21. Constant Coefficient partial differential equations

Suppose that $p(\xi) = \sum_{|\alpha| \le k} a_\alpha \xi^\alpha$ with $a_\alpha \in \mathbb{C}$ and

$$L = p(D_x) := \Sigma_{|\alpha| \le N} a_\alpha D_x^\alpha = \Sigma_{|\alpha| \le N} a_\alpha \left(\frac{1}{i}\partial_x\right)^\alpha. \tag{21.1}$$

Then for $f \in \mathcal{S}$

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

that is to say the Fourier transform takes a constant coefficient partial differential operator to multiplication by a polynomial. This fact can often be used to solve constant coefficient partial differential equation. For example suppose $g : \mathbb{R}^n \to \mathbb{C}$ is a given function and we want to find a solution to the equation $Lf = g$. Taking the Fourier transform of both sides of the equation $Lf = g$ would imply $p(\xi)\hat{f}(\xi) = \hat{g}(\xi)$ and therefore $\hat{f}(\xi) = \hat{g}(\xi)/p(\xi)$ provided $p(\xi)$ is never zero. (We will discuss what happens when $p(\xi)$ has zeros a bit more later on.) So we should expect

$$f(x) = \mathcal{F}^{-1}\left(\frac{1}{p(\xi)}\hat{g}(\xi)\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{p(\xi)}\right) \bigstar g(x).$$

Definition 21.1. Let $L = p(D_x)$ as in Eq. (21.1). Then we let $\sigma(L) :=\mathrm{Ran}(p) \subset \mathbb{C}$ and call $\sigma(L)$ the **spectrum** of L. Given a measurable function $G : \sigma(L) \to \mathbb{C}$, we define (a possibly unbounded operator) $G(L) : L^2(\mathbb{R}^n, m) \to L^2(\mathbb{R}^n, m)$ by

$$G(L)f := \mathcal{F}^{-1} M_{G\circ p} \mathcal{F}$$

where $M_{G\circ p}$ denotes the operation on $L^2(\mathbb{R}^n, m)$ of multiplication by $G \circ p$, i.e.

$$M_{G\circ p} f = (G \circ p)\, f$$

with domain given by those $f \in L^2$ such that $(G \circ p)\, f \in L^2$.

At a formal level we expect

$$G(L)f = \mathcal{F}^{-1}(G \circ p) \bigstar g.$$

21.0.3. *Elliptic examples.* As a specific example consider the equation

$$\left(-\Delta + m^2\right) f = g \tag{21.2}$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the usual Laplacian on $\mathbb{R}^n$. By Corollary 20.16 (i.e. taking the Fourier transform of this equation), solving Eq. (21.2) with $f, g \in L^2$ is equivalent to solving

$$\left(|\xi|^2 + m^2\right) \hat{f}(\xi) = \hat{g}(\xi). \tag{21.3}$$

The unique solution to this latter equation is

$$\hat{f}(\xi) = \left(|\xi|^2 + m^2\right)^{-1} \hat{g}(\xi)$$

and therefore,

$$f(x) = \mathcal{F}^{-1}\left(\left(|\xi|^2 + m^2\right)^{-1} \hat{g}(\xi)\right)(x) =: \left(-\Delta + m^2\right)^{-1} g(x).$$

We expect

$$\mathcal{F}^{-1}\left(\left(|\xi|^2 + m^2\right)^{-1} \hat{g}(\xi)\right)(x) = G_m \bigstar g(x) = \int_{\mathbb{R}^n} G_m(x - y) g(y) \mathbf{d}y,$$

where

$$G_m(x) := \mathcal{F}^{-1}\left(|\xi|^2+m^2\right)^{-1}(x) = \int_{\mathbb{R}^n} \frac{1}{m^2+|\xi|^2} e^{i\xi\cdot x}\mathbf{d}\xi.$$

At the moment $\mathcal{F}^{-1}\left(|\xi|^2+m^2\right)^{-1}$ only makes sense when $n = 1, 2$, or 3 because only then is $\left(|\xi|^2+m^2\right)^{-1} \in L^2(\mathbb{R}^n)$.

For now we will restrict our attention to the one dimensional case, $n = 1$, in which case

$$G_m(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{(\xi+mi)(\xi-mi)} e^{i\xi x} d\xi. \tag{21.4}$$

The function G_m may be computed using standard complex variable contour integration methods to find, for $x \geq 0$,

$$G_m(x) = \frac{1}{\sqrt{2\pi}} 2\pi i \frac{e^{i^2 mx}}{2im} = \frac{1}{2m}\sqrt{2\pi} e^{-mx}$$

and since G_m is an even function,

$$G_m(x) = \mathcal{F}^{-1}\left(|\xi|^2+m^2\right)^{-1}(x) = \frac{\sqrt{2\pi}}{2m} e^{-m|x|}. \tag{21.5}$$

This result is easily verified to be correct, since

$$\begin{aligned}
\mathcal{F}\left[\frac{\sqrt{2\pi}}{2m} e^{-m|x|}\right](\xi) &= \frac{\sqrt{2\pi}}{2m} \int_{\mathbb{R}} e^{-m|x|} e^{-ix\cdot\xi}\mathbf{d}x \\
&= \frac{1}{2m}\left(\int_0^\infty e^{-mx}e^{-ix\cdot\xi}dx + \int_{-\infty}^0 e^{mx}e^{-ix\cdot\xi}dx\right) \\
&= \frac{1}{2m}\left(\frac{1}{m+i\xi} + \frac{1}{m-i\xi}\right) = \frac{1}{m^2+\xi^2}.
\end{aligned}$$

Hence in conclusion we find that $\left(-\Delta+m^2\right) f = g$ has solution given by

$$f(x) = G_m \bigstar g(x) = \frac{\sqrt{2\pi}}{2m} \int_{\mathbb{R}} e^{-m|x-y|} g(y)\mathbf{d}y = \frac{1}{2m}\int_{\mathbb{R}} e^{-m|x-y|} g(y) dy.$$

Question. Why do we get a unique answer here given that $f(x) = A\sinh(x) + B\cosh(x)$ solves

$$\left(-\Delta+m^2\right) f = 0?$$

The answer is that such an f is not in L^2 unless $f = 0$! More generally it is worth noting that $A\sinh(x) + B\cosh(x)$ is not in $\mathcal{P}$ unless $A = B = 0$.

