $w \in C^1((b(x) - \delta, b(x) + \delta), U)$ such that

$$\dot{w}(t) = Z(t, w(t)) \quad \text{and } w(b(x)) = y(b(x)-).$$

Now define $\tilde{y} : (a, b(x) + \delta) \to U$ by

$$\tilde{y}(t) = \begin{cases} y(t) & \text{if } t \in J_x \\ w(t) & \text{if } t \in (b(x) - \delta, b(x) + \delta) \end{cases}.$$

By uniqueness of solutions to ODE's $\tilde{y}$ is well defined, $\tilde{y} \in C^1((a(x), b(x) + \delta), X)$ and $\tilde{y}$ solves the ODE in Eq. 5.1. But this violates the maximality of y and hence we must have that $y(b(x)-) \notin U$. The assertions for t near $a(x)$ are proved similarly. ■

Remark 5.17. In general it is **not** true that the functions a and b are continuous. For example, let U be the region in $\mathbb{R}^2$ described in polar coordinates by $r > 0$ and $0 < \theta < 3\pi/4$ and $Z(x, y) = (0, -1)$ as in Figure 12 below. Then $b(x, y) = y$ for all $x, y > 0$ while $b(x, y) = \infty$ for all $x < 0$ and $y \in \mathbb{R}$ which shows b is discontinuous. On the other hand notice that

$$\{b > t\} = \{x < 0\} \cup \{(x, y) : x \ge 0, y > t\}$$

is an open set for all $t > 0$.

Theorem 5.18 (Global Continuity)**.** *Let $Z \in C(J \times U, X)$ be a locally Lipschitz function in x. Then $\mathcal{D}(Z)$ is an open subset of $J \times U$ and the functions $\phi : \mathcal{D}(Z) \to U$ and $\dot{\phi} : \mathcal{D}(Z) \to U$ are continuous. More precisely, for all $x_0 \in U$ and all*

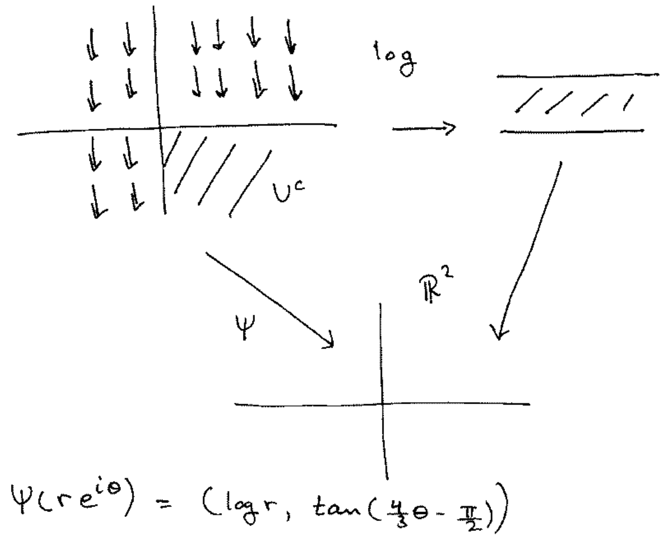

FIGURE 12. An example of a vector field for which $b(x)$ is discontinuous. This is given in the top left hand corner of the figure. The map ψ would allow the reader to find an example on $\mathbb{R}^2$ if so desired. Some calculations shows that Z transfered to $\mathbb{R}^2$ by the map ψ is given by

$$\tilde{Z}(x,y) = -e^{-x}\left(\sin\left(\frac{3\pi}{8} + \frac{3}{4}\tan^{-1}(y)\right), \cos\left(\frac{3\pi}{8} + \frac{3}{4}\tan^{-1}(y)\right)\right).$$

open intervals J_0 *such that* $0 \in J_0 \sqsubset\sqsubset J_{x_0}$ *there exists* $\delta = \delta(x_0, J_0) > 0$ *and* $C = C(x_0, J_0) < \infty$ *such that* $J_0 \subset J_y$ *and*

$$\|\phi(\cdot, x) - \phi(\cdot, x_0)\|_{BC(J_0,U)} \le C\|x - x_0\| \text{ for all } x \in B(x_0, \delta). \tag{5.29}$$

