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Undergraduate Analysis Tools

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Numbers

Natural, integer, and rational Numbers

Notation 1.1 Let $\mathbb{N} = \{1, 2, \dots\}$ denote the **natural numbers**, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$,

$$\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\} = \{\pm n : n \in \mathbb{N}_0\}$$

be the **integers**, and

$$\mathbb{Q} := \left\{ \frac{m}{n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$$

be the **rational numbers**.

I am going to assume that the reader is familiar with all the standard arithmetic operations (addition, multiplication, inverses, etc.) on $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{Q}$. However let us review the important induction axiom of the natural numbers.

Induction Axiom If $S \subset \mathbb{N}$ is a subset such that $1 \in S$ and $n + 1 \in S$ whenever $n \in S$, then $S = \mathbb{N}$.

This axiom leads takes on two other useful forms which we describe in the next Propositions.

Proposition 1.2 (Strong form of Induction). Suppose $S \subset \mathbb{N}$ is a subset such that $1 \in S$ and $n + 1 \in S$ whenever $\{1, 2, \dots, n\} \subset S$, then $S = \mathbb{N}$.

Proof. Let $T := \{n \in \mathbb{N} : \{1, 2, \dots, n\} \subset S\}$. Then $1 \in T$ and if $n \in T$ then $n + 1 \in T$ by assumption. Therefore by the induction axiom, $T = \mathbb{N}$ so that $\{1, 2, \dots, n\} \subset S$ in for all $n \in \mathbb{N}$. This suffices to show $S = \mathbb{N}$. ■

Proposition 1.3 (Well ordering principle). Suppose $S \subset \mathbb{N}$ is a non-empty subset, then there exists a smallest element m of S .

Proof. Let S be a subset of $\mathbb{N}$ for which there is no smallest element, $m \in S$. Let

$$T = \{n \in \mathbb{N} : n < s \text{ for all } s \in S\}.$$

If $1 \notin T$, then $1 \in S$ and 1 would be a smallest element of S . Hence we must have $1 \in T$. Now suppose that $n \in T$ so that $n < s$ for all $s \in S$. If $n + 1 \notin T$ then there exists $s \in S$ such that $n < s \leq n + 1$ which would force $s = n + 1 \in S$. But we would then have $n + 1$ is the minimal element of S which is assumed not to exist. So we have shown if $n \in T$ then $n + 1 \in T$. So by the induction axiom of $\mathbb{N}$ it follows that $T = \mathbb{N}$ and therefore $n \notin S$ for all $n \in \mathbb{N}$, i.e. $S = \emptyset$. ■

Remark 1.4. Let us further observe that the well ordering principle implies the induction axiom. Indeed, suppose that $S \subset \mathbb{N}$ is a subset such that $1 \in S$ and $n + 1 \in S$ whenever $n \in S$. For sake of contradiction suppose that $S \neq \mathbb{N}$ so that $T := \mathbb{N} \setminus S$ is not empty. By the well ordering principle there T has a unique minimal element m and in particular $T \subset \{m, m + 1, \dots\}$. This implies that $\{1, 2, \dots, m - 1\} \subset S$ and then by assumption that $\{1, 2, \dots, m\} \subset S$. But this then implies $T \subset \{m + 1, \dots\}$ and therefore $m \notin T$ which violates m being the minimal element of T . We have arrived at the desired contradiction and therefore conclude that $S = \mathbb{N}$.

Remark 1.5. Recall that, for $q \in \mathbb{Q}$, we define¹

$$|q| = \begin{cases} q & \text{if } q \geq 0 \\ -q & \text{if } q \leq 0. \end{cases}$$

Recall that, for all $a, b \in \mathbb{Q}$,

$$|a + b| \leq |a| + |b|, \quad |ab| = |a| |b|, \quad \text{and} \quad \left| \frac{1}{a} \right| = \frac{1}{|a|} \text{ when } a \neq 0.$$

It is also often useful to keep in mind that the following statements are equivalent for $a, b \in \mathbb{Q}$ with $b \geq 0$;

1. $|a| \leq b$,
2. $-b \leq a \leq b$, and
3. $\pm a \leq b$, i.e. $a \leq b$ and $-a \leq b$.

Lemma 1.6. If $a, b \in \mathbb{Q}$, then

$$||b| - |a|| \leq |b - a|. \quad (1.1)$$

Proof. Since both sides of Eq. (1.1) are symmetric in a and b , we may assume that $|b| \geq |a|$ so that $||b| - |a|| = |b| - |a|$. Since

$$|b| = |b - a + a| \leq |b - a| + |a|,$$

it follows that

¹ Absolute values will be discussed in more generality in Section 2.2 below.

$$||b| - |a|| = |b| - |a| \leq |b - a|.$$

■

The proof of the previous lemma illustrates one of the **key techniques of adding 0 to an expression**. In this case we added 0 in the form of $-a + a$ to b . The next remark records a couple of other very important “tricks” in this subject. Taking to heart the following remarks will greatly aid the student in real analysis.

Remark 1.7 (Some basic philosophies of real analysis). Let a, b, ε be numbers (i.e. in $\mathbb{Q}$ or later real numbers). We will often prove;

1. $a \leq b$ by showing that $a \leq b + \varepsilon$ for all $\varepsilon > 0$. (See the next theorem.)
2. $a = b$ by proving $a \leq b$ and $b \leq a$ or
3. prove $a = b$ by showing $|b - a| \leq \varepsilon$ for all $\varepsilon > 0$.

Theorem 1.8. *The rationale² numbers have the following properties;*

1. For any $p \in \mathbb{Q}$ there exists $N \in \mathbb{N}$ such that $p < N$.
2. For any $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \varepsilon$.
3. If $a, b \in \mathbb{Q}$ and $a \leq b + \varepsilon$ for all $\varepsilon > 0$, then $a \leq b$.

Proof. 1. If $p \leq 0$ we may take $N = 1$. So suppose that $p = \frac{m}{n}$ with $m, n \in \mathbb{N}$. In this case let $N = m$.

2. Write $\varepsilon = \frac{m}{n}$ with $m, n \in \mathbb{N}$ and then take $N = 2n$.

3. If $a \leq b$ is false happens iff $a > b$ which is equivalent to $a - b > 0$. If we now let $\varepsilon := \frac{a-b}{2} > 0$, then

$$a = b + (b - a) > b + \varepsilon$$

which would violate the assumption that $a \leq b + \varepsilon$ for all $\varepsilon > 0$. ■

1.1 Limits in $\mathbb{Q}$

In this course we will often use the abbreviations, **i.o.** and **a.a.** which stand for **infinitely often** and **almost always** respectively. For example $a_n \leq b_n$ a.a. n means there exists an $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N$ while $a_n \leq b_n$ i.o. n means for all $N \in \mathbb{N}$ there exists a $n \geq N$ such that $a_n \leq b_n$. So for example, $1/n \leq 1/100$ for a.a. n while and $(-1)^n \geq 0$ i.o. By the way, it should be clear that if something happens for a.a. n then it also happens i.o. n .

² We will see that the real numbers have these same properties as well.

Definition 1.9. A sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ **converges to** $0 \in \mathbb{Q}$ if for all $\varepsilon > 0$ in $\mathbb{Q}$ there exists $N \in \mathbb{N}$ such that $|a_n| \leq \varepsilon$ for all $n \geq N$. Alternatively put, for all $\varepsilon > 0$ we have $|a_n| \leq \varepsilon$ for a.a. n . This may also be stated as for all $M \in \mathbb{N}$, $|a_n| \leq \frac{1}{M}$ for a.a. n .

Definition 1.10. A sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ **converges to** $a \in \mathbb{Q}$ if $|a - a_n| \rightarrow 0$ as $n \rightarrow \infty$, i.e. if for all $N \in \mathbb{N}$, $|a - a_n| \leq \frac{1}{N}$ for a.a. n . As usual if $\{a_n\}_{n=1}^{\infty}$ converges to a we will write $a_n \rightarrow a$ as $n \rightarrow \infty$ or $a = \lim_{n \rightarrow \infty} a_n$.

Proposition 1.11. If $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ converges to $a \in \mathbb{Q}$, then $\lim_{n \rightarrow \infty} |a_n| = |a|$.