What about when $m = 0$ in which case $m^2+\xi^2$ becomes ξ^2 which has a zero at 0. Noting that constants are solutions to $\Delta f = 0$, we might look at

$$\lim_{m\downarrow 0}(G_m(x) - 1) = \lim_{m\downarrow 0} \frac{\sqrt{2\pi}}{2m}(e^{-m|x|} - 1) = -\frac{\sqrt{2\pi}}{2}|x|.$$

as a solution, i.e. we might conjecture that

$$f(x) := -\frac{1}{2}\int_{\mathbb{R}} |x-y|\, g(y) dy$$

solves the equation $-f'' = g$. To verify this we have

$$f(x) := -\frac{1}{2}\int_{-\infty}^x (x-y)\, g(y) dy - \frac{1}{2}\int_x^\infty (y-x)\, g(y) dy$$

so that

$$f'(x) = -\frac{1}{2}\int_{-\infty}^{x} g(y)dy + \frac{1}{2}\int_{x}^{\infty} g(y)dy \text{ and}$$
$$f''(x) = -\frac{1}{2}g(x) - \frac{1}{2}g(x).$$

21.0.4. *Poisson Semi-Group.* Let us now consider the problems of finding a function $(x_0, x) \in [0,\infty) \times \mathbb{R}^n \to u(x_0, x) \in \mathbb{C}$ such that

$$\left(\frac{\partial^2}{\partial x_0^2} + \Delta\right) u = 0 \text{ with } u(0,\cdot) = f \in L^2(\mathbb{R}^n). \tag{21.6}$$

Let $\hat{u}(x_0, \xi) := \int_{\mathbb{R}^n} u(x_0, x)e^{-ix\cdot\xi}\mathbf{d}x$ denote the Fourier transform of u in the $x \in \mathbb{R}^n$ variable. Then Eq. (21.6) becomes

$$\left(\frac{\partial^2}{\partial x_0^2} - |\xi|^2\right) \hat{u}(x_0, \xi) = 0 \text{ with } \hat{u}(0,\xi) = \hat{f}(\xi) \tag{21.7}$$

and the general solution to this differential equation ignoring the initial condition is of the form

$$\hat{u}(x_0, \xi) = A(\xi)e^{-x_0|\xi|} + B(\xi)e^{x_0|\xi|} \tag{21.8}$$

for some function $A(\xi)$ and $B(\xi)$. Let us now impose the extra condition that $u(x_0,\cdot) \in L^2(\mathbb{R}^n)$ or equivalently that $\hat{u}(x_0,\cdot) \in L^2(\mathbb{R}^n)$ for all $x_0 \geq 0$. The solution in Eq. (21.8) will not have this property unless $B(\xi)$ decays very rapidly at ∞. The simplest way to achieve this is to assume $B = 0$ in which case we now get a unique solution to Eq. (21.7), namely

$$\hat{u}(x_0, \xi) = \hat{f}(\xi)e^{-x_0|\xi|}.$$

Applying the inverse Fourier transform gives

$$u(x_0, x) = \mathcal{F}^{-1}\left[\hat{f}(\xi)e^{-x_0|\xi|}\right](x) =: \left(e^{-x_0\sqrt{-\Delta}}f\right)(x)$$

and moreover

$$\left(e^{-x_0\sqrt{-\Delta}}f\right)(x) = P_{x_0} * f(x)$$

where $P_{x_0}(x) = (2\pi)^{-n/2}\left(\mathcal{F}^{-1}e^{-x_0|\xi|}\right)(x)$. From Exercise 21.1,

$$P_{x_0}(x) = (2\pi)^{-n/2}\left(\mathcal{F}^{-1}e^{-x_0|\xi|}\right)(x) = c_n\frac{x_0}{(x_0^2 + |x|^2)^{(n+1)/2}}$$

where

$$c_n = (2\pi)^{-n/2}\frac{\Gamma((n+1)/2)}{\sqrt{\pi}2^{n/2}} = \frac{\Gamma((n+1)/2)}{2^n\pi^{(n+1)/2}}.$$

Hence we have proved the following proposition.

Proposition 21.2. *For $f \in L^2(\mathbb{R}^n)$,*

$$e^{-x_0\sqrt{-\Delta}}f = P_{x_0} * f \textit{ for all } x_0 \geq 0$$

and the function $u(x_0, x) := e^{-x_0\sqrt{-\Delta}}f(x)$ is C^∞ for $(x_0, x) \in (0,\infty) \times \mathbb{R}^n$ and solves Eq. (21.6).

21.0.5. *Heat Equation on $\mathbb{R}^n$.* The heat equation for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ is the partial differential equation

$$\left(\partial_t - \frac{1}{2}\Delta\right) u = 0 \text{ with } u(0,x) = f(x), \tag{21.9}$$

where f is a given function on $\mathbb{R}^n$. By Fourier transforming Eq. (21.9) in the x – variables only, one finds that (21.9) implies that

$$\left(\partial_t + \frac{1}{2}|\xi|^2\right) \hat{u}(t,\xi) = 0 \text{ with } \hat{u}(0,\xi) = \hat{f}(\xi). \tag{21.10}$$

and hence that $\hat{u}(t,\xi) = e^{-t|\xi|^2/2}\hat{f}(\xi)$. Inverting the Fourier transform then shows that

$$u(t,x) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\hat{f}(\xi)\right)(x) = \left(\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right)\bigstar f\right)(x) =: e^{t\Delta/2}f(x).$$

From Example 20.4,

$$\mathcal{F}^{-1}\left(e^{-t|\xi|^2/2}\right)(x) = p_t(x) = t^{-n/2}e^{-\frac{1}{2t}|x|^2}$$

and therefore,

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)\mathbf{d}y.$$

This suggests the following theorem.

Theorem 21.3. *Let*

$$\rho(t,x,y) := (2\pi t)^{-n/2}\, e^{-|x-y|^2/2t} \tag{21.11}$$

be the ***heat kernel*** *on $\mathbb{R}^n$. Then*

$$\left(\partial_t - \frac{1}{2}\Delta_x\right)\rho(t,x,y) = 0 \text{ and } \lim_{t\downarrow 0}\rho(t,x,y) = \delta_x(y), \tag{21.12}$$

where δ_x is the δ – function at x in $\mathbb{R}^n$. More precisely, if f is a continuous bounded (can be relaxed considerably) function on $\mathbb{R}^n$, then $u(t,x) = \int_{\mathbb{R}^n}\rho(t,x,y)f(y)dy$ is a solution to Eq. (21.9) where $u(0,x) := \lim_{t\downarrow 0}u(t,x)$.

Proof. Direct computations show that $\left(\partial_t - \frac{1}{2}\Delta_x\right)\rho(t,x,y) = 0$ and an application of Theorem 11.21 shows $\lim_{t\downarrow 0}\rho(t,x,y) = \delta_x(y)$ or equivalently that $\lim_{t\downarrow 0}\int_{\mathbb{R}^n}\rho(t,x,y)f(y)dy = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$. This shows that $\lim_{t\downarrow 0}u(t,x) = f(x)$ uniformly on compact subsets of $\mathbb{R}^n$. ∎

This notation suggests that we should be able to compute the solution to g to $(\Delta - m^2)g = f$ using

$$g(x) = \left(m^2 - \Delta\right)^{-1}f(x) = \int_0^\infty \left(e^{-\left(m^2-\Delta\right)t}f\right)(x)dt = \int_0^\infty \left(e^{-m^2t}p_{2t}\bigstar f\right)(x)dt,$$

a fact which is easily verified using the Fourier transform. This gives us a method to compute $G_m(x)$ from the previous section, namely

$$G_m(x) = \int_0^\infty e^{-m^2t}p_{2t}(x)dt = \int_0^\infty (2t)^{-n/2}e^{-m^2t-\frac{1}{4t}|x|^2}dt.$$