Proof. Let $|J_0| = b_0 - a_0$, $I = \bar{J}_0$ and $E := y(\bar{J}_0)$ – a compact subset of U and let $\epsilon > 0$ and $K < \infty$ be given as in Lemma 5.13, i.e. K is the Lipschitz constant for Z on E_ϵ. Suppose that $x \in E_\epsilon$, then by Corollary 5.9,

$$\|\phi(t,x) - \phi(t,x_0)\| \le \|x - x_0\|e^{K|t|} \le \|x - x_0\|e^{K|J_0|} \tag{5.30}$$

for all $t \in J_0 \cap J_x$ such that $\phi(t,x) \in E_\epsilon$. Letting $\delta := \epsilon e^{-K|J_0|}/2$, and assuming $x \in B(x_0, \delta)$, the previous equation implies

$$\|\phi(t,x) - \phi(t,x_0)\| \le \epsilon/2 < \epsilon \text{ for all } t \in J_0 \cap J_x.$$

This estimate further shows that $\phi(t,x)$ remains bounded and strictly away from the boundary of U for all $t \in J_0 \cap J_x$. Therefore, it follows from Proposition 5.14 that $J_0 \subset J_x$ and Eq. (5.30) is valid for all $t \in J_0$. This proves Eq. (5.29) with $C := e^{K|J_0|}$.

Suppose that $(t_0, x_0) \in \mathcal{D}(Z)$ and let $0 \in J_0 \sqsubset\sqsubset J_{x_0}$ such that $t_0 \in J_0$ and δ be as above. Then we have just shown $J_0 \times B(x_0, \delta) \subset \mathcal{D}(Z)$ which proves $\mathcal{D}(Z)$ is open. Furthermore, since the evaluation map

$$(t_0, y) \in J_0 \times BC(J_0, U) \xrightarrow{e} y(t_0) \in X$$

is continuous (as the reader should check) it follows that $\phi = e \circ (x \to \phi(\cdot, x)) : J_0 \times B(x_0, \delta) \to U$ is also continuous; being the composition of continuous maps. The continuity of $\dot{\phi}(t_0, x)$ is a consequences of the continuity of ϕ and the differential equation 5.1

Alternatively using Eq. (5.2),

$$\|\phi(t_0,x) - \phi(t,x_0)\| \le \|\phi(t_0,x) - \phi(t_0,x_0)\| + \|\phi(t_0,x_0) - \phi(t,x_0)\|$$
$$\le C\,\|x - x_0\| + \left| \int_t^{t_0} \|Z(\tau, \phi(\tau, x_0))\|\, d\tau \right| \le C\,\|x - x_0\| + M\,|t_0 - t|$$

where C is the constant in Eq. (5.29) and $M = \sup_{\tau \in J_0} \|Z(\tau, \phi(\tau, x_0))\| < \infty$. This clearly shows ϕ is continuous. ∎

5.6. **Semi-Group Properties of time independent flows.** To end this chapter we investigate the semi-group property of the flow associated to the vector-field Z. It will be convenient to introduce the following suggestive notation. For $(t,x) \in \mathcal{D}(Z)$, set $e^{tZ}(x) = \phi(t,x)$. So the path $t \to e^{tZ}(x)$ is the maximal solution to

$$\frac{d}{dt} e^{tZ}(x) = Z(e^{tZ}(x)) \text{ with } e^{0Z}(x) = x.$$

This exponential notation will be justified shortly. It is convenient to have the following conventions.

Notation 5.19. We write $f : X \to X$ to mean a function defined on some open subset $D(f) \subset X$. The open set $D(f)$ will be called the domain of f. Given two functions $f : X \to X$ and $g : X \to X$ with domains $D(f)$ and $D(g)$ respectively, we define the composite function $f \circ g : X \to X$ to be the function with domain

$$D(f \circ g) = \{x \in X : x \in D(g) \text{ and } g(x) \in D(f)\} = g^{-1}(D(f))$$

given by the rule $f \circ g(x) = f(g(x))$ for all $x \in D(f \circ g)$. We now write $f = g$ iff $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f) = D(g)$. We will also write $f \subset g$ iff $D(f) \subset D(g)$ and $g|_{D(f)} = f$.

