13. Abstract Wave Equation

In the next section we consider

$$u_{tt} - \triangle u = 0 \text{ with } u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \text{ for } x \in \mathbb{R}^n. \tag{13.1}$$

Before working with this explicit equation we will work out an abstract Hilbert space theory first.

Theorem 13.1 (Existence)**.** *Suppose $A : H \to H$ is a self-adjoint non-positive operator, i.e. $A^* = A$ and $A \le 0$ and $f \in D(A)$ and $g \in g \in D\left(\sqrt{-A}\right)$ are given. Then*

$$u(t) = \cos(t\sqrt{-A})f + \frac{\sin(t\sqrt{-A})}{\sqrt{-A}}g \tag{13.2}$$

satisfies:

(1) $\dot{u}(t) = \cos(t\sqrt{-A})\sqrt{-A}\ f + \sin(t\sqrt{-A})g$ *exists and is continuous.*
(2) $\ddot{u}(t)$ *exists and is continuous*

$$\ddot{u}(t) = Au \text{ with } u(0) = f \text{ and } \dot{u}(0) = g. \tag{13.3}$$

(3) $\frac{d}{dt}\ \sqrt{-A}\ u(t) = -\cos(t\sqrt{-A})A\ f + \sin(t\sqrt{-A})\sqrt{-A}g$ *exists and is continuous.*

Eq. (13.3) is Newton's equation of motion for an infinite dimensional harmonic oscillation. Given any solution u to Eq. (13.3) it is natural to define its energy by

$$E(t,u) := \frac{1}{2}\ [\|\dot{u}(t)\|^2 + \|\omega u(t)\|^2] = \text{K.E.} + \text{ P.E.}$$

where $\omega := \sqrt{-A}$. Notice that Eq. (13.3) becomes $\ddot{u} + \omega^2 u = 0$ with this definition of ω.

Lemma 13.2 (Conservation of Energy)**.** *Suppose u is a solution to Eq. (13.3) such that $\frac{d}{dt}\ \sqrt{-A}u(t)$ exists and is continuous. Then $\dot{E}(t) = 0$.*

Proof.

$$\dot{E}(t) = \mathrm{Re}(\dot{u}, \ddot{u}) + \mathrm{Re}(\omega u, \omega \dot{u}) = \mathrm{Re}(\dot{u}, -\omega^2 u) - \mathrm{Re}(\omega^2 u, \dot{u}) = 0.$$

■

Theorem 13.3 (Uniqueness of Solutions)**.** *The only function $u \in C^2(\mathbb{R}, H)$ satisfying 1) $u(t) \in D(A)$ for all t and 2)*

$$\ddot{u} = Au \text{ with } u(0) = 0 = \dot{u}(0)$$

is the $u(t) \equiv 0$ for all t.

Proof. Let $\chi_M(x) = 1_{|x| \le M}$ and define $P_M = \chi_M(A)$ so that P_M is orthogonal projection onto the spectral subspace of H where $-M \le A \le 0$. Then for all $f \in D(A)$ we have $P_M A f = A\, P_M f$ and for all $f \in H$ we have $P_M f \in D((-A)^\alpha)$ for any $\alpha \ge 0$. Let $u_M(t) := P_M u(t)$, then $u_M \in C^2(\mathbb{R}, H)$, $u_M(t) \in D((-A)^\alpha)$ for all t and α, $t \to \sqrt{-A}u_M(t)$ is continuous and

$$\ddot{u}_M = \frac{d^2}{dt^2}(P_M u) = P_M \ddot{u} = P_M A u = A P_M u = A u_M$$

with $u_M(0) = 0 = \dot{u}_M(0)$. By Lemma 13.2,

$$\frac{1}{2}\ [\|\dot{u}_M(t)\|^2 + \|\omega u_M(t)\|^2] = \frac{1}{2}\ [\|\dot{u}_M(0)\|^2 + \|\omega u_M(0)\|^2] = 0$$

for all t. In particular this implies $\dot{u}_M(t) = 0$ and hence $P_M u(t) = u_M(t) \equiv 0$. Letting $M \to \infty$ then shows $u(t) \equiv 0$. ■

Corollary 13.4. *Any solution to* $\ddot{u} = Au$ *with* $u(0) \in D(A)$ *and* $\dot{u}(0) \in D(\sqrt{-A})$ *must satisfy* $t \to \sqrt{-A}u(t)$ *is* C^1.