Proof. From Lemma 1.6 we have,

$$||a| - |a_n|| \leq |a - a_n|.$$

Thus if $\varepsilon > 0$ is given, by definition of $a_n \rightarrow a$ there exists $N \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq N$. From the previously displayed equation, it follows that $||a| - |a_n|| < \varepsilon$ for all $n \geq N$ and hence we may conclude that $\lim_{n \rightarrow \infty} |a_n|$ exists and is equal to $|a|$. ■

Lemma 1.12 (Convergent sequences are bounded). If $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ converges to $a \in \mathbb{Q}$, then there exists $M \in \mathbb{Q}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Proof. Taking $\varepsilon = 1$ in the definition of $a = \lim_{n \rightarrow \infty} a_n$ implies there exists $N \in \mathbb{N}$ such that $|a_n - a| \leq 1$ for all $n \geq N$. Therefore,

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a| \text{ for } n \geq N.$$

We may now take $M := \max \left(\left\{ |a_n|_{n=1}^N \right\} \cup \{1 + |a|\} \right)$. ■

Theorem 1.13. If $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ converges to $a \in \mathbb{Q} \setminus \{0\}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}. \quad (1.2)$$

It is possible that $a_n = 0$ for small n so that $\frac{1}{a_n}$ is not defined but for large n this can not happen and therefore it makes sense to talk about the limit which only depends on the tail of the sequences.

Proof. Since $a \neq 0$ we know that $|a| > 0$. Hence, there exists $M := M_{|a|} \in \mathbb{N}$ such that $|a_n - a| < \frac{|a|}{2}$ for all $n \geq M$. Therefore for $n \geq M$

$$|a| = |a - a_n + a_n| \leq |a - a_n| + |a_n| < \frac{|a|}{2} + |a_n|$$

from which it follows³ that $|a_n| > \frac{|a|}{2}$ for all $n \geq M$. If $\varepsilon > 0$ is given arbitrarily, we may choose $N \geq M$ such that $|a - a_n| < \varepsilon$ for all $n \geq M$. Then for $n \geq N$ we have,

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right| = \frac{|a - a_n|}{|a_n| |a|} < \frac{\varepsilon}{\frac{|a|}{2} \cdot |a|} = \frac{2\varepsilon}{|a|^2}.$$

As $\varepsilon > 0$ is arbitrary it follows that $\frac{2\varepsilon}{|a|^2} > 0$ is arbitrarily small as well (replace ε by $\varepsilon|a|^2/2$ if you feel it is necessary), and hence we may conclude that Eq. (1.2) holds.

Variation on the method. In order to make these arguments more routing, it is often a good idea to write $a_n = a + \delta_n$ where $\delta_n := a_n - a$ is the error between a_n and a . By assumption, $\lim_{n \rightarrow \infty} \delta_n = 0$ and so for any $\delta > 0$ given there exists $N(\delta) \in \mathbb{N}$ such that $|\delta_n| \leq \delta$. With this notation we have,

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{1}{a + \delta_n} - \frac{1}{a} \right| = \left| \frac{-\delta_n}{a + \delta_n} \cdot \frac{1}{a} \right| \leq \frac{\delta}{|a|(|a| - \delta)}.$$

So if we assume that $\delta \leq |a|/2$ we find that

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| \leq \frac{2}{|a|^2} \delta \text{ for all } n \geq N(\delta). \quad (1.3)$$

Taking $\delta = \delta(\varepsilon) = \min(|a|/2, |a|^2 \varepsilon/2)$ in Eq. (1.3) shows for $n \geq N(\delta(\varepsilon))$ that $\left| \frac{1}{a_n} - \frac{1}{a} \right| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary we may conclude that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}$. ■

- End of Lecture 1, 9/28/2012

Definition 1.14. A sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ is **Cauchy** if $|a_n - a_m| \rightarrow 0$ as $m, n \rightarrow \infty$. More precisely we require for each $\varepsilon > 0$ in $\mathbb{Q}$ that $|a_m - a_n| \leq \varepsilon$ for a.a. pairs (m, n) , i.e. there should exist $N \in \mathbb{N}$ such that $|a_m - a_n| \leq \varepsilon$ for all $m, n \geq N$.

Exercise 1.1. Show that all convergent sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ are Cauchy.

Exercise 1.2. Show all Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ are bounded – i.e. there exists $M \in \mathbb{N}$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

³ The idea is very simple here. If a_n is near a and $a \neq 0$ then a_n must stay away from zero. You should draw the picture to go along with the proof.

Exercise 1.3. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^{\infty}$ and $\{a_n \cdot b_n\}_{n=1}^{\infty}$ are Cauchy.

Exercise 1.4. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^{\infty}$ and $\{a_n \cdot b_n\}_{n=1}^{\infty}$ are convergent in $\mathbb{Q}$ and

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \text{ and } \lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n.$$

Exercise 1.5. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$ such that $a_n \leq b_n$ for all n . Show $A := \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n =: B$.

Exercise 1.6 (Sandwich Theorem). Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$ such that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$. If $\{x_n\}_{n=1}^{\infty}$ is another sequence in $\mathbb{Q}$ which satisfies $a_n \leq x_n \leq b_n$ for all n , then

$$\lim_{n \rightarrow \infty} x_n = a := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

Please note that the main part of the problem is to show that $\lim_{n \rightarrow \infty} x_n$ exists in $\mathbb{Q}$. **Hint:** start by showing; if $a \leq x \leq b$ then $|x| \leq \max(|a|, |b|)$.

Definition 1.15 (Subsequence). We say a sequence, $\{y_k\}_{k=1}^{\infty}$ is a **subsequence** of another sequence, $\{x_n\}_{n=1}^{\infty}$, provided there exists a strictly increasing function, $\mathbb{N} \ni k \rightarrow n_k \in \mathbb{N}$ such that $y_k = x_{n_k}$ for all $k \in \mathbb{N}$. [Example, $n_k = k^2 + 3$, and $\{y_k := x_{k^2+3}\}_{k=1}^{\infty}$ would be a subsequence of $\{x_n\}_{n=1}^{\infty}$.]

Exercise 1.7. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$ (or $\mathbb{R}$) which has a convergent subsequence, $\{y_k = x_{n_k}\}_{k=1}^{\infty}$ in $\mathbb{Q}$ (or $\mathbb{R}$). Show that $\lim_{n \rightarrow \infty} x_n$ exists and is equal to $\lim_{k \rightarrow \infty} y_k$.

1.2 The Problem with $\mathbb{Q}$

The problem with $\mathbb{Q}$ is that it is full of “holes.” To be more precise, $\mathbb{Q}$ is not “complete,” i.e. not all Cauchy sequences are convergent. In fact, according to Corollary 5.21 below, “most” Cauchy sequences of rational numbers do not converge to a rational number. Let us demonstrate some examples pointing out this flaw. We first pause to recall how to sum geometric series.

Lemma 1.16 (Geometric Series). Let $\alpha \in \mathbb{Q}$, $m, n \in \mathbb{Z}$ with $n \leq m$, and $S := \sum_{k=n}^m \alpha^k$. Then

$$S = \begin{cases} m - n + 1 & \text{if } \alpha = 1 \\ \frac{\alpha^{m+1} - \alpha^n}{\alpha - 1} & \text{if } \alpha \neq 1 \end{cases}.$$

Moreover if $0 \leq \alpha < 1$, then

$$\sum_{k=n}^m \alpha^k = \alpha^n \frac{1 - \alpha^{m-n+1}}{1 - \alpha} \leq \frac{\alpha^n}{1 - \alpha}. \quad (1.4)$$

Proof. When $\alpha = 1$,

$$S = \sum_{k=n}^m 1^k = m - n + 1.$$

If $\alpha \neq 1$, then

$$\alpha S - S = \alpha^{m+1} - \alpha^n.$$

Solving for S gives

$$S = \sum_{k=n}^m \alpha^k = \frac{\alpha^{m+1} - \alpha^n}{\alpha - 1} \text{ if } \alpha \neq 1. \quad (1.5)$$

■

Example 1.17. Let $S_n := \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q}$ for all $n \in \mathbb{N}$. For $n > m$ in $\mathbb{N}$ we have,

$$\begin{aligned} 0 \leq S_n - S_m &= \sum_{k=m+1}^n \frac{1}{k!} = \sum_{j=1}^{n-m} \frac{1}{(m+j)!} \\ &= \frac{1}{(m+1)!} + \cdots + \frac{1}{n!} \\ &\leq \frac{1}{m!} \left[\frac{1}{m+1} + \left(\frac{1}{m+1} \right)^2 + \cdots + \left(\frac{1}{m+1} \right)^{n-m} \right] \\ &\leq \frac{1}{m!} \frac{1}{m+1} \frac{1}{1 - \frac{1}{m+1}} = \frac{1}{m \cdot m!}, \end{aligned} \quad (1.6)$$

wherein we have used Eq. (1.4) for the last inequality. From this inequality it follows that $\{S_n\}_{n=0}^\infty$ is a Cauchy sequence and we also have,

$$\frac{1}{(m+1)!} \leq S_n - S_m \leq \frac{1}{m \cdot m!} \text{ for all } n > m. \quad (1.7)$$