Theorem 5.20. *For fixed $t \in \mathbb{R}$ we consider e^{tZ} as a function from X to X with domain $D(e^{tZ}) = \{x \in U : (t,x) \in \mathcal{D}(Z)\}$, where $D(\phi) = \mathcal{D}(Z) \subset \mathbb{R} \times U$, $\mathcal{D}(Z)$ and ϕ are defined in Notation 5.15. Conclusions:*

1. *If $t, s \in \mathbb{R}$ and $t \cdot s \ge 0$, then $e^{tZ} \circ e^{sZ} = e^{(t+s)Z}$.*
2. *If $t \in \mathbb{R}$, then $e^{tZ} \circ e^{-tZ} = Id_{D(e^{-tZ})}$.*
3. *For arbitrary $t, s \in \mathbb{R}$, $e^{tZ} \circ e^{sZ} \subset e^{(t+s)Z}$.*

Proof. Item 1. For simplicity assume that $t, s \ge 0$. The case $t, s \le 0$ is left to the reader. Suppose that $x \in D(e^{tZ} \circ e^{sZ})$. Then by assumption $x \in D(e^{sZ})$ and $e^{sZ}(x) \in D(e^{tZ})$. Define the path $y(\tau)$ via:

$$y(\tau) = \begin{cases} e^{\tau Z}(x) & \text{if } 0 \le \tau \le s \\ e^{(\tau - s)Z}(x) & \text{if } s \le \tau \le t + s \end{cases}.$$

It is easy to check that y solves $\dot{y}(\tau) = Z(y(\tau))$ with $y(0) = x$. But since, $e^{\tau Z}(x)$ is the maximal solution we must have that $x \in D(e^{(t+s)Z})$ and $y(t+s) = e^{(t+s)Z}(x)$. That is $e^{(t+s)Z}(x) = e^{tZ} \circ e^{sZ}(x)$. Hence we have shown that $e^{tZ} \circ e^{sZ} \subset e^{(t+s)Z}$.

To finish the proof of item 1. it suffices to show that $D(e^{(t+s)Z}) \subset D(e^{tZ} \circ e^{sZ})$. Take $x \in D(e^{(t+s)Z})$, then clearly $x \in D(e^{sZ})$. Set $y(\tau) = e^{(\tau+s)Z}(x)$ defined for $0 \leq \tau \leq t$. Then y solves

$$\dot{y}(\tau) = Z(y(\tau)) \text{ with } y(0) = e^{sZ}(x).$$

But since $\tau \to e^{\tau Z}(e^{sZ}(x))$ is the maximal solution to the above initial valued problem we must have that $y(\tau) = e^{\tau Z}(e^{sZ}(x))$, and in particular at $\tau = t$, $e^{(t+s)Z}(x) = e^{tZ}(e^{sZ}(x))$. This shows that $x \in D(e^{tZ} \circ e^{sZ})$ and in fact $e^{(t+s)Z} \subset e^{tZ} \circ e^{sZ}$.

Item 2. Let $x \in D(e^{-tZ})$ – again assume for simplicity that $t \geq 0$. Set $y(\tau) = e^{(\tau-t)Z}(x)$ defined for $0 \leq \tau \leq t$. Notice that $y(0) = e^{-tZ}(x)$ and $\dot{y}(\tau) = Z(y(\tau))$. This shows that $y(\tau) = e^{\tau Z}(e^{-tZ}(x))$ and in particular that $x \in D(e^{tZ} \circ e^{-tZ})$ and $e^{tZ} \circ e^{-tZ}(x) = x$. This proves item 2.

Item 3. I will only consider the case that $s < 0$ and $t + s \geq 0$, the other cases are handled similarly. Write u for $t + s$, so that $t = -s + u$. We know that $e^{tZ} = e^{uZ} \circ e^{-sZ}$ by item 1. Therefore

$$e^{tZ} \circ e^{sZ} = (e^{uZ} \circ e^{-sZ}) \circ e^{sZ}.$$

Notice in general, one has $(f \circ g) \circ h = f \circ (g \circ h)$ (you prove). Hence, the above displayed equation and item 2. imply that

$$e^{tZ} \circ e^{sZ} = e^{uZ} \circ (e^{-sZ} \circ e^{sZ}) = e^{(t+s)Z} \circ I_{D(e^{sZ})} \subset e^{(t+s)Z}.$$

■

The following result is trivial but conceptually illuminating partial converse to Theorem 5.20.