13.1. **Corresponding first order O.D.E..** Let $v(t) = \dot{u}(t)$, and

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

then

$$\dot{x} = \begin{pmatrix} \dot{u} \\ \ddot{u} \end{pmatrix} = \begin{pmatrix} v \\ Au \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} x = Bx \text{ with}$$
$$x(0) = \begin{pmatrix} f \\ g \end{pmatrix},$$

where

$$B := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\omega^2 & 0 \end{pmatrix}.$$

Note formally that

$$e^{tB}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t\ f + \frac{\sin t\omega}{\omega} g \\ -\omega \sin\omega t\ f + \cos\omega t\ g \end{pmatrix}$$
$$= \begin{pmatrix} \cos\omega t & \frac{\sin t\omega}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \tag{13.4}$$

and this suggests that

$$e^{tB} = \begin{pmatrix} \cos\omega t & \frac{\sin t\omega}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}$$

which is formally correct since

$$\frac{d}{dt}e^{tB} = \begin{pmatrix} -\omega\sin\omega t & \cos\omega t \\ -\omega^2\cos\omega t & -\omega\sin\omega t \end{pmatrix}$$
$$= \begin{pmatrix} 0 & I \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} \cos\omega t & \frac{\sin t\omega}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix} = Be^{tB}.$$

Since the energy form $E(t) = \|\dot{u}\|^2 + \|\omega\ u\|^2$ is conserved, it is reasonable to let

$$K = D(\sqrt{-A}) \oplus H := \begin{pmatrix} D(\sqrt{-A}) \\ H \end{pmatrix}$$

with inner product

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix} \middle| \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} \right\rangle = (g, \tilde{g}) + (\omega f, \omega \tilde{f}).$$

For simplicity we assume $\mathrm{Nul}(\sqrt{-A}) = \mathrm{Nul}(\omega) = \{0\}$ in which case K becomes a Hilbert space and e^{tB} is a unitary evolution on K. Indeed,

$$\|e^{tB}\begin{pmatrix} f \\ g \end{pmatrix}\|_K^2 = \|\cos\omega tg - \omega\sin\omega tf\|^2 + \|\omega(\cos + \omega f) + \sin\omega tg\|^2$$
$$= \|\cos\omega tg\|^2 + \|\omega\sin\omega tf\|^2 + \|\omega\cos\omega tf\|^2 + \|\sin\omega tg\|^2$$
$$= \|\omega f\|^2 + \|g\|^2.$$

From Eq. (13.4), it easily follows that $\frac{d}{dt}\Big|_0 e^{tB}\begin{pmatrix} f \\ g \end{pmatrix}$ exists iff $g \in D(\omega)$ and $f \in D(-\omega^2) = D(A)$. Therefore we define $D(B) := D(A) \oplus D(\omega) = D(A) \oplus D\left(\sqrt{-A}\right)$and

$$B = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} : D(B) \to \begin{matrix} D(\omega) \\ \oplus \\ H \end{matrix} = K$$

Since B is the infinitesimal generator of a unitary semigroup, it follows that $B^{*K} = -B$, i.e. B is skew adjoint. This may be checked directly as well as follows.

Alternate Proof that $B^{*K} = -B$. For

$$\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(B) = D(A) \oplus D(\omega),$$

$$\begin{aligned} \langle B\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle &= \langle\begin{pmatrix} v \\ Au \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle = (Au, \tilde{v}) + (\omega v, \omega\tilde{u}) \\ &= (Au, \tilde{v}) - (Av, \tilde{u}) = (u, A\tilde{v}) - (v, A\tilde{u}) \end{aligned}$$

and similarly

$$\begin{aligned} \langle\begin{pmatrix} u \\ v \end{pmatrix}, B\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle &= \langle\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{v} \\ A\tilde{u} \end{pmatrix}\rangle = (\omega u, \omega\tilde{v}) + (v, A\tilde{u}) \\ &= (-Au, \tilde{v}) + (v, A\tilde{u}) = -\langle B\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}\rangle \end{aligned}$$

which shows $-B \subset B^*$. Conversely if $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(B^*)$ and $B^*\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$, then

$$\langle B\begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{matrix} \tilde{u} \\ \tilde{v} \end{matrix} \rangle = \langle \begin{matrix} u \\ v \end{matrix} \Big| \begin{matrix} f \\ g \end{matrix} \rangle = (v, g) + (\omega u, \omega f) \tag{13.5}$$