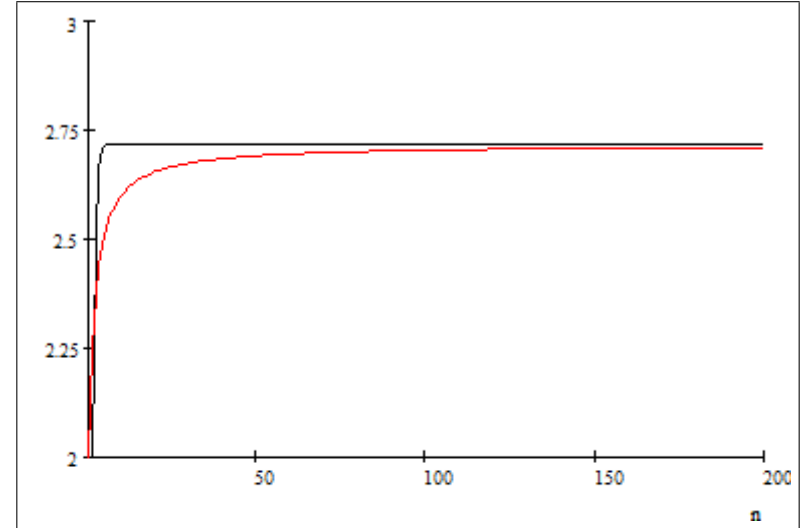
Suppose that $e := \lim_{n \rightarrow \infty} S_n$ were to exist in $\mathbb{Q}$. Then letting $n \rightarrow \infty$ in Eq. (1.7) would show,

$$0 < \frac{1}{(m+1)!} \leq e - S_m \leq \frac{1}{m \cdot m!}.$$

Multiplying this inequality by $m!$ then implies,

$$0 < m!e - m!S_m \leq \frac{1}{m}.$$

However for m sufficiently large $m!e \in \mathbb{N}$ (as e is assumed to be rational) and $m!S_m$ is always in $\mathbb{N}$ and therefore $k := m!e - m!S_m \in \mathbb{N}$. But there is no element $k \in \mathbb{N}$ such that $0 < k < \frac{1}{m}$ and hence we must conclude $\lim_{n \rightarrow \infty} S_n$ can not exist in $\mathbb{Q}$. **Moral:** the number $e = \sum_{n=0}^\infty \frac{1}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ that you learned about in calculus is not in $\mathbb{Q}$!



Plotting the partial sums $\sum_{k=0}^n \frac{1}{k!}$ (black curve) and $\left(1 + \frac{1}{n}\right)^n$ (red curve) which are both converging to “ e .”

Example 1.18 (Square roots need not exist). The square root, $\sqrt{2}$, of 2 does not exist in $\mathbb{Q}$. Indeed, if $\sqrt{2} = \frac{m}{n}$ where m and n have no common factors (in particular no common factors of 2 so that either m or n is odd), then

$$\frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

This shows that m^2 is even which would then imply that $m = 2k$ is even (since odd-odd=odd). However this implies $4k^2 = 2n^2$ from which it follows that $n^2 = 2k^2$ is even and hence n is even. But this contradicts the assumption that m and n had no common factors (of 2).

Exercise 1.8. Use the following outline to construct another Cauchy sequence $\{q_n\}_{n=1}^\infty \subset \mathbb{Q}$ which is **not** convergent in $\mathbb{Q}$.

1. Recall that there is no element $q \in \mathbb{Q}$ such that $q^2 = 2$. To each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be chosen so that

$$\frac{m_n^2}{n^2} < 2 < \frac{(m_n + 1)^2}{n^2} \quad (1.8)$$

and let $q_n := \frac{m_n}{n}$.

2. Verify that $q_n^2 \rightarrow 2$ as $n \rightarrow \infty$ and that $\{q_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{Q}$.
3. Show $\{q_n\}_{n=1}^\infty$ does not have a limit in $\mathbb{Q}$.

Example 1.19. It is also a fact that $\pi \notin \mathbb{Q}$ where

$$\begin{aligned} \pi &= 2 \int_0^\infty \frac{1}{1+x^2} dx = 2 \lim_{N \rightarrow \infty} \int_0^N \frac{1}{1+x^2} dx \\ &= 2 \lim_{N \rightarrow \infty} \sum_{k=1}^{N^2} \frac{1}{1 + \left(\frac{k}{N}\right)^2} \cdot \frac{1}{N} \\ &= \lim_{N \rightarrow \infty} \sum_{k=1}^{N^2} \frac{2N}{N^2 + k^2}. \end{aligned}$$

The point is that the basic operations from calculus tend to produce “real numbers” which are not rational even though we start with only rational numbers.

- End of Lecture 2, 10/1/2012

1.3 Peano's arithmetic (Highly Optional)

This section is for those who want to understand $\mathbb{N}$ at a more fundamental level. Here we start with Peano's rather minimalist axioms for $\mathbb{N}$ and show how they lead to all the standard properties you are used to using for $\mathbb{N}$. Here are the axioms;

non-empty $\mathbb{N}_0$ is a non-empty set which contains a distinguished element, 0.

We let $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ and call these the natural numbers.

Successor Function There is an injective⁴ function, $s : \mathbb{N}_0 \rightarrow \mathbb{N}$ and we let $1 := s(0) \in \mathbb{N}$.

Induction hypothesis If $S \subset \mathbb{N}_0$ is a set such that $0 \in S$ and $s(n) \in S$ whenever $n \in S$, then $S = \mathbb{N}_0$.

Assuming these axioms one may develop all of the properties of $\mathbb{N}_0$ that you are accustomed to seeing. I will develop the basic properties of addition, multiplication, and the ordering on $\mathbb{N}_0$ in this section. For more on this point

⁴ Injective is the same as one to one. Thus we are assuming that if $s(n) = s(m)$ then $n = m$.

and then the further construction of $\mathbb{Z}$ and $\mathbb{Q}$ from $\mathbb{N}_0$, the reader is referred to the notes; “*Numbers*” by *M. Taylor*. You may also consult E. Landau's book [1] for a very detailed (but perhaps too long winded) exposition of these topics.

Lemma 1.20. *The map $s : \mathbb{N}_0 \rightarrow \mathbb{N}$ is a bijection.*

Proof. Let $S := s(\mathbb{N}_0) \cup \{0\} \subset \mathbb{N}_0$. Then $0 \in S$ and $s(0) \in s(\mathbb{N}_0) \subset S$. Moreover if $x \in \mathbb{N} \cap S$ then $s(x) \in s(\mathbb{N}_0) \subset S$ so that $x \in S \implies s(x) \in S$ and hence $S = \mathbb{N}_0$ and therefore $s(\mathbb{N}_0) = \mathbb{N}$. ■

Theorem 1.21 (Addition). *There exists a function $p : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that $p(x, 0) = x$ for all $x \in \mathbb{N}_0$ and $p(x, s(y)) = s(p(x, y))$ for all $x, y \in \mathbb{N}_0$. Moreover, we may construct p so that $p(s(x), y) = p(x, s(y))$ for all $x, y \in \mathbb{N}_0$. This function p satisfies the following properties;*

1. $p(x, 0) = x = p(0, x)$ for all $x \in \mathbb{N}_0$,
2. $p(x, 1) = p(1, x) = s(x)$ for all $x \in \mathbb{N}_0$,
3. $p(x, y) = p(y, x)$ for all $x, y \in \mathbb{N}_0$,
4. $p(x, p(y, z)) = p(p(x, y), z)$ for all $x, y, z \in \mathbb{N}_0$.

Proof. We will construct p inductively. Let

$$S := \{x \in \mathbb{N} : \exists p_x : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \ni p_x(0) = x \text{ and } p_x(s(y)) = s(p_x(y)) \ \forall y \in \mathbb{N}_0\}.$$

Taking $p_0(y) = y$ shows $0 \in S$. Moreover if $x \in S$ we define

$$p_{s(x)}(y) := s(p_x(y)) \text{ for all } y \in \mathbb{N}_0.$$

We then have $p_{s(x)}(0) = s(p_x(0)) = s(x)$ and

$$p_{s(x)}(s(y)) := s(p_x(s(y))) = s \circ s(p_x(y)) = s(p_{s(x)}(y))$$

which shows $s(x) \in S$. Thus we may conclude $S = \mathbb{N}_0$ and we may now define $p(x, y) := p_x(y)$ for all $x, y \in \mathbb{N}_0$. By construction this function satisfies,

$$p(s(x), y) = s(p(x, y)) = p(x, s(y)).$$

We now verify the properties in items 1. – 4.

1. By construction $p(x, 0) = x$ for all $x \in \mathbb{N}_0$. Let $S = \{x \in \mathbb{N} : p(0, x) = x\}$, then $0 \in S$ and if $x \in S$ we have $p(0, s(x)) = s(p(0, x)) = s(x)$ so that $s(x) \in S$. Therefore $S = \mathbb{N}_0$ and the first item holds.
2. $p(x, 1) = p(x, s(0)) = s(p(x, 0)) = s(x)$ and $p(1, x) = p(s(0), x) = s(p(0, x)) = s(x)$ so that item 2. is proved.