We make the change of variables, $\lambda = |x|^2/4t$ ($t = |x|^2/4\lambda$, $dt = -\frac{|x|^2}{4\lambda^2}d\lambda$) to find

$$G_m(x) = \int_0^\infty (2t)^{-n/2} e^{-m^2 t - \frac{1}{4t}|x|^2} dt = \int_0^\infty \left(\frac{|x|^2}{2\lambda}\right)^{-n/2} e^{-m^2|x|^2/4\lambda - \lambda} \frac{|x|^2}{(2\lambda)^2} d\lambda$$

$$= \frac{2^{(n/2-2)}}{|x|^{n-2}} \int_0^\infty \lambda^{n/2-2} e^{-\lambda} e^{-m^2|x|^2/4\lambda} d\lambda. \tag{21.13}$$

In case $n = 3$, Eq. (21.13) becomes

$$G_m(x) = \frac{\sqrt{\pi}}{\sqrt{2}\,|x|} \int_0^\infty \frac{1}{\sqrt{\pi\lambda}} e^{-\lambda} e^{-m^2|x|^2/4\lambda} d\lambda = \frac{\sqrt{\pi}}{\sqrt{2}\,|x|} e^{-m|x|}$$

where the last equality follows from Exercise 21.1. Hence when $n = 3$ we have found

$$\left(m^2 - \Delta\right)^{-1} f(x) = G_m \bigstar f(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \frac{\sqrt{\pi}}{\sqrt{2}\,|x-y|} e^{-m|x-y|} f(y) dy$$

$$= \int_{\mathbb{R}^3} \frac{1}{4\pi\,|x-y|} e^{-m|x-y|} f(y) dy. \tag{21.14}$$

The function $\frac{1}{4\pi|x|}e^{-m|x|}$ is called the Yukawa potential.

Let us work out $G_m(x)$ for n odd. By differentiating Eq. (21.26) of Exercise 21.1 we find

$$\int_0^\infty d\lambda \lambda^{k-1/2} e^{-\frac{1}{4\lambda}x^2} e^{-\lambda m^2} = \int_0^\infty d\lambda \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{4\lambda}x^2} \left(-\frac{d}{da}\right)^k e^{-\lambda a}|_{a=m^2}$$

$$= \left(-\frac{d}{da}\right)^k \frac{\sqrt{\pi}}{\sqrt{a}} e^{-\sqrt{a}x} = p_{m,k}(x) e^{-mx}$$

where $p_{m,k}(x)$ is a polynomial in x with $\deg p_m = k$ with

$$p_{m,k}(0) = \sqrt{\pi}\left(-\frac{d}{da}\right)^k a^{-1/2}|_{a=m^2} = \sqrt{\pi}(\frac{1}{2}\frac{3}{2}\dots\frac{2k-1}{2})m^{2k+1} = m^{2k+1}\sqrt{\pi}2^{-k}(2k-1)!!.$$

Letting $k - 1/2 = n/2 - 2$ and $m = 1$ we find $k = \frac{n-1}{2} - 2 \in \mathbb{N}$ for $n = 3, 5, \dots$. and we find

$$\int_0^\infty \lambda^{n/2-2} e^{-\frac{1}{4\lambda}x^2} e^{-\lambda} d\lambda = p_{1,k}(x) e^{-x} \text{ for all } x > 0.$$

Therefore,

$$G_m(x) = \frac{2^{(n/2-2)}}{|x|^{n-2}} \int_0^\infty \lambda^{n/2-2} e^{-\lambda} e^{-m^2|x|^2/4\lambda} d\lambda = \frac{2^{(n/2-2)}}{|x|^{n-2}} p_{1,n/2-2}(m\,|x|) e^{-m|x|}.$$

Now for even m, I think we get Bessel functions in the answer. (BRUCE: look this up.) Let us at least work out the asymptotics of $G_m(x)$ for $x \to \infty$. To this end let

$$\psi(y) := \int_0^\infty \lambda^{n/2-2} e^{-(\lambda+\lambda^{-1}y^2)} d\lambda = y^{n-2} \int_0^\infty \lambda^{n/2-2} e^{-(\lambda y^2 + \lambda^{-1})} d\lambda$$

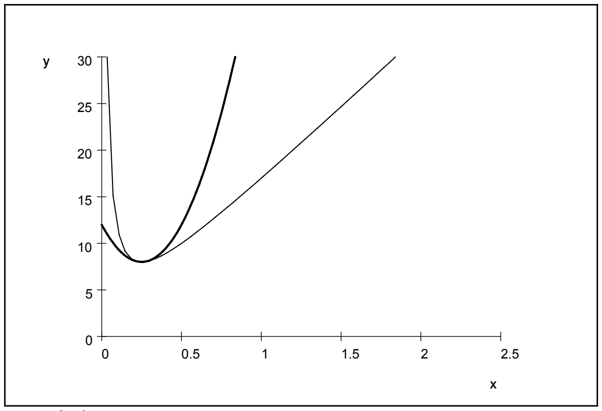
The function $f_y(\lambda) := (y^2\lambda + \lambda^{-1})$ satisfies,

$$f_y'(\lambda) = \left(y^2 - \lambda^{-2}\right) \text{ and } f_y''(\lambda) = 2\lambda^{-3} \text{ and } f_y'''(\lambda) = -6\lambda^{-4}$$

so by Taylor's theorem with remainder we learn

$$f_y(\lambda) \cong 2y + y^3(\lambda - y^{-1})^2 \text{ for all } \lambda > 0,$$

see Figure 21.0.5 below.



Plot of f_4 and its second order Taylor approximation.

So by the usual asymptotics arguments,

$$\begin{aligned}
\psi(y) &\cong y^{n-2} \int_{(-\epsilon+y^{-1},y^{-1}+\epsilon)} \lambda^{n/2-2} e^{-(\lambda y^2+\lambda^{-1})} d\lambda \\
&\cong y^{n-2} \int_{(-\epsilon+y^{-1},y^{-1}+\epsilon)} \lambda^{n/2-2} \exp\left(-2y - y^3(\lambda - y^{-1})^2\right) d\lambda \\
&\cong y^{n-2} e^{-2y} \int_{\mathbb{R}} \lambda^{n/2-2} \exp\left(-y^3(\lambda - y^{-1})^2\right) d\lambda \text{ (let } \lambda \to \lambda y^{-1}) \\
&= e^{-2y} y^{n-2} y^{-n/2+1} \int_{\mathbb{R}} \lambda^{n/2-2} \exp\left(-y(\lambda-1)^2\right) d\lambda \\
&= e^{-2y} y^{n-2} y^{-n/2+1} \int_{\mathbb{R}} (\lambda+1)^{n/2-2} \exp\left(-y\lambda^2\right) d\lambda.
\end{aligned}$$

The point is we are still going to get exponential decay at ∞.