Proposition 5.21 (Flows and Complete Vector Fields)**.** *Suppose $U \subset_o X$, $\phi \in C(\mathbb{R} \times U, U)$ and $\phi_t(x) = \phi(t, x)$. Suppose ϕ satisfies:*

1. *$\phi_0 = I_U$,*
2. *$\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$, and*
3. *$Z(x) := \dot{\phi}(0, x)$ exists for all $x \in U$ and $Z \in C(U, X)$ is locally Lipschitz.*

Then $\phi_t = e^{tZ}$.

Proof. Let $x \in U$ and $y(t) \equiv \phi_t(x)$. Then using Item 2.,

$$\dot{y}(t) = \frac{d}{ds}|_0 y(t+s) = \frac{d}{ds}|_0 \phi_{(t+s)}(x) = \frac{d}{ds}|_0 \phi_s \circ \phi_t(x) = Z(y(t)).$$

Since $y(0) = x$ by Item 1. and Z is locally Lipschitz by Item 3., we know by uniqueness of solutions to ODE's (Corollary 5.9) that $\phi_t(x) = y(t) = e^{tZ}(x)$. ■

5.7. **Exercises.**

Exercise 5.1. Find a vector field Z such that $e^{(t+s)Z}$ is not contained in $e^{tZ} \circ e^{sZ}$.

Definition 5.22. A locally Lipschitz function $Z : U \subset_o X \longrightarrow X$ is said to be a complete vector field if $\mathcal{D}(Z) = \mathbb{R} \times U$. That is for any $x \in U$, $t \longrightarrow e^{tZ}(x)$ is defined for all $t \in \mathbb{R}$.

Exercise 5.2. Suppose that $Z : X \longrightarrow X$ is a locally Lipschitz function. Assume there is a constant $C > 0$ such that

$$\|Z(x)\| \leq C(1 + \|x\|) \text{ for all } x \in X.$$

Then Z is complete. **Hint:** use Gronwall's Lemma 5.8 and Proposition 5.16.

Exercise 5.3. Suppose y is a solution to $\dot{y}(t) = |y(t)|^{1/2}$ with $y(0) = 0$. Show there exists $a, b \in [0, \infty]$ such that

$$y(t) = \begin{cases} \frac{1}{4}(t-b)^2 & \text{if} \quad t \geq b \\ 0 & \text{if} \quad -a < t < b \\ -\frac{1}{4}(t+a)^2 & \text{if} \quad t \leq -a. \end{cases}$$

Exercise 5.4. Using the fact that the solutions to Eq. (5.3) are never 0 if $x \neq 0$, show that $y(t) = 0$ is the only solution to Eq. (5.3) with $y(0) = 0$.

Exercise 5.5. Suppose that $A \in L(X)$. Show directly that:

(1) e^{tA} define in Eq. (5.14) is convergent in $L(X)$ when equipped with the operator norm.
(2) e^{tA} is differentiable in t and that $\frac{d}{dt}e^{tA} = Ae^{tA}$.

Exercise 5.6. Suppose that $A \in L(X)$ and $v \in X$ is an eigenvector of A with eigenvalue λ, i.e. that $Av = \lambda v$. Show $e^{tA}v = e^{t\lambda}v$. Also show that $X = \mathbb{R}^n$ and A is a diagonalizable $n \times n$ matrix with

$$A = SDS^{-1} \text{ with } D = diag(\lambda_1, \dots, \lambda_n)$$

then $e^{tA} = Se^{tD}S^{-1}$ where $e^{tD} = diag(e^{t\lambda_1}, \dots, e^{t\lambda_n})$.

Exercise 5.7. Suppose that $A, B \in L(X)$ and $[A, B] \equiv AB - BA = 0$. Show that $e^{(A+B)} = e^A e^B$.

Exercise 5.8. Suppose $A \in C(\mathbb{R}, L(X))$ satisfies $[A(t), A(s)] = 0$ for all $s, t \in \mathbb{R}$. Show

$$y(t) := e^{\left(\int_0^t A(\tau)d\tau\right)}x$$

is the unique solution to $\dot{y}(t) = A(t)y(t)$ with $y(0) = x$.

