Math 21D (Driver) Midterm II, November 22, 1999

Instructions: Please put your name and section number or time on your blue book. Each problems is worth 10 points except for problems 5 and 9 which are worth 15 points. You should do **all** of the problems on the test. Please explain your work! **GOOD LUCK!!**

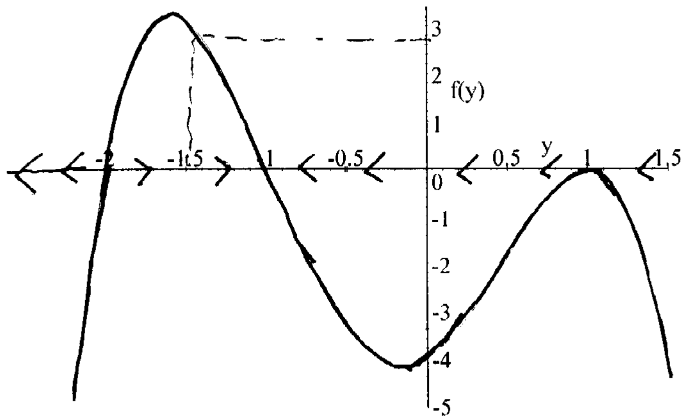


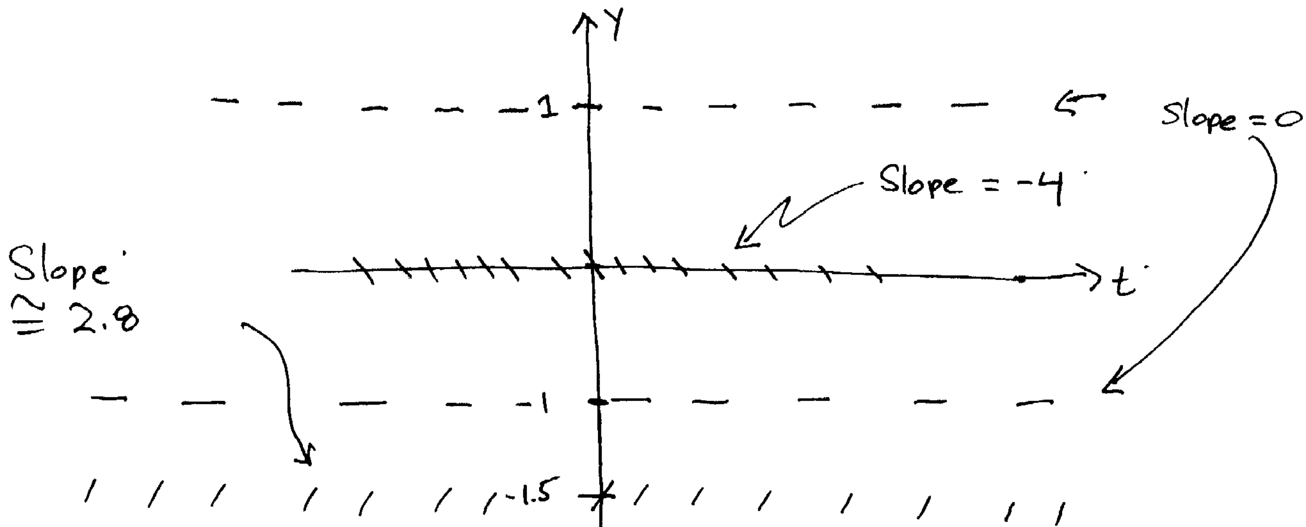
Figure 1. Plot of $f(y)$ versus y.

The zeros of the function f are located at $y = -2, -1,$ and 1.

1. Draw a rough sketch of the direction fields at heights $y = -1.5, -1, 0, 1$ for the differential equation

$$y'(t) = f(y(t))$$

where f is the function graphed in Figure 1 above. (The coordinate axis of your picture should be labeled by y on the vertical axis and t on the horizontal axis.)



2. (No explanation need be given for this problem but you must get the correct answer.) Let $y(t)$ be a solution to the differential equation

$$y'(t) = f(y(t))$$

where f is the function graphed in Figure 1 above.

(a) Assuming the $y(0) = -1.5$, what is $\lim_{t\to\infty} y(t)$? Ans. $\lim_{t\to\infty} y(t) = -1$.
(b) Assuming the $y(0) = 0$, what is $\lim_{t\to\infty} y(t)$? Ans. $\lim_{t\to\infty} y(t) = -1$.
(c) Assuming the $y(0) = 1$, what is $\lim_{t\to\infty} y(t)$? Ans. $\lim_{t\to\infty} y(t) = 1$.

3. Find the most general solution to

$$\dot{y}(t) + \frac{2}{t} y(t) = t \qquad \text{for } t > 0.$$

Solution, let $\mu(t) := \exp \int \frac{2}{t} dt = \exp(2 \ln t) = t^2$ is the integrating factor, so that

$$y(t) = \frac{1}{t^2}\left(\int t^2 t dt + C\right) = \frac{t^2}{4} + Ct^{-2}.$$

4. Describe **implicitly** the general solution to the differential equation

$$\dot{y}(x) = \cos(x) e^{y(x)}.$$

Do not try to solve explicitly for $y(x)$.