3. Let $S = \{x \in \mathbb{N}_0 : p(x, \cdot) = p(\cdot, x)\}$. Then by items 1 and 2. it follows that $0, 1 \in S$. Moreover if $x \in S$, then for all $y \in \mathbb{N}_0$ we find,

$$p(s(x), y) = s(p(x, y)) = s(p(y, x)) = p(y, s(x))$$

which shows $s(x) \in S$. Therefore $S = \mathbb{N}_0$ and item 3. is proved.

4. Let

$$S := \{x \in \mathbb{N}_0 : p(x, p(y, z)) = p(p(x, y), z) \quad \forall y, z \in \mathbb{N}_0\}.$$

Then $0 \in S$ and if $x \in S$ we find,

$$\begin{aligned} p(s(x), p(y, z)) &= s(p(x, p(y, z))) = s(p(p(x, y), z)) \\ &= p(s(p(x, y)), z) = p(p(s(x), y), z) \end{aligned}$$

which shows that $s(x) \in S$ and therefore $S = \mathbb{N}_0$ and item 4. is proved. ■

Notation 1.22 We now write $x + y$ for $p(x, y)$ and refer to the symmetric binary operator, $+$, as addition.

To summarize we have now shown addition satisfies for all $x, y, z \in \mathbb{N}_0$;

1. $x + 0 = 0 + x = x$,
2. $s(x) = x + 1 = 1 + x$,
3. $x + y = y + x$,
4. $(x + y) + z = x + (y + z)$.
5. The induction hypothesis may now be written as; if $S \subset \mathbb{N}_0$ is a subset such that $0 \in S$ and $n + 1 \in S$ whenever $n \in S$, then $S = \mathbb{N}_0$.

Proposition 1.23 (Additive Cancellation). If $x, y, z \in \mathbb{N}_0$ and $x + z = y + z$, then $x = y$.

Proof. Let S be those $z \in \mathbb{N}_0$ for which the statement $x + z = y + z$ implies $x = y$ holds. It is clear that $0 \in S$. Moreover if $z \in S$ and $x + (z + 1) = y + (z + 1)$ then $(x + 1) + z = (y + 1) + z$ and so by the inductive hypothesis $s(x) = x + 1 = y + 1 = s(y)$. Recall that s is one to one by assumption and therefore we may conclude $x = y$ and we have shown $s(z) \in S$. Therefore $S = \mathbb{N}_0$ and the proposition is proved. ■

Definition 1.24. Given $x, y \in \mathbb{N}_0$, we say $x < y$ iff $y = x + n$ for some $n \in \mathbb{N}$ and $x \leq y$ iff $y = x + n$ for some $n \in \mathbb{N}_0$. We further let

$$R_x := \{x + n : n \in \mathbb{N}_0\}$$

so that $y \geq x$ iff $y \in R_x$.

Proposition 1.25. If $x, y \in \mathbb{N}_0$ and $x \leq y$ and $y \leq x$ then $x = y$. Moreover if $x \leq y$ then either $x < y$ or $x = y$.

Proof. By assumption there exists $m, n \in \mathbb{N}_0$ such that $y = x + m$ and $x = y + n$ and therefore $y = y + (m + n)$. Hence by cancellation it follows that $m + n = 0$. If $n \neq 0$ then $n = s(x)$ for some $x \in \mathbb{N}_0$ and we have $m + n = m + s(x) = s(m + x) \in \mathbb{N}$ which would imply $m + n \neq 0$. Thus we conclude that $m = 0 = n$ and therefore $x = y$.

If $x \leq y$ and $x \neq y$ then $y = x + n$ for some $n \in \mathbb{N}_0$ with $n \neq 0$, i.e. $x < y$. ■

Proposition 1.26. If $x, y \in \mathbb{N}_0$ then precisely one of the following three choices must hold, 1) $x < y$, 2) $x = y$, 3) $y < x$.

Proof. Suppose that $x \leq y$ does not hold, i.e. $y \notin R_x$. We wish to show that $y < x$, i.e. that $x = y + n$ for some $n \in \mathbb{N}$. We do this by induction on y . That is let S be the set of $y \in \mathbb{N}_0$ such that the statement $y \notin R_x$ implies $y < x$ holds. If $y = 0 \notin R_x$ implies $n := x \neq 0$ so that $x = y + n$, i.e. $y = 0 < x$. This shows $0 \in S$. Now suppose that $y \in S$ and that $y + 1 \notin R_x = \{x + m : m \in \mathbb{N}_0\}$. It follows that $y + 1 \neq x + m + 1$ for all $m \in \mathbb{N}_0$ and hence that $y \neq x + m$ for all $m \in \mathbb{N}_0$, i.e. $y \notin R_x$. So by induction $y < x$ and therefore $x = y + k$ for some $k \in \mathbb{N}$. Since $k \in \mathbb{N}$ we know there exists $k' \in \mathbb{N}_0$ such that $k = k'$ and it follows that $x = y + 1 + k'$, i.e. $y + 1 \leq x$. Since $y + 1 \notin R_x$ we may conclude that in fact $y + 1 < x$ and therefore $y + 1 \in S$. So by induction $S = \mathbb{N}_0$ and we have shown if $x < y$ does not hold iff $y \leq x$. Combining this statement with the Proposition 1.25 completes the proof. ■

We have now set up a satisfactory addition operations and ordering on $\mathbb{N}_0$. Our next goal is to define multiplication on $\mathbb{N}_0$.

Theorem 1.27. There exists a function $M : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that $M(x, 0) = 0$ for all $x \in \mathbb{N}_0$ and $M(x, y + 1) = M(x, y) + x$ for all $x, y \in \mathbb{N}_0$. This function M satisfies the following properties;

1. $M(x, 0) = 0 = M(0, x)$ for all $x \in \mathbb{N}_0$,
2. $M(x, 1) = M(1, x) = x$ for all $x \in \mathbb{N}_0$,
3. $M(x, y) = M(y, x)$ for all $x, y \in \mathbb{N}_0$,
4. $M(x, y + z) = M(x, y) + M(x, z)$ for all $x, y, z \in \mathbb{N}_0$.
5. $M(x, M(y, z)) = M(M(x, y), z)$ for all $x, y, z \in \mathbb{N}_0$.

Proof. Let S denote those $x \in \mathbb{N}_0$ such that there exists a function $M_x : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ satisfying $M_x(0) = 0$ and $M_x(y + 1) = M_x(y) + x$ for all $y \in \mathbb{N}_0$. Taking $M_0(y) := 0$ shows $0 \in S$. Moreover if $x \in S$ we define $M_{x+1}(y) := M_x(y) + x$. Then $M_{x+1}(0) = 0$ and

$$M_{x+1}(y + 1) = M_x(y + 1) + x + 1 = M_x(y) + x + y + 1$$

while

$$M_{x+1}(y) + (x+1) = M_x(y) + y + x + 1 = M_{x+1}(y+1).$$

This shows that $x+1 \in S$ and so by induction $S = \mathbb{N}_0$ and we may now define $M(x, y) := M_x(y)$ for all $x, y \in \mathbb{N}_0$. We now prove the properties of M stated above.

1. By construction $M(x, 0) = 0$ for all x . Let $S := \{x \in \mathbb{N}_0 : M(0, x) = 0\}$. Then $0 \in S$ and if $x \in S$ we have

$$M(0, x+1) = M(0, x) + 0 = 0 + 0 = 0$$

which shows $x+1 \in S$. Therefore by induction $S = \mathbb{N}_0$ and $M(0, x) = 0$ for all $x \in \mathbb{N}_0$.

2. $M(x, 1) = M(x, 0+1) = M(x, 0) + x = 0 + x = x$ for all $x \in \mathbb{N}_0$. Let $S := \{x \in \mathbb{N}_0 : M(1, x) = x\}$. Then $0 \in S$ and if $x \in S$ we have

$$M(1, x+1) = M(1, x) + 1 = x + 1$$

which shows $x+1 \in S$. Therefore $S = \mathbb{N}_0$ and $M(1, x) = x$ for all $x \in \mathbb{N}_0$.

3. Let $S := \{x \in \mathbb{N}_0 : M(x, \cdot) = M(\cdot, x)\}$. Then by items 1. and 2. we know that $0, 1 \in S$. Now suppose that $x \in S$, then by construction,

$$M(x+1, y) = M_{x+1}(y) = M(x, y) + y$$

while

$$M(y, x+1) = M(y, x) + y.$$

The last two displayed equations along with the induction hypothesis shows $x+1 \in S$ and therefore $S = \mathbb{N}_0$ and item 3. is proved.

4. Let S denotes those $x \in S$ such that $M(x, y+z) = M(x, y) + M(x, z)$ for all $y, z \in \mathbb{N}_0$. Then $0, 1 \in S$ and if $x \in S$ we have,

$$\begin{aligned} M(x+1, y+z) &= M(x, y+z) + y+z \\ &= M(x, y) + M(x, z) + y+z \\ &= M(x, y) + y + M(x, z) + z \\ &= M(x+1, y) + M(x+1, z) \end{aligned}$$

which shows $x+1 \in S$. Therefore $S = \mathbb{N}_0$ and we have proved item 4.