Exercise 5.9. Compute e^{tA} when

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and use the result to prove the formula

$$\cos(s+t) = \cos s \cos t - \sin s \sin t.$$

Hint: Sum the series and use $e^{tA}e^{sA} = e^{(t+s)A}$.

Exercise 5.10. Compute e^{tA} when

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Use your result to compute $e^{t(\lambda I + A)}$ where $\lambda \in \mathbb{R}$ and I is the 3×3 identity matrix. **Hint:** Sum the series.

Exercise 5.11. Prove Theorem 5.7 using the following outline.

(1) First show $t \in [0, \infty) \to T_t \in L(X)$ is continuos.
(2) For $\epsilon > 0$, let $S_\epsilon := \frac{1}{\epsilon}\int_0^\epsilon T_\tau d\tau \in L(X)$. Show $S_\epsilon \to I$ as $\epsilon \downarrow 0$ and conclude from this that S_ϵ is invertible when $\epsilon > 0$ is sufficiently small. For the remainder of the proof fix such a small $\epsilon > 0$.

(3) Show
$$T_t S_\epsilon = \frac{1}{\epsilon}\int_t^{t+\epsilon} T_\tau d\tau$$
and conclude from this that
$$\lim_{t\downarrow 0} t^{-1}\left(T_t - I\right) S_\epsilon = \frac{1}{\epsilon}\left(T_\epsilon - Id_X\right).$$
(4) Using the fact that S_ϵ is invertible, conclude $A = \lim_{t\downarrow 0} t^{-1}\left(T_t - I\right)$ exists in $L(X)$ and that
$$A = \frac{1}{\epsilon}\left(T_\epsilon - I\right) S_\epsilon^{-1}.$$
(5) Now show using the semigroup property and step 4. that $\frac{d}{dt}T_t = AT_t$ for all $t > 0$.
(6) Using step 5, show $\frac{d}{dt}e^{-tA}T_t = 0$ for all $t > 0$ and therefore $e^{-tA}T_t = e^{-0A}T_0 = I$.

Exercise 5.12 (Higher Order ODE)**.** Let X be a Banach space, , $\mathcal{U} \subset_o X^n$ and $f \in C\left(J \times \mathcal{U}, X\right)$ be a Locally Lipschitz function in $\mathbf{x} = (x_1, \ldots, x_n)$. Show the n^{th} ordinary differential equation,
(5.31)
$$y^{(n)}(t) = f(t, y(t), \dot{y}(t), \ldots y^{(n-1)}(t)) \text{ with } y^{(k)}(0) = y_0^k \text{ for } k = 0, 1, 2 \ldots, n-1$$
where $(y_0^0, \ldots, y_0^{n-1})$ is given in $\mathcal{U}$, has a unique solution for small $t \in J$. **Hint**: let $\mathbf{y}(t) = \left(y(t), \dot{y}(t), \ldots y^{(n-1)}(t)\right)$ and rewrite Eq. (5.31) as a first order ODE of the form
$$\dot{\mathbf{y}}(t) = Z(t, \mathbf{y}(t)) \text{ with } \mathbf{y}(0) = (y_0^0, \ldots, y_0^{n-1}).$$

Exercise 5.13. Use the results of Exercises 5.10 and 5.12 to solve
$$\ddot{y}(t) - 2\dot{y}(t) + y(t) = 0 \text{ with } y(0) = a \text{ and } \dot{y}(0) = b.$$
Hint: The 2×2 matrix associated to this system, A, has only one eigenvalue 1 and may be written as $A = I + B$ where $B^2 = 0$.

Exercise 5.14. Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $U, V : \mathbb{R} \to L(X)$ are the unique solution to the linear differential equations
$$\dot{V}(t) = A(t)V(t) \text{ with } V(0) = I$$
and
$$\dot{U}(t) = -U(t)A(t) \text{ with } U(0) = I. \tag{5.32}$$
Prove that $V(t)$ is invertible and that $V^{-1}(t) = U(t)$. **Hint:** 1) show $\frac{d}{dt}\left[U(t)V(t)\right] = 0$ (which is sufficient if $\dim(X) < \infty$) and 2) show compute $y(t) := V(t)U(t)$ solves a linear differential ordinary differential equation that has $y \equiv 0$ as an obvious solution. Then use the uniqueness of solutions to ODEs. (The fact that $U(t)$ must be defined as in Eq. (5.32) is the content of Exercise 22.2 below.)