Ans. The equation is separable,

$$-e^{-y} = \int e^{-y} dy = \int \cos(x) dx + C = \sin(x) + C$$

Hence $e^{y(x)} = -\sin(x) - C$. Actually, this one is easy to solve for $y(x)$ to find

$$y(x) = \ln(K - \sin(x))$$

where $K = -C$ is a constant.

5. (15 pts) Consider the differential form:

$$(y + 3x^2)dx + (x + 2y)dy = 0. \qquad (*)$$

(a) Verify that this expression is exact. Ans. Let $M = (y + 3x^2)$ and $N = (x + 2y)$. The test for exactness is

$$1 = M_y \stackrel{?}{=} N_x = 1$$

which does hold in this case. So (*) is exact.

(b) Find a function $\psi(x,y)$ so that any solution to the differential equation

$$(y+3x^2)+(x+2y)\frac{dy}{dx}=0$$

satisfies $\psi(x,y(x))=C$, for some constant C.
Ans. We have to solve $\psi_x = M = (y+3x^2)$ and $\psi_y = N = (x+2y)$. Integrating the first equation relative to x implies that $\psi(x,y) = yx + x^3 + C(y)$. Plugging this into the second equation gives

$$x + C'(y) = \psi_y = N = (x+2y)$$

and hence that $C'(y) = 2y$. This implies that $C(y) = y^2 + K$. Therefore,

$$\psi(x,y) = yx + x^3 + y^2 + K.$$

6. Determine the largest interval in which the initial value problem,

$$(t-3)(t+1)\ddot{y}(t) - \ln|t|\dot{y}(t) + y(t) = 2, \qquad y(1)=3, \qquad \dot{y}(1) = -1,$$

is certain to have a unique twice differentiable solution. (**Do not** try to solve the equation!!)

Ans. we rewrite the equations as

$$\ddot{y}(t) - \frac{\ln|t|}{(t-3)(t+1)}\dot{y}(t) + \frac{1}{(t-3)(t+1)}y(t) = \frac{2}{(t-3)(t+1)}, \qquad y(1)=3, \qquad \dot{y}(1)=-1.$$

So the coefficients have singularities at $t=-1, 0$ and 3. Since the initial condition is at $t=1$, we are guaranteed a unique solution for $0<t<3$.

7. Compute the Wronskian of $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. Are the functions y_1 and y_2 linearly independent?

The Wronskian is

$$\begin{aligned} W(t) &= \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t)+t\cos(t) \end{vmatrix} = \sin(t)\left(\sin(t)+t\cos(t)\right) - t\sin(t)\cos(t) \\ &= \sin^2(t). \end{aligned}$$

$W(\pi/2) \neq 0$ implies that y_1 and y_2 are linearly independent. Alternatively, $y_2(t)/y_1(t) = t$ is not the constant function, hence again y_1 and y_2 are linearly independent.

8. Find the solution to the initial value problem:

$$L(y) = \ddot{y} - 4\dot{y} + 5y = 0, \qquad y(0)=0, \qquad \dot{y}(0)=1.$$

Ans. The characteristic polynomial is $\lambda^2 - 4\lambda + 5$ which has roots

$$\lambda = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = 2 \pm \sqrt{4 - 5} = 2 \pm i.$$

Therefore, the general solution is

$$y(t) = e^{2t}(C_1 \cos(t) + C_2 \sin(t)).$$

Determine the constants from the initial conditions

$$0 = y(0) = C_1$$

which shows that $y(t) = C_2 e^{2t} \sin(t)$. Since,

$$\begin{aligned} 1 &= \dot{y}(0) = C_2(2e^{2t} \sin(t) + e^{2t} \cos(t))|_{t=0} \\ &= C_2 \end{aligned}$$

we learn that the solution is

$$y(t) = e^{2t} \sin(t).$$

9. (15 pts) Use the method of undetermined coefficients to find a particular solution to the differential equation

$$L(y) = \ddot{y} - 4\dot{y} + 5y = e^t + 5t - 4.$$

We will solve this in parts. First $L(e^t) = (1 - 4 + 5)e^t = 2e^t$ so that

$$L(\frac{1}{2}e^t) = e^t.$$

Also we have that

$$\begin{aligned} L(A + Bt) &= -4B + 5(A + Bt) = 5A - 4B + 5Bt \\ &\overset{\text{set}}{=} 5t - 4. \end{aligned}$$

This shows that $5B = 5$ and $5A - 4B = -4$, so that $B = 1$ and $A = 0$. Thus

$$L(t) = 5t - 4.$$

From the previous equations and linearity of L it follows that

$$L(\frac{1}{2}e^t + t) = L(\frac{1}{2}e^t) + L(t) = e^t + 5t - 4$$

so a particular solution is

$$Y(t) = \frac{1}{2}e^t + t.$$