5. Let

$$S := \{x \in \mathbb{N}_0 : M(x, M(y, z)) = M(M(x, y), z) \quad \forall y, z \in \mathbb{N}_0\}.$$

Then $0 \in S$ and if $x \in S$ we find,

$$M(x+1, M(y, z)) = M(x, M(y, z)) + M(y, z)$$

while

$$M(M(x+1, y), z) = M(M(x, y) + y, z) = M(M(x, y), z) + M(y, z).$$

The last two equations along with the induction hypothesis shows $x+1 \in S$ and therefore $S = \mathbb{N}_0$ and item 5. is proved. ■

Notation 1.28 We now write $x \cdot y$ for $M(x, y)$ and refer to the symmetric binary operator, $\cdot$, as multiplication..

To summarize Theorem 1.27 we have shown multiplication satisfies for all $x, y, z \in \mathbb{N}_0$;

1. $x \cdot 0 = 0 = 0 \cdot x$,
2. $x \cdot 1 = x = 1 \cdot x$,
3. $x \cdot y = y \cdot x$,
4. $x \cdot (y+z) = x \cdot y + x \cdot z$,
5. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

Proposition 1.29 (Multiplicative Cancellation). If $x, y \in \mathbb{N}_0$ and $z \in \mathbb{N}$ such that $x \cdot z = y \cdot z$, then $x = y$.

Proof. If $x \neq y$, say $x < y$, then $y = x + n$ for some $n \in \mathbb{N}$ and therefore

$$y \cdot z = (x + n) \cdot z = x \cdot z + n \cdot z.$$

Hence if $x \cdot z = y \cdot z$, then by additive cancellation we must have $n \cdot z = 0$. As $n, x \in \mathbb{N}$ we may write $n = n' + 1$ and $z = z' + 1$ with $n', z' \in \mathbb{N}_0$ and therefore,

$$0 = n \cdot z = (n' + 1) \cdot (z' + 1) = n' \cdot z' + n' + z' + 1 \neq 0$$

which is a contradiction. ■

Remark 1.30 (Base 10 counting). The typical method of counting is to use base 10 enumeration of $\mathbb{N}_0$. The rules are;

$$\begin{aligned} 0 &= 0, \quad 1 = 1, \quad 2 := 1 + 1, \quad 3 := 2 + 1, \quad 4 := 3 + 1, \quad 5 := 4 + 1 \\ 6 &:= 5 + 1, \quad 7 := 6 + 1, \quad 8 := 7 + 1, \quad 9 := 8 + 1, \quad \text{and } 10 := 9 + 1. \end{aligned}$$

Once these element of $\mathbb{N}_0$ have been defined, then given $a_0, \dots, a_n \in \{0, 1, \dots, 9\}$ with $a_n \neq 0$, we let

$$a_n a_{n-1} \dots a_0 := \sum_{k=0}^n a_k 10^k.$$

For example, $35 = 3 \cdot 10 + 5 = 34 + 1$, etc.

As mentioned above one can formalize $\mathbb{Z}$ and $\mathbb{Q}$ using $\mathbb{N}_0$ constructed above. I will omit the details here and refer the reader to the references already mentioned.

Fields

The basic question we want to eventually address is; What are the real numbers? Our answer is going to be; the real numbers is the essentially unique complete ordered field, see Theorem 3.3 below. In order to make sense of this answer we need to explain the terms, “complete,” “ordered,” and “field.” We will start with the notion of a field which loosely stated means something that can reasonably be interpreted a “numbers.”

Definition 2.1 (Fields, i.e. “ numbers”). *A field is a non-empty set $\mathbb{F}$ equipped with two operations called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the following axioms hold; (subtraction and division are defined implicitly in terms of the inverse operations of addition and multiplication:)*

1. **Closure** of $\mathbb{F}$ under addition and multiplication. *For all a, b in $\mathbb{F}$, both $a + b$ and $a \cdot b$ are in $\mathbb{F}$ (or more formally, $+$ and $\cdot$ are binary operations on $\mathbb{F}$).*
2. **Associativity of addition and multiplication**. *For all a, b , and c in $\mathbb{F}$, the following equalities hold: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*
3. **Commutativity of addition and multiplication**. *For all a and b in $\mathbb{F}$, the following equalities hold: $a + b = b + a$ and $a \cdot b = b \cdot a$.*
4. **Additive and multiplicative identity**. *There exists an element of $\mathbb{F}$, called the additive identity element and denoted by $0 = 0_{\mathbb{F}}$, such that for all a in $\mathbb{F}$, $a + 0 = a$. Likewise, there is an element, called the multiplicative identity element and denoted by $1 = 1_{\mathbb{F}}$, such that for all a in $\mathbb{F}$, $a \cdot 1 = a$. It is assumed that $0_{\mathbb{F}} \neq 1_{\mathbb{F}}$.*
5. **Additive and multiplicative inverses**. *For every a in $\mathbb{F}$, there exists an element $-a$ in $\mathbb{F}$, such that $a + (-a) = 0$. Similarly, for any $a \in \mathbb{F}$ other than 0 , there exists an element a^{-1} in $\mathbb{F}$, such that $a \cdot a^{-1} = 1$. (The elements $a + (-b)$ and $a \cdot b^{-1}$ are also denoted $a - b$ and a/b , respectively.) In other words, subtraction and division operations exist.*
6. **Distributivity of multiplication over addition**. *For all a, b and c in $\mathbb{F}$, the following equality holds: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.*

(Note that all but the last axiom are exactly the axioms for a commutative group, while the last axiom is a compatibility condition between the two operations.)

2.1 Basic Properties of Fields

Here are some sample properties about fields. For more information about Fields see 5-8 of Rudin.

Lemma 2.2. *Let $\mathbb{F}$ be a field, then;*

1. *There is only one additive and multiplicative inverses.*
2. *If $x, y, z \in \mathbb{F}$ with $x \neq 0$ and $xy = xz$ then $y = z$.*
3. *$0 \cdot x = 0$ for all $x \in \mathbb{F}$.*
4. *If $x, y \in \mathbb{F}$ such that $xy = 0$ then $x = 0$ or $y = 0$.*
5. *$(-x)y = -(xy)$.*
6. *$-(-x) = x$ for all $x \in \mathbb{F}$.*
7. *$(-x)(-y) = xy$ or all $x, y \in \mathbb{F}$.*

Proof. We take each item in turn.

1. Suppose that $x + y = 0 = x + y'$, then adding $-x$ to both sides of this equation shows $y = y'$. Taking $y = -x$ then shows $y = -x = y'$, i.e. additive inverses are unique. Similarly if $x \neq 0$ and $xy = 1$ then multiplying this equation by x^{-1} shows $y = x^{-1}$ and so there is only one multiplicative inverse.
2. If $xy = xz$ then multiplying this equation by x^{-1} shows $y = z$.
- 3.

$$0 \cdot x + x = 0 \cdot x + 1 \cdot x = (0 + 1) \cdot x = 1 \cdot x = x.$$

Adding $-x$ to both side of this equation using associativity and commutativity of addition then implies $0 \cdot x = 0$.

4. If $x \in \mathbb{F} \setminus \{0\}$ and $y \in \mathbb{F}$ such that $xy = 0$, then

$$0 = x^{-1} \cdot 0 = x^{-1}(xy) = (x^{-1}x)y = 1y = y.$$

5. $(-x)y + xy = (-x + x) \cdot y = 0 \cdot y = 0 \implies (-x)y = -(xy)$.
6. Since $(-x) + x = 0$ we have $-(-x) = x$.
7. $(-x)(-y) = -(x \cdot (-y)) = -(-(xy)) = xy$ by 6.

■

Example 2.3. Here are a few examples of Fields;

1. $\mathbb{F}_2 = \{0, 1\}$ with $0 + 0 = 0 = 1 + 1$, and $0 + 1 = 1 + 0 = 0$ and $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$ and $1 \cdot 1 = 1$. In this case $-1 = 1$, $1^{-1} = 1$ and $-0 = 0$.
2. $\mathbb{Q}$ – the rational numbers with the usual addition and multiplication of fractions. $\left(\frac{m}{n}\right)^{-1} = \frac{n}{m}$ if $m \neq 0$ and $-\frac{m}{n} = \frac{-m}{n}$.
3. $\mathbb{F} = \mathbb{Q}(t)$ where

$$\mathbb{Q}(t) = \left\{ \frac{p(t)}{q(t)} : p(t) \text{ and } q(t) \text{ are polynomials over } \mathbb{Q} \ni q(t) \neq 0 \right\}.$$

Again the multiplication and addition are as usual.