When $m = 0$, Eq. (21.13) becomes

$$G_0(x) = \frac{2^{(n/2-2)}}{|x|^{n-2}} \int_0^\infty \lambda^{n/2-1} e^{-\lambda} \frac{d\lambda}{\lambda} = \frac{2^{(n/2-2)}}{|x|^{n-2}} \Gamma(n/2-1)$$

where $\Gamma(x)$ in the gamma function defined in Eq. (8.30). Hence for "reasonable" functions f (and $n \neq 2$)

$$\begin{aligned}
(-\Delta)^{-1} f(x) = G_0 \bigstar f(x) &= 2^{(n/2-2)} \Gamma(n/2-1)(2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy \\
&= \frac{1}{4\pi^{n/2}} \Gamma(n/2-1) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy.
\end{aligned}$$

The function

$$\tilde{G}_0(x,y) := \frac{1}{4\pi^{n/2}} \Gamma(n/2-1) \frac{1}{|x-y|^{n-2}}$$

is a "Green's function" for $-\Delta$. Recall from Exercise 8.16 that, for $n = 2k$, $\Gamma(\frac{n}{2} - 1) = \Gamma(k-1) = (k-2)!$, and for $n = 2k+1$,

$$\Gamma(\frac{n}{2} - 1) = \Gamma(k - 1/2) = \Gamma(k - 1 + 1/2) = \sqrt{\pi}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-3)}{2^{k-1}}$$
$$= \sqrt{\pi}\frac{(2k-3)!!}{2^{k-1}} \text{ where } (-1)!! \equiv 1.$$

Hence

$$\tilde{G}_0(x,y) = \frac{1}{4}\frac{1}{|x-y|^{n-2}}\begin{cases} \frac{1}{\pi^k}(k-2)! & \text{if} \quad n = 2k \\ \frac{1}{\pi^k}\frac{(2k-3)!!}{2^{k-1}} & \text{if} \quad n = 2k+1 \end{cases}$$

and in particular when $n = 3$,

$$\tilde{G}_0(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}$$

which is consistent with Eq. (21.14) with $m = 0$.

21.0.6. *Wave Equation on* $\mathbb{R}^n$. Let us now consider the wave equation on $\mathbb{R}^n$,

$$0 = \left(\partial_t^2 - \Delta\right) u(t,x) \text{ with}$$
$$u(0,x) = f(x) \text{ and } u_t(0,x) = g(x). \tag{21.15}$$

Taking the Fourier transform in the x variables gives the following equation

$$0 = \hat{u}_{tt}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) \text{ with}$$
$$\hat{u}(0,\xi) = \hat{f}(\xi) \text{ and } \hat{u}_t(0,\xi) = \hat{g}(\xi). \tag{21.16}$$

The solution to these equations is

$$\hat{u}(t,\xi) = \hat{f}(\xi)\cos(t|\xi|) + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|}$$

and hence we should have

$$u(t,x) = \mathcal{F}^{-1}\left(\hat{f}(\xi)\cos(t|\xi|) + \hat{g}(\xi)\frac{\sin t|\xi|}{|\xi|}\right)(x)$$
$$= \mathcal{F}^{-1}\cos(t|\xi|)\bigstar f(x) + \mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}\bigstar g(x)$$
$$= \frac{d}{dt}\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]\bigstar f(x) + \mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]\bigstar g(x). \tag{21.17}$$

The question now is how interpret this equation. In particular what are the inverse Fourier transforms of $\mathcal{F}^{-1}\cos(t|\xi|)$ and $\mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}$. Since $\frac{d}{dt}\mathcal{F}^{-1}\frac{\sin t|\xi|}{|\xi|}\bigstar f(x) = \mathcal{F}^{-1}\cos(t|\xi|)\bigstar f(x)$, it really suffices to understand $\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]$. The problem we immediately run into here is that $\frac{\sin t|\xi|}{|\xi|} \in L^2(\mathbb{R}^n)$ iff $n = 1$ so that is the case we should start with.

Again by complex contour integration methods one can show

$$\left(\mathcal{F}^{-1}\xi^{-1}\sin t\xi\right)(x) = \frac{\pi}{\sqrt{2\pi}}\left(1_{x+t>0} - 1_{(x-t)>0}\right)$$
$$= \frac{\pi}{\sqrt{2\pi}}\left(1_{x>-t} - 1_{x>t}\right) = \frac{\pi}{\sqrt{2\pi}}1_{[-t,t]}(x)$$

where in writing the last line we have assume that $t \geq 0$. Again this easily seen to be correct because

$$\mathcal{F}\left[\frac{\pi}{\sqrt{2\pi}}1_{[-t,t]}(x)\right](\xi) = \frac{1}{2}\int_{\mathbb{R}} 1_{[-t,t]}(x)e^{-i\xi\cdot x}dx = \frac{1}{-2i\xi}e^{-i\xi\cdot x}|_{-t}^{t}$$
$$= \frac{1}{2i\xi}\left[e^{i\xi t} - e^{-i\xi t}\right] = \xi^{-1}\sin t\xi.$$

Therefore,

$$\left(\mathcal{F}^{-1}\xi^{-1}\sin t\xi\right)\bigstar f(x) = \frac{1}{2}\int_{-t}^{t} f(x-y)dy$$

and the solution to the one dimensional wave equation is

$$u(t,x) = \frac{d}{dt}\frac{1}{2}\int_{-t}^{t} f(x-y)dy + \frac{1}{2}\int_{-t}^{t} g(x-y)dy$$
$$= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{-t}^{t} g(x-y)dy$$
$$= \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

We can arrive at this same solution by more elementary means as follows. We first note in the one dimensional case that wave operator factors, namely

$$0 = \left(\partial_t^2 - \partial_x^2\right)u(t,x) = \left(\partial_t - \partial_x\right)\left(\partial_t + \partial_x\right)u(t,x).$$

Let $U(t,x) := \left(\partial_t + \partial_x\right)u(t,x)$, then the wave equation states $\left(\partial_t - \partial_x\right)U = 0$ and hence by the chain rule $\frac{d}{dt}U(t, x-t) = 0$. So

$$U(t,x-t) = U(0,x) = g(x) + f'(x)$$

and replacing x by $x+t$ in this equation shows

$$\left(\partial_t + \partial_x\right)u(t,x) = U(t,x) = g(x+t) + f'(x+t).$$

Working similarly, we learn that

$$\frac{d}{dt}u(t,x+t) = g(x+2t) + f'(x+2t)$$

which upon integration implies

$$u(t,x+t) = u(0,x) + \int_0^t \left\{g(x+2\tau) + f'(x+2\tau)\right\}d\tau$$
$$= f(x) + \int_0^t g(x+2\tau)d\tau + \frac{1}{2}f(x+2\tau)|_0^t$$
$$= \frac{1}{2}\left(f(x) + f(x+2t)\right) + \int_0^t g(x+2\tau)d\tau.$$

Replacing $x \to x-t$ in this equation gives

$$u(t,x) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \int_0^t g(x-t+2\tau)d\tau$$

and then letting $y = x - t + 2\tau$ in the last integral shows again that

$$u(t,x) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy.$$

When $n > 3$ it is necessary to treat $\mathcal{F}^{-1}\left[\frac{\sin t|\xi|}{|\xi|}\right]$ as a "distribution" or "generalized function," see Section 27 below. So for now let us take $n = 3$, in which case from Example 20.18 it follows that

$$\mathcal{F}^{-1}\left[\frac{\sin t\,|\xi|}{|\xi|}\right] = \frac{t}{4\pi t^2}\sigma_t = t\bar{\sigma}_t \tag{21.18}$$

where $\bar{\sigma}_t$ is $\frac{1}{4\pi t^2}\sigma_t$, the surface measure on S_t normalized to have total measure one. Hence from Eq. (21.17) the solution to the three dimensional wave equation should be given by

$$u(t,x) = \frac{d}{dt}\left(t\bar{\sigma}_t \bigstar f(x)\right) + t\bar{\sigma}_t \bigstar g\left(x\right). \tag{21.19}$$