$(Au, \tilde{v}) + (\omega v, \omega\tilde{u})$ for all $u \in D(A), v \in D(\omega)$. Take $u = 0$ implies $(\omega v, \omega\tilde{u}) = (v, g)$ for all $v \in D(\omega)$ which then implies $\omega\tilde{u} \in D(\omega^*) = D(\omega)$ and hence $-A\tilde{u} = \omega^2\tilde{u} = g$. (Note $\tilde{u} \in D(A)$.) Taking $v = 0$ in Eq. (13.5) implies $(Au, \tilde{v}) = (\omega u, \omega f) = (-Au, f)$. Since

$$\overline{\mathrm{Ran}(A)} = \mathrm{Nul}(A)^\perp = \{0\}^\perp = H,$$

we find that $f = -\tilde{v} \in D(\omega)$ since $f \in D(\omega)$. Therefore $D(B^*) \subset D(B)$ and for $(\tilde{u}, \tilde{v}) \in D(B^*)$ we have

$$B^*\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = -\begin{pmatrix} -\tilde{v} \\ -A\tilde{u} \end{pmatrix} = -B\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

13.2. **Du Hamel's Principle.** Consider

$$\ddot{u} = Au + f(t) \text{ with } u(0) = g \text{ and } \dot{u}(0) = h. \tag{13.6}$$

Eq. (13.6) implies, with $v = \dot{u}$, that

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \ddot{u} \end{pmatrix} = \begin{pmatrix} v \\ Au + f \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{t\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}}\begin{pmatrix} g \\ h \end{pmatrix} + \int_0^t e^{(t-\tau)\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}}\begin{pmatrix} 0 \\ f(\tau) \end{pmatrix} d\tau$$

hence

$$u(t) = \cos(t\sqrt{-A})g + \frac{\sin(t\sqrt{-A})}{\sqrt{-A}}h + \int_0^t \frac{\sin((t-\tau)\sqrt{-A})}{\sqrt{-A}}f(\tau)d\tau.$$

Theorem 13.5. *Suppose $f(t) \in D(\sqrt{-A})$ for all t and that $f(t)$ is continuous relative to $\|f\|_A := \|f\| + \|\sqrt{-A}\, f\|$. Then*

$$u(t) := \int_0^t \frac{\sin((t-\tau)\sqrt{-A})}{\sqrt{-A}}\ f(\tau)d\tau$$

solves $\ddot{u} = Au + f$ with $u(0) = 0, \dot{u}(0) = 0$.

Proof. $\dot{u}(t) = \int_0^t \cos((t-\tau)\sqrt{-A})f(\tau)d\tau$.

$$\begin{aligned}\ddot{u}(t) &= f(t) - \int_0^t \sin((t-\tau)\sqrt{-A})\sqrt{-A}\ f(\tau)d\tau \\ &= f(t) - A\int_0^t \frac{\sin((t-\tau)\sqrt{-A})}{\sqrt{-A}}\ f(\tau)d\tau.\end{aligned}$$

So $\dot{u} = Au + f$. Note $u(0) = 0 = \dot{u}(0)$.

Alternate. Let $\omega := \sqrt{-A}$, then

$$\begin{aligned}u(t) &= \int_0^t \frac{\sin((t-\tau)\omega)}{\omega}\ f(\tau)d\tau \\ &= \int_0^t \frac{\sin\omega t\cos\omega\tau - \sin\omega\tau\cos\omega t}{\omega}\ f(\tau)d\tau\end{aligned}$$

and hence

$$\begin{aligned}\dot{u}(t) &= \frac{\sin\omega t\cos\omega t - \sin\omega t\cos\omega t}{\omega}\ f(t) \\ &\quad + \int_0^t (\cos\omega t\cos\omega\tau + \sin\omega\tau\sin\omega t)\ f(\tau)d\tau \\ &= \int_0^t (\cos\omega t\cos\omega\tau + \sin\omega\tau\sin\omega t)\ f(\tau)d\tau.\end{aligned}$$

Similarly,

$$\begin{aligned}\ddot{u}(t) &= (\cos\omega t\cos\omega t + \sin\omega t\sin\omega t)\ f(t) \\ &\quad + \int_0^t \omega\left(-\sin\omega t\cos\omega\tau + \sin\omega\tau\cos\omega t\right)\ f(\tau)d\tau \\ &= f(t) - \int_0^t \sin((t-\tau)\omega)\ \omega f(\tau)d\tau = f(t) - \omega^2 u(t) \\ &= Au(t) + f(t).\end{aligned}$$

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