Example 2.4. $\mathbb{Z}$ is not a field. For example, 2 has no multiplicative inverse in $\mathbb{Z}$. The inverse to 2, 2^{-1} , should be $\frac{1}{2}$ but this is not in $\mathbb{Z}$.

Definition 2.5. We say a map $\varphi : \mathbb{Z} \rightarrow \mathbb{F}$ is a (ring) homomorphism iff $\varphi(1) = 1_{\mathbb{F}}$, $\varphi(0) = 0_{\mathbb{F}}$, and for all $x, y \in \mathbb{Z}$;

$$\varphi(x + y) = \varphi(x) + \varphi(y) \text{ and } \varphi(xy) = \varphi(x)\varphi(y).$$

[The assumption that $\varphi(0) = 0_{\mathbb{F}}$ is redundant since $\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0)$ and therefore $\varphi(0) = 0_{\mathbb{F}}$.]

Lemma 2.6. For every field $\mathbb{F}$ there a unique (ring) homomorphism, $\varphi : \mathbb{Z} \rightarrow \mathbb{F}$. In fact, $\varphi(n) = n1_{\mathbb{F}}$ for all $n \in \mathbb{Z}$ where $0 \cdot 1_{\mathbb{F}} = 0_{\mathbb{F}}$,

$$\begin{aligned} n1_{\mathbb{F}} &:= \overbrace{1_{\mathbb{F}} + \cdots + 1_{\mathbb{F}}}^{n \text{ times}} \text{ if } n \in \mathbb{N} \text{ and} \\ (-n)1_{\mathbb{F}} &:= -(n1_{\mathbb{F}}) \text{ if } n \in \mathbb{N}. \end{aligned}$$

[The map φ need not be injective as is seen by taking $\mathbb{F} = \mathbb{F}_2$.]

Proof. Let us first work on $\mathbb{N}_0 \subset \mathbb{Z}$. We must define $\varphi(0) = 0$ and $\varphi(1) = 1$ and then φ inductively by $\varphi(n + 1) = \varphi(n) + \varphi(1) = \varphi(n) + 1_{\mathbb{F}}$ so that

$$\varphi(n) = \overbrace{1_{\mathbb{F}} + \cdots + 1_{\mathbb{F}}}^{n \text{ times}}.$$

We now write $n1_{\mathbb{F}}$ for $\varphi(n)$ with the convention that $01_{\mathbb{F}} = 0_{\mathbb{F}}$. For $n \in \mathbb{N}$ we must set $\varphi(-n) = -\varphi(n) = -(n1_{\mathbb{F}})$. Thus we have $\varphi(n) = n1_{\mathbb{F}}$ for all $n \in \mathbb{Z}$. We now must show φ is a homomorphism.

Additive homomorphism: First suppose that $m, n \in \mathbb{N}_0$ and let

$$S := \{m \in \mathbb{N}_0 : \varphi(m + n) = \varphi(m) + \varphi(n) \text{ for all } n \in \mathbb{N}_0\}.$$

One easily sees that $0 \in S$ and that $1 \in S$ by construction. Moreover if $m \in S$, then

$$\begin{aligned} \varphi((m + 1) + n) &= \varphi(m + n + 1) = \varphi(m) + \varphi(n + 1) \\ &= \varphi(m) + \varphi(n) + 1_{\mathbb{F}} \\ &= \varphi(m) + 1_{\mathbb{F}} + \varphi(n) = \varphi(m + 1) + \varphi(n) \end{aligned}$$

which shows $m + 1 \in S$. Therefore by induction, $S = \mathbb{N}_0$ and $\varphi(m + n) = \varphi(m) + \varphi(n)$ for all $m, n \in \mathbb{N}_0$.

If $m \in \mathbb{N}_0$ we have $\varphi(-m) = -\varphi(m)$ by construction. If $n > m \in \mathbb{N}_0$, then

$$\varphi(n + (-m)) + \varphi(m) = \varphi(n - m) + \varphi(m) = \varphi(n)$$

so that

$$\varphi(n + (-m)) = \varphi(n) + (-\varphi(m)) = \varphi(n) + \varphi(-m).$$

If $n < m \in \mathbb{N}_0$, then

$$\varphi(n + (-m)) = -\varphi(m - n) = -[\varphi(m) - \varphi(n)] = \varphi(n) + \varphi(-m)$$

and if $m, n \in \mathbb{N}_0$, then

$$\begin{aligned} \varphi(-n + (-m)) &= \varphi(-(n + m)) = -\varphi(n + m) \\ &= -[\varphi(n) + \varphi(m)] = -\varphi(n) - \varphi(m) \\ &= \varphi(-n) + \varphi(-m). \end{aligned}$$

Putting all of this together shows φ is an additive homomorphism.

Multiplicative homomorphism: First suppose that $m, n \in \mathbb{N}_0$ and let

$$S := \{m \in \mathbb{N}_0 : \varphi(mn) = \varphi(m)\varphi(n) \text{ for all } n \in \mathbb{N}_0\}.$$

It is easily seen that $0, 1 \in S$. Moreover if $m \in S$ and $n \in \mathbb{N}_0$, then

$$\begin{aligned} \varphi((m + 1)n) &= \varphi(mn + n) = \varphi(mn) + \varphi(n) \\ &= \varphi(m)\varphi(n) + \varphi(n) = (\varphi(m) + 1_{\mathbb{F}})\varphi(n) \\ &= \varphi(m + 1)\varphi(n), \end{aligned}$$

which shows $m + 1 \in S$. Therefore by induction, $S = \mathbb{N}_0$ and $\varphi(mn) = \varphi(m)\varphi(n)$ for all $m, n \in \mathbb{N}_0$.

If $m, n \in \mathbb{N}_0$, then

$$\varphi((-m)n) = \varphi(-mn) = -\varphi(mn) = -[\varphi(m)\varphi(n)] = [-\varphi(m)]\varphi(n) = \varphi(-m)\varphi(n)$$

and

$$\varphi((-m)(-n)) = \varphi(mn) = \varphi(mn) = (-\varphi(m))(-\varphi(n)) = -\varphi(m)\varphi(-n)$$

which completes the verification that φ is a multiplicative homomorphism. ■

2.2 Ordered Fields

Definition 2.7 (Ordered Field). We say $\mathbb{F}$ is an **ordered field** if there exists, $P \subset \mathbb{F}$, called the positive elements, such that

- Ord 1. $\mathbb{F}$ is the disjoint union of P , $\{0\}$, and $-P$, i.e. if $x \in \mathbb{F}$ then precisely one of following happens; $x \in P$, $x = 0$, or $-x \in P$.
Ord 2. $P + P \subset P$ and $P \cdot P \subset P$.

Lemma 2.8. Let $(\mathbb{F}, P)$ be an ordered field, then;

1. For all $x \in \mathbb{F} \setminus \{0\}$, $x^2 \in P$. In particular $1 = 1^2 \in P$.
2. If $x \in P$ and $y \in -P$ then $xy \in -P$.
3. If $x \in P$ then $x^{-1} \in P$.

Proof. If $x \in P$ then $x^2 \in P \cdot P \subset P$ while if $x \in -P$ then $-x \in P$ and $x^2 = (-x)^2 \in P$. For item 3. we have $x \cdot x^{-1} = 1$ ■

Example 2.9. The field $\mathbb{F} = \{0, 1\}$ is not ordered. The only possible choice for P is $P = \{1\}$ which does not work since $1 + 1 = 0 \notin P$.

Example 2.10. Take $\mathbb{F} = \mathbb{Q}$ and $P = \{\frac{m}{n} : m, n > 0\}$. This is in fact the unique choice we can make for P in this case. Indeed suppose that P is any order on $\mathbb{Q}$. By Lemma 2.8, we know $1 \in P$ and then by induction it follows that $\mathbb{N} \subset P$. Then again by Lemma 2.8 we must have $m \cdot n^{-1} \in P$ for all $m, n \in \mathbb{Q}$.

Example 2.11. Take $\mathbb{F} = \mathbb{Q}(t)$ and

$$P = \left\{ \frac{p(t)}{q(t)} \in \mathbb{F} : \frac{p(t)}{q(t)} > 0 \text{ for } t > 0 \text{ large} \right\},$$

i.e. $\frac{p(t)}{q(t)} \in P$ iff the highest order coefficients of $p(t)$ and $q(t)$ have the same sign. For example $\frac{t^2-25t+7}{t^4-10^{26}} \in P$ while $\frac{-t^2+25t+7}{t^4-10^{26}} \in -P$.

Notice that $t > n$ for all $n \in \mathbb{N}$ and $\frac{1}{t} < \frac{1}{n}$ for all $n \in \mathbb{N}$. This is kind of strange and explains why you have to prove the “obvious” in this course!!!
Moral: obvious statements are often false.