Exercise 5.15 (Duhamel' s Principle I)**.** Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $V : \mathbb{R} \to L(X)$ is the unique solution to the linear differential equation in Eq. (22.28). Let $x \in X$ and $h \in C(\mathbb{R}, X)$ be given. Show that the unique solution to the differential equation:
$$\dot{y}(t) = A(t)y(t) + h(t) \text{ with } y(0) = x \tag{5.33}$$

is given by

$$y(t) = V(t)x + V(t)\int_0^t V(\tau)^{-1}h(\tau)\,d\tau. \tag{5.34}$$

Hint: compute $\frac{d}{dt}[V^{-1}(t)y(t)]$ when y solves Eq. (5.33).

Exercise 5.16 (Duhamel' s Principle II)**.** Suppose that $A : \mathbb{R} \to L(X)$ is a continuous function and $V : \mathbb{R} \to L(X)$ is the unique solution to the linear differential equation in Eq. (22.28). Let $W_0 \in L(X)$ and $H \in C(\mathbb{R}, L(X))$ be given. Show that the unique solution to the differential equation:

$$\dot{W}(t) = A(t)W(t) + H(t) \text{ with } W(0) = W_0 \tag{5.35}$$

is given by

$$W(t) = V(t)W_0 + V(t)\int_0^t V(\tau)^{-1}H(\tau)\,d\tau. \tag{5.36}$$

Exercise 5.17 (Non-Homogeneous ODE)**.** Suppose that $U \subset_o X$ is open and $Z : \mathbb{R} \times U \to X$ is a continuous function. Let $J = (a, b)$ be an interval and $t_0 \in J$. Suppose that $y \in C^1(J, U)$ is a solution to the "non-homogeneous" differential equation:

$$\dot{y}(t) = Z(t, y(t)) \text{ with } y(t_o) = x \in U. \tag{5.37}$$

Define $Y \in C^1(J - t_0, \mathbb{R} \times U)$ by $Y(t) \equiv (t + t_0, y(t + t_0))$. Show that Y solves the "homogeneous" differential equation

$$\dot{Y}(t) = \tilde{Z}(Y(t)) \text{ with } Y(0) = (t_0, y_0), \tag{5.38}$$

where $\tilde{Z}(t, x) \equiv (1, Z(x))$. Conversely, suppose that $Y \in C^1(J - t_0, \mathbb{R} \times U)$ is a solution to Eq. (5.38). Show that $Y(t) = (t + t_0, y(t + t_0))$ for some $y \in C^1(J, U)$ satisfying Eq. (5.37). (In this way the theory of non-homogeneous ode's may be reduced to the theory of homogeneous ode's.)

Exercise 5.18 (Differential Equations with Parameters)**.** Let W be another Banach space, $U \times V \subset_o X \times W$ and $Z \in C(U \times V, X)$ be a locally Lipschitz function on $U \times V$. For each $(x, w) \in U \times V$, let $t \in J_{x,w} \to \phi(t, x, w)$ denote the maximal solution to the ODE

$$\dot{y}(t) = Z(y(t), w) \text{ with } y(0) = x. \tag{5.39}$$

Prove

$$\mathcal{D} := \{(t, x, w) \in \mathbb{R} \times U \times V : t \in J_{x,w}\} \tag{5.40}$$

is open in $\mathbb{R} \times U \times V$ and ϕ and $\dot{\phi}$ are continuous functions on $\mathcal{D}$.

Hint: If $y(t)$ solves the differential equation in (5.39), then $v(t) \equiv (y(t), w)$ solves the differential equation,

$$\dot{v}(t) = \tilde{Z}(v(t)) \text{ with } v(0) = (x, w), \tag{5.41}$$

where $\tilde{Z}(x, w) \equiv (Z(x, w), 0) \in X \times W$ and let $\psi(t, (x, w)) := v(t)$. Now apply the Theorem 5.18 to the differential equation (5.41).