Using this definition in Eq. (21.19) gives

$$\begin{aligned} u(t,x) &= \frac{d}{dt}\left\{t\int_{S_t} f(x-y)d\bar{\sigma}_t(y)\right\} + t\int_{S_t} g(x-y)d\bar{\sigma}_t(y) \\ &= \frac{d}{dt}\left\{t\int_{S_1} f(x-t\omega)d\bar{\sigma}_1(\omega)\right\} + t\int_{S_1} g(x-t\omega)d\bar{\sigma}_1(\omega) \\ &= \frac{d}{dt}\left\{t\int_{S_1} f(x+t\omega)d\bar{\sigma}_1(\omega)\right\} + t\int_{S_1} g(x+t\omega)d\bar{\sigma}_1(\omega). \end{aligned} \tag{21.20}$$

Proposition 21.4. *Suppose $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$, then $u(t,x)$ defined by Eq. (21.20) is in $C^2\left(\mathbb{R}\times\mathbb{R}^3\right)$ and is a classical solution of the wave equation in Eq. (21.15).*

Proof. The fact that $u \in C^2\left(\mathbb{R}\times\mathbb{R}^3\right)$ follows by the usual differentiation under the integral arguments. Suppose we can prove the proposition in the special case that $f \equiv 0$. Then for $f \in C^3(\mathbb{R}^3)$, the function $v(t,x) = +t\int_{S_1} g(x+t\omega)d\bar{\sigma}_1(\omega)$ solves the wave equation $0 = \left(\partial_t^2 - \Delta\right)v(t,x)$ with $v(0,x) = 0$ and $v_t(0,x) = g(x)$. Differentiating the wave equation in t shows $u = v_t$ also solves the wave equaton with $u(0,x) = g(x)$ and $u_t(0,x) = v_{tt}(0,x) = -\Delta_x v(0,x) = 0$.

These remarks reduced the problems to showing u in Eq. (21.20) with $f \equiv 0$ solves the wave equation. So let

$$u(t,x) := t\int_{S_1} g(x+t\omega)d\bar{\sigma}_1(\omega). \tag{21.21}$$

We now give two proofs the u solves the wave equation.

Proof 1. Since solving the wave equation is a local statement and $u(t,x)$ only depends on the values of g in $B(x,t)$ we it suffices to consider the case where $g \in C_c^2\left(\mathbb{R}^3\right)$. Taking the Fourier transform of Eq. (21.21) in the x variable shows

$$\begin{aligned} \hat{u}(t,\xi) &= t\int_{S_1} d\bar{\sigma}_1(\omega)\int_{\mathbb{R}^3} g(x+t\omega)e^{-i\xi\cdot x}\mathbf{d}x \\ &= t\int_{S_1} d\bar{\sigma}_1(\omega)\int_{\mathbb{R}^3} g(x)e^{-i\xi\cdot x}e^{it\omega\cdot\xi}\mathbf{d}x = \hat{g}(\xi)t\int_{S_1} e^{it\omega\cdot\xi}d\bar{\sigma}_1(\omega) \\ &= \hat{g}(\xi)t\frac{\sin|tk|}{|tk|} = \hat{g}(\xi)\frac{\sin\left(t\,|\xi|\right)}{|\xi|} \end{aligned}$$

wherein we have made use of Example 20.18. This completes the proof since $\hat{u}(t,\xi)$ solves Eq. (21.16) as desired.

Proof 2. Differentiating

$$S(t,x) := \int_{S_1} g(x+t\omega)d\bar{\sigma}_1(\omega)$$

in t gives

$$\begin{aligned} S_t(t,x) &= \frac{1}{4\pi}\int_{S_1} \nabla g(x+t\omega)\cdot\omega d\sigma(\omega) = \frac{1}{4\pi}\int_{B(0,1)} \nabla_\omega \cdot \nabla g(x+t\omega)dm(\omega) \\ &= \frac{t}{4\pi}\int_{B(0,1)} \Delta g(x+t\omega)dm(\omega) = \frac{1}{4\pi t^2}\int_{B(0,t)} \Delta g(x+y)dm(y) \\ &= \frac{1}{4\pi t^2}\int_0^t dr\ r^2 \int_{|y|=r} \Delta g(x+y)d\sigma(y) \end{aligned}$$

where we have used the divergence theorem, made the change of variables $y = t\omega$ and used the disintegration formula in Eq. (8.27),

$$\int_{\mathbb{R}^d} f(x)dm(x) = \int_{[0,\infty)\times S^{n-1}} f(r\ \omega)\ d\sigma(\omega)r^{n-1}dr = \int_0^\infty dr \int_{|y|=r} f(y)d\sigma(y).$$

Since $u(t,x) = tS(t,x)$ if follows that

$$\begin{aligned} u_{tt}(t,x) &= \frac{\partial}{\partial t}\left[S(t,x)+tS_t(t,x)\right] \\ &= S_t(t,x) + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_0^t dr\ r^2 \int_{|y|=r} \Delta g(x+y)d\sigma(y)\right] \\ &= S_t(t,x) - \frac{1}{4\pi t^2}\int_0^t dr \int_{|y|=r} \Delta g(x+y)d\sigma(y) + \frac{1}{4\pi t}\int_{|y|=t} \Delta g(x+y)d\sigma(y) \\ &= S_t(t,x) - S_t(t,x) + \frac{t}{4\pi t^2}\int_{|y|=1} \Delta g(x+t\omega)d\sigma(\omega) = t\Delta u(t,x) \end{aligned}$$

as required. ■

The solution in Eq. (21.20) exhibits a basic property of wave equations, namely finite propagation speed. To exhibit the finite propagation speed, suppose that $f = 0$ (for simplicity) and g has compact support near the origin, for example think of $g = \delta_0(x)$. Then $x + tw = 0$ for some w iff $|x| = t$. Hence the "wave front" propagates at unit speed and the wave front is sharp. See Figure 38 below.

The solution of the two dimensional wave equation may be found using "Hadamard's method of decent" which we now describe. Suppose now that f and g are functions on $\mathbb{R}^2$ which we may view as functions on $\mathbb{R}^3$ which happen not to depend on the third coordinate. We now go ahead and solve the three dimensional wave equation using Eq. (21.20) and f and g as initial conditions. It is easily seen that the solution $u(t,x,y,z)$ is again independent of z and hence is a solution to the two dimensional wave equation. See figure 39 below.

Notice that we still have finite speed of propagation but no longer sharp propagation. The explicit formula for u is given in the next proposition.

Proposition 21.5. *Suppose $f \in C^3(\mathbb{R}^2)$ and $g \in C^2(\mathbb{R}^2)$, then*

$$u(t,x) := \frac{\partial}{\partial t}\left[\frac{t}{2\pi}\iint_{D_1} \frac{f(x+tw)}{\sqrt{1-|w|^2}}dm(w)\right] + \frac{t}{2\pi}\iint_{D_1} \frac{g(x+tw)}{\sqrt{1-|w|^2}}dm(w)$$

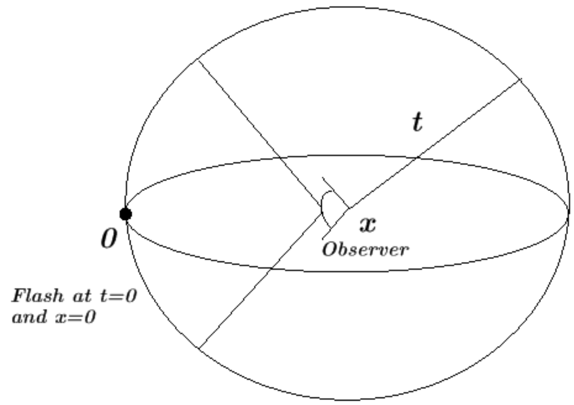


FIGURE 38. The geometry of the solution to the wave equation in three dimensions. The observer sees a flash at $t = 0$ and $x = 0$ only at time $t = |x|$. The wave progates sharply with speed 1.

