ANS KEY. OCT. 25, 1999

MIDTERM I SOLUTIONS (DRIVER MATH 21D, FALL 99)

(1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{4n^4 + \text{L.O.T.}}{5\cdot 4\, n^4 + \text{L.O.T.}}$ ← Lower ORDER TERMS

$= \boxed{\dfrac{1}{5}}$

(2) $\ln(a_n) = \dfrac{1}{n}\ln(1+3n)$

$\lim_{n\to\infty} \dfrac{1}{n}\ln(1+3n) = \lim_{n\to\infty} \dfrac{\frac{3}{1+3n}}{1}$ L'hopital's rule or $\ln(x)$ grows slower than x.

$= 0$

So $a_n = e^{\ln(a_n)} \xrightarrow[n\to\infty]{} e^0 = \boxed{1}$.

$\boxed{\lim_{n\to\infty} a_n = 1}$,

(3) $S_2 = (a_2 - a_0) + (a_3 - a_1) + (a_4 - a_2)$

$= -a_0 - a_1 + a_4 + a_3$

$S_3 = S_2 + (a_5 - a_3) = -a_0 - a_1 + a_5 + a_4$

$S_4 = S_3 + (a_6 - a_4) = -a_0 - a_1 + a_6 + a_5$

$\boxed{S_N = -a_0 - a_1 + a_{N+2} + a_{N+1}}$

(4) Converges by alternating series test, notice that $b_n = f(n) = \dfrac{1}{2\sqrt{n}+1}$ is decreasing.

(5) Let $a_n = \dfrac{e^{1/n}}{n^2+n}$ $b_n = \dfrac{1}{n^2}$.

$\lim_{n\to\infty} \dfrac{b_n}{a_n} = \lim_{n\to\infty} \dfrac{n^2}{n^2+n} e^{\frac{1}{n}} = 1 \cdot 1 = 1$

Now $\sum \frac{1}{n^2} < \infty$ (p series with $p = 2 > 1$).
So by the limit comparison theorem

$$\sum \frac{e^{1/n}}{n^2+n} < \infty \quad \text{as well, i.e. } \boxed{\text{Convergent.}}$$

(6)

$$|S - S_{25}| = \left| \sum_{n=26}^{\infty} \frac{\cos(e^{\sin(n)})}{n^2} \right| \le \sum_{n=26}^{\infty} \frac{|\cos(e^{\sin(n)})|}{n^2}$$

$$\le \sum_{n=26}^{\infty} \frac{1}{n^2} < \infty.$$ So the sum is

absolutely convergent.

Also $|S - S_{25}| \le \int_{25}^{\infty} \frac{1}{x^2}\,dx = \boxed{\frac{1}{25}}$ by the Integral test.

(7)

Let $a_n = \frac{n^2-2n}{3^n}(x-5)^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)^2 - 2(n+1)}{3^{n+1}} (x-5)^{n+1} \cdot \frac{3^n}{(n^2-2n)(x-5)^n} \right|$$

$$= \left|\frac{x-5}{3}\right| \left| \frac{(n+1)^2 - 2(n+1)}{n^2-2n} \right| \xrightarrow[n\to\infty]{} \left|\frac{x-5}{3}\right|.$$

Which is less than 1 $\iff$ $|x-5| < 3$.

So $\boxed{R = 3}$ by the Ratio test

(8) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$

$f(x) = 1 + \frac{3}{1!}(x-1) + \frac{\cancel{6}^{\,3}}{\cancel{2!}}(x-1)^2 + \frac{\cancel{6}^{\,1}}{\cancel{3!}}(x-1)^3$

	at $x=1$
$f(x) = x^3$	1
$f' = 3x^2$	3
$f'' = 3!\,x$	6
$f''' = 3!$	6
$f^{(n)} = 0 \quad \forall n > 3$	0
	⋮
	0

(8) Ans $\boxed{x^3 = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3.}$

(9) $R_N(17) = \dfrac{\sin^{(N+1)}(z)\,(17-7)^{N+1}}{(N+1)!}$

for some z between 7 and 17.

Since $\left|\sin^{(N+1)}(z)\right| = \begin{cases} |\cos(z)| & N+1 \text{ odd} \\ |\sin(z)| & N+1 \text{ even} \end{cases} \le 1,$

$|R_N(17)| \le \dfrac{10^{N+1}}{(N+1)!} \longrightarrow 0$ as $N \to \infty$.

$\left(\text{Since } \sum_{N=0}^{\infty} \dfrac{10^{N+1}}{(N+1)!} < \infty, \quad \dfrac{10^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty \right)$

(10) $f(x) = e^x \sin(x) = \left(1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \cdots\right)\left(x - \dfrac{x^3}{6} + \cdots\right)$

$= x + x^2 + x^3\left(\dfrac{1}{2} - \dfrac{1}{6}\right) + \cdots$

$= x + x^2 + x^3 \dfrac{2}{6} + \cdots$

$\boxed{f(x) = x + x^2 + \tfrac{1}{3}x^3 + \cdots}$

Alt $f'(x) = e^x[\sin(x) + \cos(x)]$, $f''(x) = e^x[\sin + \cos + \cos - \sin]$
$= e^x\, 2\cos(x)$

$f'''(x) = 2e^x[\cos(x) - \sin(x)]$

So $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 2$.

So $f(x) = 0 + \dfrac{1x}{1!} + \dfrac{2x^2}{2!} + \dfrac{2x^3}{3!} + \cdots$

$\boxed{f(x) = x + x^2 + \tfrac{1}{3}x^3 + \cdots}$