Notation 2.12 (Max and Min) We will often use the following notation in the sequel. If a, b are elements of an ordered field, let

$$a \wedge b := \min(a, b) = \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b \leq a \end{cases}$$

and

$$a \vee b := \max(a, b) = \begin{cases} b & \text{if } a \leq b \\ a & \text{if } b \leq a \end{cases}.$$

More generally if $\{a_i\}_{i=1}^n \subset \mathbb{F}$ we let

$$\begin{aligned} a_1 \wedge \cdots \wedge a_n &:= \min(a_1, \dots, a_n) \text{ and} \\ a_1 \vee \cdots \vee a_n &:= \max(a_1, \dots, a_n) \end{aligned}$$

be the smallest and largest element in the finite list $(a_1, \dots, a_n)$.

Definition 2.13. Suppose that $\mathbb{F}$ and $\mathbb{G}$ are fields. A map, $\varphi : \mathbb{F} \rightarrow \mathbb{G}$ is a (field) homomorphism iff $\varphi(1_{\mathbb{F}}) = 1_{\mathbb{G}}$, $\varphi(0_{\mathbb{F}}) = 0_{\mathbb{G}}$, and for all $x, y \in \mathbb{F}$;

$$\varphi(x + y) = \varphi(x) + \varphi(y) \text{ and } \varphi(xy) = \varphi(x) \varphi(y).$$

Lemma 2.14 ($\mathbb{Q}$ embeds into an ordered field). For every ordered field $(\mathbb{F}, P)$, there a unique field homomorphism, $\varphi : \mathbb{Q} \rightarrow \mathbb{F}$. In fact,

$$\varphi\left(\frac{m}{n}\right) = \frac{m}{n} \cdot 1_{\mathbb{F}} := m1_{\mathbb{F}} \cdot (n1_{\mathbb{F}})^{-1} \quad (2.1)$$

where $n1_{\mathbb{F}} := \overbrace{1_{\mathbb{F}} + \cdots + 1_{\mathbb{F}}}^{n \text{ times}}$ and $(-n)1_{\mathbb{F}} := -(n1_{\mathbb{F}})$ for $n \in \mathbb{N}$ and $0 \cdot 1_{\mathbb{F}} = 0_{\mathbb{F}}$. Moreover;

1. $\varphi(x) \in P$ whenever $x > 0$,
2. and φ is injective. Thus we may identify $\mathbb{Q}$ with $\varphi(\mathbb{Q})$ and consider $\mathbb{Q}$ as a sub-field of $\mathbb{F}$.

[In particular, ordered fields must be fields with an infinite number of elements in it.]

Proof. From Lemma 2.6 we know there is a unique ring homomorphism, $\varphi : \mathbb{Z} \rightarrow \mathbb{F}$, given by $\varphi(m) = m \cdot 1_{\mathbb{F}}$. So for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we must have

$$\varphi\left(\frac{m}{n}\right) \cdot n1_{\mathbb{F}} = \varphi\left(\frac{m}{n}\right) \cdot \varphi(n) = \varphi\left(\frac{m}{n} \cdot n\right) = \varphi(m) = m1_{\mathbb{F}}$$

which forces us to define φ as in Eq. (2.1). Notice that is easy to verify by induction that $n1_{\mathbb{F}} = \varphi(n) \in P$ for all $n \in \mathbb{N}$ and in particular $n1_{\mathbb{F}} \neq 0$ for $n \in \mathbb{N}$. In particular if $x = m/n > 0$ then $\varphi\left(\frac{m}{n}\right) = m1_{\mathbb{F}} \cdot (n1_{\mathbb{F}})^{-1} \in P$ by Lemma 2.8. We must still check that φ is well defined homomorphism.

Well defined. Suppose that $k \in \mathbb{N}$, we must show

$$(km)1_{\mathbb{F}} \cdot ((kn)1_{\mathbb{F}})^{-1} = m1_{\mathbb{F}} \cdot (n1_{\mathbb{F}})^{-1}.$$

By cross multiplying, this will happen iff

$$(km)1_{\mathbb{F}} \cdot (n1_{\mathbb{F}}) = ((kn)1_{\mathbb{F}}) \cdot m1_{\mathbb{F}}$$

which is the case as $\varphi : \mathbb{Z} \rightarrow \mathbb{F}$ is a ring homomorphism.

Homomorphism property. We have

$$\begin{aligned}\varphi\left(\frac{m}{n} + \frac{p}{n}\right) &= \varphi\left(\frac{m+p}{n}\right) = \varphi(m+p) \cdot \varphi(n)^{-1} \\ &= [\varphi(m) + \varphi(p)] \cdot \varphi(n)^{-1} \\ &= \varphi(m) \cdot \varphi(n)^{-1} + \varphi(p) \cdot \varphi(n)^{-1} \\ &= \varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{n}\right)\end{aligned}$$

and

$$\begin{aligned}\varphi\left(\frac{m}{n}\right) \varphi\left(\frac{q}{p}\right) &= \varphi(m) \varphi(n)^{-1} \varphi(q) \varphi(p)^{-1} \\ &= \varphi(m) \varphi(q) [\varphi(n) \varphi(p)]^{-1} \\ &= \varphi(mq) [\varphi(np)]^{-1} = \varphi\left(\frac{mq}{np}\right).\end{aligned}$$

Injectivity. If $0 = \varphi\left(\frac{m}{n}\right)$ then

$$0 = \varphi(m) \cdot \varphi(n)^{-1}$$

which implies $\varphi(m) = 0$ which happens iff $m = 0$, i.e. $m/n = 0$. ■

- End of Lecture 3, 10/3/2012

Notation 2.15 If $(\mathbb{F}, P)$ is an ordered field we write $x > y$ iff $x - y \in P$. We also write $x \geq y$ iff $x > y$ or $x = y$.

Notice that if $x, y \in \mathbb{F}$ then either $x - y = 0$ (i.e. $x = y$), or $x - y \in P$ (i.e. $x > y$), or $x - y \in -P$ (i.e. $y - x \in P$ and $y > x$). Also in this notation we have $P = \{x \in \mathbb{F} : x > 0\}$,

$$-P = \{x \in \mathbb{F} : x < 0 \text{ (i.e. } 0 > x)\}.$$

Lemma 2.16. Suppose that $x < y$ and $y < z$ and $a > 0$. Then $x < z$ and $ax < ay$.

Proof. By assumption $y - x \in P$ and $z - y \in P$, therefore $z - x = (y - x) + (z - y) \in P$, i.e. $z > x$. Moreover, $a \in P$ and $(y - x) \in P$ implies

$$P \ni a(y - x) = ay - ax.$$

That is $ay > ax$. ■

Exercise 2.1. Let $(\mathbb{F}, P)$ be an ordered field and $x, y \in \mathbb{F}$ with $y > x$. Show;

1. $y + a > x + a$ for all $a \in \mathbb{F}$,
2. $-x > -y$,
3. if we further suppose $x > 0$, show $\frac{1}{x} > \frac{1}{y}$.

Definition 2.17. Given $x \in \mathbb{F}$, we say that $y \in \mathbb{F}$ is a square root of x if $y^2 = x$. [From Lemma 2.8, it follows that if $x \in \mathbb{F}$ has a square root then $x \geq 0$.]

Lemma 2.18. Suppose $x, y \in \mathbb{F}$ with $x^2 = y^2$, then either $x = y$ or $x = -y$. In particular, there are at most 2 square roots of any number $x \geq 0$ in $\mathbb{F}$.

Proof. Observe that

$$\begin{aligned}(x - y)(x + y) &= (x - y)x + (x - y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2 = 0.\end{aligned}$$

Thus it follows that either $x - y = 0$ or $x + y = 0$, i.e. $x = y$ or $x = -y$. ■

Definition 2.19. If $x > 0$ admits a square root we let $\sqrt{x}$ be the unique positive root. We also define $\sqrt{0} = 0$.

Lemma 2.20. Suppose that $0 < x < y$, i.e. $x, y - x \in P$, then $x^2 < y^2$.

Proof. By Lemma 2.16, we know $x \cdot x < x \cdot y$ and $x \cdot y < y \cdot y$ and therefore $x^2 < y^2$. ■

Corollary 2.21. If $0 \leq x < y$ and $\sqrt{x}$ and $\sqrt{y}$ exists, then $0 \leq \sqrt{x} < \sqrt{y}$.

Proof. If $\sqrt{x} = \sqrt{y}$ then $x = (\sqrt{x})^2 = (\sqrt{y})^2 = y$ which is impossible. Similarly if $\sqrt{x} > \sqrt{y}$ then

$$x = (\sqrt{x})^2 > (\sqrt{y})^2 = y$$

which is again false.