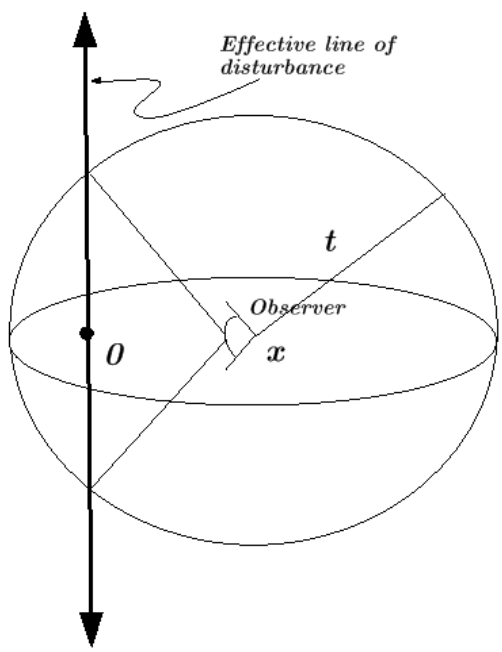


FIGURE 39. The geometry of the solution to the wave equation in two dimensions. A flash at $0 \in \mathbb{R}^2$ looks like a line of flashes to the fictitious $3 - \mathrm{d}$ observer and hence she sees the effect of the flash for $t \geq |x|$. The wave still propagates with speed 1. However there is no longer sharp propagation of the wave front, similar to water waves.

is in $C^2\left(\mathbb{R}\times\mathbb{R}^2\right)$ *and solves the wave equation in Eq. (21.15).*

Proof. As usual it suffices to consider the case where $f\equiv 0$. By symmetry u may be written as

$$u(t,x)=2t\int_{S_t^+}g(x-y)d\bar{\sigma}_t(y)=2t\int_{S_t^+}g(x+y)d\bar{\sigma}_t(y)$$

where S_t^+ is the portion of S_t with $z\geq 0$. The surface S_t^+ may be parametrized by $R(u,v)=(u,v,\sqrt{t^2-u^2-v^2})$ with $(u,v)\in D_t:=\left\{(u,v):u^2+v^2\leq t^2\right\}$. In these coordinates we have

$$\begin{aligned}4\pi t^2 d\bar{\sigma}_t &= \left|\left(-\partial_u\sqrt{t^2-u^2-v^2},-\partial_v\sqrt{t^2-u^2-v^2},1\right)\right|dudv\\ &=\left|\left(\frac{u}{\sqrt{t^2-u^2-v^2}},\frac{v}{\sqrt{t^2-u^2-v^2}},1\right)\right|dudv\\ &=\sqrt{\frac{u^2+v^2}{t^2-u^2-v^2}+1}dudv=\frac{|t|}{\sqrt{t^2-u^2-v^2}}dudv\end{aligned}$$

and therefore,

$$\begin{aligned}u(t,x)&=\frac{2t}{4\pi t^2}\int_{D_t}g(x+(u,v,\sqrt{t^2-u^2-v^2}))\frac{|t|}{\sqrt{t^2-u^2-v^2}}dudv\\ &=\frac{1}{2\pi}\operatorname{sgn}(t)\int_{D_t}\frac{g(x+(u,v))}{\sqrt{t^2-u^2-v^2}}dudv.\end{aligned}$$

This may be written as

$$\begin{aligned}u(t,x)&=\frac{1}{2\pi}\operatorname{sgn}(t)\iint_{D_t}\frac{g(x+w)}{\sqrt{t^2-|w|^2}}dm(w)=\frac{1}{2\pi}\operatorname{sgn}(t)\frac{t^2}{|t|}\iint_{D_1}\frac{g(x+tw)}{\sqrt{1-|w|^2}}dm(w)\\ &=\frac{1}{2\pi}t\iint_{D_1}\frac{g(x+tw)}{\sqrt{1-|w|^2}}dm(w)\end{aligned}$$

■

21.1. **Elliptic Regularity.** The following theorem is a special case of the main theorem (Theorem 21.10) of this section.

Theorem 21.6. *Suppose that* $M\subset_o\mathbb{R}^n$, $v\in C^\infty(M)$ *and* $u\in L^1_{loc}(M)$ *satisfies* $\Delta u=v$ *weakly, then* u *has a (necessarily unique) version* $\tilde{u}\in C^\infty(M)$.

Proof. We may always assume $n\geq 3$, by embedding the $n=1$ and $n=2$ cases in the $n=3$ cases. For notational simplicity, assume $0\in M$ and we will show u is smooth near 0. To this end let $\theta\in C_c^\infty(M)$ such that $\theta=1$ in a neighborhood of 0 and $\alpha\in C_c^\infty(M)$ such that $\operatorname{supp}(\alpha)\subset\{\theta=1\}$ and $\alpha=1$ in a neighborhood of 0 as well. Then formally, we have with $\beta:=1-\alpha$,

$$\begin{aligned}G*(\theta v)&=G*(\theta\Delta u)=G*(\theta\Delta(\alpha u+\beta u))\\ &=G*(\Delta(\alpha u)+\theta\Delta(\beta u))=\alpha u+G*(\theta\Delta(\beta u))\end{aligned}$$

so that

$$u(x)=G*(\theta v)(x)-G*(\theta\Delta(\beta u))(x)$$

for $x \in \operatorname{supp}(\alpha)$. The last term is formally given by

$$G*(\theta\Delta(\beta u))(x) = \int_{\mathbb{R}^n} G(x-y)\theta(y)\Delta(\beta(y)u(y))dy$$
$$= \int_{\mathbb{R}^n} \beta(y)\Delta_y\left[G(x-y)\theta(y)\right]\cdot u(y)dy$$

which makes sense for x near 0. Therefore we find

$$u(x) = G*(\theta v)(x) - \int_{\mathbb{R}^n} \beta(y)\Delta_y\left[G(x-y)\theta(y)\right]\cdot u(y)dy.$$

Clearly all of the above manipulations were correct if we know u were C^2 to begin with. So for the general case, let $u_n = u*\delta_n$ with $\{\delta_n\}_{n=1}^\infty$ – the usual sort of δ – sequence approximation. Then $\Delta u_n = v*\delta_n =: v_n$ away from ∂M and

$$u_n(x) = G*(\theta v_n)(x) - \int_{\mathbb{R}^n} \beta(y)\Delta_y\left[G(x-y)\theta(y)\right]\cdot u_n(y)dy. \tag{21.22}$$

Since $u_n \to u$ in $L^1_{loc}(\mathcal{O})$ where $\mathcal{O}$ is a sufficiently small neighborhood of 0, we may pass to the limit in Eq. (21.22) to find $u(x) = \tilde{u}(x)$ for a.e. $x \in \mathcal{O}$ where

$$\tilde{u}(x) := G*(\theta v)(x) - \int_{\mathbb{R}^n} \beta(y)\Delta_y\left[G(x-y)\theta(y)\right]\cdot u(y)dy.$$

This concluded the proof since $\tilde{u}$ is smooth for x near 0. ■

Definition 21.7. We say $L = p(D_x)$ as defined in Eq. (21.1) is **elliptic** if $p_k(\xi) := \sum_{|\alpha|=k} a_\alpha \xi^\alpha$ is zero iff $\xi = 0$. We will also say the polynomial $p(\xi) := \sum_{|\alpha|\le k} a_\alpha \xi^\alpha$ is **elliptic** if this condition holds.

Remark 21.8. If $p(\xi) := \sum_{|\alpha|\le k} a_\alpha \xi^\alpha$ is an elliptic polynomial, then there exists $A < \infty$ such that $\inf_{|\xi|\ge A} |p(\xi)| > 0$. Since $p_k(\xi)$ is everywhere non-zero for $\xi \in S^{n-1}$ and $S^{n-1} \subset \mathbb{R}^n$ is compact, $\epsilon := \inf_{|\xi|=1} |p_k(\xi)| > 0$. By homogeneity this implies

$$|p_k(\xi)| \ge \epsilon\,|\xi|^k \text{ for all } \xi \in \mathbb{A}^n.$$

Since

$$|p(\xi)| = \left|p_k(\xi) + \sum_{|\alpha|<k} a_\alpha\xi^\alpha\right| \ge |p_k(\xi)| - \left|\sum_{|\alpha|<k} a_\alpha\xi^\alpha\right|$$
$$\ge \epsilon\,|\xi|^k - C\left(1+|\xi|^{k-1}\right)$$

for some constant $C < \infty$ from which it is easily seen that for A sufficiently large,

$$|p(\xi)| \ge \frac{\epsilon}{2}\,|\xi|^k \text{ for all } |\xi| \ge A.$$

For the rest of this section, let $L = p(D_x)$ be an elliptic operator and $M \subset_0 \mathbb{R}^n$. As mentioned at the beginning of this section, the formal solution to $Lu = v$ for $v \in L^2(\mathbb{R}^n)$ is given by

$$u = L^{-1}v = G*v$$

where

$$G(x) := \int_{\mathbb{R}^n} \frac{1}{p(\xi)} e^{ix\cdot\xi}\mathbf{d}\xi.$$

Of course this integral may not be convergent because of the possible zeros of p and the fact $\frac{1}{p(\xi)}$ may not decay fast enough at infinity. We we will introduce a smooth cut off function $\chi(\xi)$ which is 1 on $C_0(A) := \{x \in \mathbb{R}^n : |x| \le A\}$ and $\operatorname{supp}(\chi) \subset C_0(2A)$ where A is as in Remark 21.8. Then for $M > 0$ let

$$G_M(x) = \int_{\mathbb{R}^n} \frac{(1-\chi(\xi))\,\chi(\xi/M)}{p(\xi)} e^{ix\cdot\xi}\mathbf{d}\xi, \tag{21.23}$$

$$\delta(x) := \chi^\vee(x) = \int_{\mathbb{R}^n} \chi(\xi) e^{ix\cdot\xi}\mathbf{d}\xi, \text{ and } \delta_M(x) = M^n\delta(Mx). \tag{21.24}$$

Notice $\int_{\mathbb{R}^n}\delta(x)dx = \mathcal{F}\delta(0) = \chi(0) = 1$, $\delta \in \mathcal{S}$ since $\chi \in \mathcal{S}$ and

$$\begin{aligned} LG_M(x) &= \int_{\mathbb{R}^n} (1-\chi(\xi))\,\chi(\xi/M) e^{ix\cdot\xi}\mathbf{d}\xi = \int_{\mathbb{R}^n} [\chi(\xi/M) - \chi(\xi)]\, e^{ix\cdot\xi}\mathbf{d}\xi \\ &= \delta_M(x) - \delta(x) \end{aligned}$$

provided $M > 2$.

Proposition 21.9. *Let p be an elliptic polynomial of degree m. The function G_M defined in Eq. (21.23) satisfies the following properties,*

1. *$G_M \in \mathcal{S}$ for all $M > 0$.*
2. *$LG_M(x) = M^n\delta(Mx) - \delta(x)$.*
3. *There exists $G \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ such that for all multi-indecies α, $\lim_{M\to\infty}\partial^\alpha G_M(x) = \partial^\alpha G(x)$ uniformly on compact subsets in $\mathbb{R}^n \setminus \{0\}$.*

Proof. We have already proved the first two items. For item 3., we notice that

$$\begin{aligned} (-x)^\beta D^\alpha G_M(x) &= \int_{\mathbb{R}^n} \frac{(1-\chi(\xi))\,\chi(\xi/M)\xi^\alpha}{p(\xi)} (-D)_\xi^\beta e^{ix\cdot\xi}\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} D_\xi^\beta\left[\frac{(1-\chi(\xi))\,\xi^\alpha}{p(\xi)}\chi(\xi/M)\right] e^{ix\cdot\xi}\mathbf{d}\xi \\ &= \int_{\mathbb{R}^n} D_\xi^\beta\frac{(1-\chi(\xi))\,\xi^\alpha}{p(\xi)}\cdot\chi(\xi/M)e^{ix\cdot\xi}\mathbf{d}\xi + R_M(x) \end{aligned}$$

where

$$R_M(x) = \sum_{\gamma<\beta}\binom{\beta}{\gamma}M^{|\gamma|-|\beta|}\int_{\mathbb{R}^n} D_\xi^\gamma\frac{(1-\chi(\xi))\,\xi^\alpha}{p(\xi)}\cdot\left(D^{\beta-\gamma}\chi\right)(\xi/M)e^{ix\cdot\xi}\mathbf{d}\xi.$$

Using

$$\left|D_\xi^\gamma\left[\frac{\xi^\alpha}{p(\xi)}(1-\chi(\xi))\right]\right| \le C\,|\xi|^{|\alpha|-m-|\gamma|}$$

and the fact that

$$\operatorname{supp}(\left(D^{\beta-\gamma}\chi\right)(\xi/M)) \subset \{\xi\in\mathbb{R}^n : A \le |\xi|/M \le 2A\} = \{\xi \in \mathbb{R}^n : AM \le |\xi| \le 2AM\}$$

we easily estimate

$$\begin{aligned} |R_M(x)| &\le C\sum_{\gamma<\beta}\binom{\beta}{\gamma}M^{|\gamma|-|\beta|}\int_{\{\xi\in\mathbb{R}^n: AM\le|\xi|\le 2AM\}} |\xi|^{|\alpha|-m-|\gamma|}\,\mathbf{d}\xi \\ &\le C\sum_{\gamma<\beta}\binom{\beta}{\gamma}M^{|\gamma|-|\beta|}M^{|\alpha|-m-|\gamma|+n} = CM^{|\alpha|-|\beta|-m+n}. \end{aligned}$$

Therefore, $R_M \to 0$ uniformly in x as $M \to \infty$ provided $|\beta| > |\alpha| - m + n$. It follows easily now that $G_M \to G$ in $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and furthermore that

$$(-x)^\beta D^\alpha G(x) = \int_{\mathbb{R}^n} D_\xi^\beta\frac{(1-\chi(\xi))\,\xi^\alpha}{p(\xi)}\cdot e^{ix\cdot\xi}\mathbf{d}\xi$$

provided β is sufficiently large. In particular we have shown,

$$D^\alpha G(x) = \frac{1}{|x|^{2k}} \int_{\mathbb{R}^n} (-\Delta_\xi)^k \frac{(1-\chi(\xi))\,\xi^\alpha}{p(\xi)} \cdot e^{ix\cdot\xi} \mathbf{d}\xi$$

provided $m - |\alpha| + 2k > n$, i.e. $k > (n - m + |\alpha|)/2$.

We are now ready to use this result to prove elliptic regularity for the constant coefficient case. ■

Theorem 21.10. *Suppose $L = p(D_\xi)$ is an elliptic differential operator on $\mathbb{R}^n$, $M \subset_o \mathbb{R}^n$, $v \in C^\infty(M)$ and $u \in L^1_{loc}(M)$ satisfies $Lu = v$ weakly, then u has a (necessarily unique) version $\tilde{u} \in C^\infty(M)$.*

Proof. For notational simplicity, assume $0 \in M$ and we will show u is smooth near 0. To this end let $\theta \in C_c^\infty(M)$ such that $\theta = 1$ in a neighborhood of 0 and $\alpha \in C_c^\infty(M)$ such that $\text{supp}(\alpha) \subset \{\theta = 1\}$, and $\alpha = 1$ in a neighborhood of 0 as well. Then formally, we have with $\beta := 1 - \alpha$,

$$\begin{aligned} G_M * (\theta v) &= G_M * (\theta L u) = G_M * (\theta L(\alpha u + \beta u)) \\ &= G_M * (L(\alpha u) + \theta L(\beta u)) = \delta_M * (\alpha u) - \delta * (\alpha u) + G_M * (\theta L(\beta u)) \end{aligned}$$

so that

$$\delta_M * (\alpha u)(x) = G_M * (\theta v)(x) - G_M * (\theta L(\beta u))(x) + \delta * (\alpha u). \tag{21.25}$$

Since

$$\begin{aligned} \mathcal{F}\left[G_M * (\theta v)\right](\xi) &= \hat{G}_M(\xi)\,(\theta v)\hat{}\,(\xi) = \frac{(1-\chi(\xi))\,\chi(\xi/M)}{p(\xi)} (\theta v)\hat{}\,(\xi) \\ &\to \frac{(1-\chi(\xi))}{p(\xi)} (\theta v)\hat{}\,(\xi) \text{ as } M \to \infty \end{aligned}$$

with the convergence taking place in L^2 (actually in $\mathcal{S}$), it follows that

$$\begin{aligned} G_M * (\theta v) \to \text{“}G * (\theta v)\text{”}(x) &:= \int_{\mathbb{R}^n} \frac{(1-\chi(\xi))}{p(\xi)} (\theta v)\hat{}\,(\xi) e^{ix\cdot\xi} \mathbf{d}\xi \\ &= \mathcal{F}^{-1}\left[\frac{(1-\chi(\xi))}{p(\xi)} (\theta v)\hat{}\,(\xi)\right](x) \in \mathcal{S}. \end{aligned}$$

So passing the the limit, $M \to \infty$, in Eq. (21.25) we learn for almost every $x \in \mathbb{R}^n$,

$$u(x) = G * (\theta v)(x) - \lim_{M\to\infty} G_M * (\theta L(\beta u))(x) + \delta * (\alpha u)(x)$$

for a.e. $x \in \text{supp}(\alpha)$. Using the support properties of θ and β we see for x near 0 that $(\theta L(\beta u))(y) = 0$ unless $y \in \text{supp}(\theta)$ and $y \notin \{\alpha = 1\}$, i.e. unless y is in an annulus centered at 0. So taking x sufficiently close to 0, we find $x - y$ stays away from 0 as y varies through the above mentioned annulus, and therefore

$$\begin{aligned} G_M * (\theta L(\beta u))(x) &= \int_{\mathbb{R}^n} G_M(x-y)(\theta L(\beta u))(y)\mathbf{d}y \\ &= \int_{\mathbb{R}^n} L_y^* \left\{\theta(y) G_M(x-y)\right\} \cdot (\beta u)(y)\mathbf{d}y \\ &\to \int_{\mathbb{R}^n} L_y^* \left\{\theta(y) G(x-y)\right\} \cdot (\beta u)(y)\mathbf{d}y \text{ as } M \to \infty. \end{aligned}$$

Therefore we have shown,

$$u(x) = G * (\theta v)(x) - \int_{\mathbb{R}^n} L_y^* \{\theta(y) G(x-y)\} \cdot (\beta u)(y) \mathbf{d}y + \delta * (\alpha u)(x)$$

for almost every x in a neighborhood of 0. (Again it suffices to prove this equation and in particular Eq. (21.25) assuming $u \in C^2(M)$ because of the same convolution argument we have use above.) Since the right side of this equation is the linear combination of smooth functions we have shown u has a smooth version in a neighborhood of 0. ■

Remarks 21.11. We could avoid introducing $G_M(x)$ if $\deg(p) > n$, in which case $\frac{(1-\chi(\xi))}{p(\xi)} \in L^1$ and so

$$G(x) := \int_{\mathbb{R}^n} \frac{(1-\chi(\xi))}{p(\xi)} e^{ix\cdot\xi} \mathbf{d}\xi$$

is already well defined function with $G \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap BC(\mathbb{R}^n)$. If $\deg(p) < n$, we may consider the operator $L^k = [p(D_x)]^k = p^k(D_x)$ where k is chosen so that $k \cdot \deg(p) > n$. Since $Lu = v$ implies $L^k u = L^{k-1} v$ weakly, we see to prove the hypoellipticity of L it suffices to prove the hypoellipticity of L^k.

21.2. **Exercises.**

Exercise 21.1. Using

$$\frac{1}{|\xi|^2 + m^2} = \int_0^\infty e^{-\lambda(|\xi|^2 + m^2)} d\lambda,$$

the identity in Eq. (21.5) and Example 20.4, show for $m > 0$ and $x \geq 0$ that

$$e^{-mx} = \frac{m}{\sqrt{\pi}} \int_0^\infty d\lambda \frac{1}{\sqrt{\lambda}} e^{-\frac{1}{4\lambda}x^2} e^{-\lambda m^2} \text{ (let } \lambda \to \lambda/m^2) \tag{21.26}$$

$$= \int_0^\infty d\lambda \frac{1}{\sqrt{\pi\lambda}} e^{-\lambda} e^{-\frac{m^2}{4\lambda}x^2}. \tag{21.27}$$

Use this formula and Example 20.4 to show, in dimension n, that

$$\mathcal{F}\left[e^{-m|x|}\right](\xi) = 2^{n/2} \frac{\Gamma((n+1)/2)}{\sqrt{\pi}} \frac{m}{(m^2 + |\xi|^2)^{(n+1)/2}}$$

where $\Gamma(x)$ in the gamma function defined in Eq. (8.30). (I am not absolutely positive I have got all the constants exactly right, but they should be close.)