Alternatively: starting with $y^2 - x^2 = (y - x)(y + x)$ and then replacing y and x by $\sqrt{y}$ and $\sqrt{x}$ respectively (assuming they exist) shows,

$$y - x = (\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) \implies \sqrt{y} - \sqrt{x} = (y - x)(\sqrt{y} + \sqrt{x})^{-1}$$

from which it follows that $\sqrt{y} - \sqrt{x} \in P$ if $(y - x) \in P$. More importantly this shows $\sqrt{y}$ depends “continuously” in on y . ■

Definition 2.22. The absolute value, $|x|$, of x in ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively we may define

$$|x| = \sqrt{x^2}.$$

Proposition 2.23. For all $x, y \in \mathbb{F}$, then

1. $|x| \geq 0$
2. $|xy| = |x| |y|$
3. $|x + y| \leq |x| + |y|$.

Proof. 1. holds by definition since $-x > 0$ if $x < 0$.

2. As $|x| |y| \geq 0$ and $(|x| |y|)^2 = |x|^2 |y|^2 = x^2 y^2 = (xy)^2$, we have

$$|x| |y| = \sqrt{(xy)^2} = |xy|.$$

3. It suffices to show $|x + y|^2 \leq (|x| + |y|)^2$. However,

$$\begin{aligned} |x + y|^2 &= (x + y)^2 = x^2 + y^2 + 2xy \\ &\leq x^2 + y^2 + 2|xy| \quad (xy \leq |xy|) \\ &= |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2. \end{aligned}$$

■

Definition 2.24. Let $(\mathbb{F}, P)$ be an ordered field and S be a subset of $\mathbb{F}$.

1. We say that $S \subset \mathbb{F}$ is **bounded from above (below)** if there exists $x \in \mathbb{F}$ such that $x \geq s$ ($x \leq s$) for all $s \in S$. Any such x is called an **upper (lower) bound** of S .
2. If S is bounded from above (below), we say that $y \in \mathbb{F}$ is a **least upper bound (greatest lower bound)** for S if y is an upper (lower) bound for S and $y \leq x$ ($y \geq x$) for any other upper (lower) bound, x , of S .

Notice that least upper bounds and greatest lower bounds are unique if they exists. We will write and

$$y = \text{l.u.b.}(S) = \sup(S)$$

if y is the least upper bound for S and

$$y = \text{g.l.b.}(S) = \inf(S)$$

if y is the greatest lower bound for S .

Example 2.25. Let $\mathbb{F} = \mathbb{Q}$, then;

1. $\max(a, b)$ and $\min(a, b)$ are least upper respectively greatest lower bounds respectively for $S = \{a, b\}$. More generally, if $S = \{a_1, \dots, a_n\}$, then

$$\begin{aligned} \sup(S) &= a_1 \vee \dots \vee a_n := \max(a_1, \dots, a_n) \quad \text{and} \\ \inf(S) &= a_1 \wedge \dots \wedge a_n := \min(a_1, \dots, a_n). \end{aligned}$$

2. $S = \mathbb{N}$ is not bounded from above while $\inf(S) = 1$.
3. $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ is bounded from above and $1 = \sup(S)$ while $\inf(S) = \frac{1}{2}$.
4. Let

$$S = \{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, \dots\}$$

where I am getting these digits from the decimal expansion of $\sqrt{2}$;

$$\sqrt{2} \cong 1.41421356237309504880168872420969807856967187537694807317667973799.$$

In this case S is bounded above by 2, or 1.42, or 1.415, etc. Nevertheless $\sqrt{2} = \sup(S)$ does not exists in $\mathbb{Q}$.

Example 2.26. Now let $\mathbb{F} = \mathbb{Q}(t)$ be the field of rational functions described in Example 2.11, then; $S = \mathbb{N}$ is bounded from above. For example t is an upper bound but there is not least upper bound. For example $\frac{1}{m}t$ is also an upper bound for S .

- End of Lecture 4, 10/5/2012

Definition 2.27 (Dedekind Cuts). A subset $\alpha \subset \mathbb{Q}$ is called a **cut** (see [2, p. 17]) if;

1. α is a proper subset of $\mathbb{Q}$, i.e. $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$,
2. if $p \in \alpha$ and $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$,
3. if $p \in \alpha$, then there exists $r \in \alpha$ with $r > p$.

Example 2.28. To each $a \in \mathbb{Q}$, let $\alpha_a := \{q \in \mathbb{Q} : q < a\}$. Then α_a is a cut and a is the least upper bound of α_a in $\mathbb{Q}$.

Example 2.29. Let $\{S_n\}_{n=0}^{\infty} \subset \mathbb{Q}$ be any bounded sequence such that $S_n \leq S_{n+1}$ for all n . Then

$$\alpha := \cup_{n=0}^{\infty} \alpha_{S_n} = \{q \in \mathbb{Q} : q < S_n \text{ a.a. } n\}$$

is a cut as the reader should verify. Let us further suppose that $\lim_{n \rightarrow \infty} S_n$ does not exist in $\mathbb{Q}$. [For example from Example 1.17 we may take $S_n := \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q}$.] If $m \in \mathbb{Q}$ is an upper bound for α , then $m \geq S_n$ for all n since if $m < S_n$ for some n then $q := \frac{1}{2}(m + S_n) \in \alpha$ with $q > m$. Since $\lim_{n \rightarrow \infty} S_n \neq m$ as $m \in \mathbb{Q}$ there must exists $\varepsilon > 0$ such that

$$m - S_n = |m - S_n| \geq \varepsilon \text{ i.o. } n.$$

As $m - S_n$ is decreasing we may conclude that $m - S_n \geq \varepsilon$ for all n , i.e. $S_n \leq m - \varepsilon$ for all n . From this it now follows that $m - \varepsilon$ is an upper bound for α which is strictly smaller than m . So there can be no least upper bound.

Real Numbers

As we saw in Section 1.2, $\mathbb{Q}$ is full of holes and calculus tends to produce answers which live in these holes. So it is imperative that we fill the holes. Doing so will lead to the real numbers provided we fill in the holes without adding too much extra filler along the way. One good answer to the question, What are the real numbers?, is contained in the statement of Theorem 3.3.

Definition 3.1. An *order preserving field isomorphism* between two ordered fields, $(\mathbb{F}_1, P_1)$ and $(\mathbb{F}_2, P_2)$, is a bijection, $f : \mathbb{F}_1 \rightarrow \mathbb{F}_2$ such that $f(0) = 0$, $f(1) = 1$, $f(P_1) = P_2$, and

$$f(x + y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y) \text{ for all } x, y \in \mathbb{F}_1.$$

Definition 3.2. An ordered field $(\mathbb{F}, P)$ is has the *least upper bound property* (or is *complete*) if every non-empty subset, $S \subset \mathbb{F}$, which is bounded from above possesses a least upper bound in $\mathbb{F}$. [As we have seen in examples above, $\mathbb{Q}$ does not have the least upper bound property.]

Theorem 3.3 (The real numbers). Up to order preserving field isomorphism (see Definition 3.1), there is exactly one complete ordered field. It is this field that we refer to as the *real numbers* and denote by $\mathbb{R}$.

Definition 3.4. We say two Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of rational numbers are equivalent and write $\{a_n\}_{n=1}^{\infty} \sim \{b_n\}_{n=1}^{\infty}$ iff

$$\lim_{n \rightarrow \infty} |a_n - b_n| = 0.$$

We then define $\alpha := [\{a_n\}_{n=1}^{\infty}]$ to be the equivalence class of the Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ and refer to the collection of these equivalence classes as the *real numbers*. The set of real numbers will be denoted by $\mathbb{R}$.

Notation 3.5 Let $i : \mathbb{Q} \rightarrow \mathbb{R}$ be defined by $i(a) := [(a, a, a, \dots)]$, i.e. $i(a)$ is the equivalence class of the constant sequence a .

Notice that if $i(a) = i(b)$ iff $a = \lim_{n \rightarrow \infty} a = \lim_{n \rightarrow \infty} b = b$. Thus the map, $i : \mathbb{Q} \rightarrow \mathbb{R}$ is injective and we will often simply identify a with $i(a)$ and in this way consider $\mathbb{Q}$ as a subset of $\mathbb{R}$.

Theorem 3.6. Let $\mathbb{R}$ be as in Theorem 3.3. For $\alpha := [\{a_n\}_{n=1}^{\infty}]$ and $\beta := [\{\beta_n\}_{n=1}^{\infty}]$ in $\mathbb{R}$ we define

$$\alpha + \beta = [\{a_n + b_n\}_{n=1}^{\infty}] \text{ and } \alpha \cdot \beta = [\{a_n \cdot b_n\}_{n=1}^{\infty}].$$

1. With these definitions, $\mathbb{R}$, satisfies the axioms of a field.
2. Moreover, we can make this into an ordered field by setting $P := \{\alpha \in \mathbb{R} : \alpha > 0\}$ where we say $\alpha > 0$ iff there exists an $N \in \mathbb{N}$ such that $a_n > \frac{1}{N}$ for a.a. n .¹
3. The ordered field $(\mathbb{R}, P)$ is complete, i.e. has the least upper bound property.

The proof of Theorem 3.6 and Theorem 3.3 will be relegated to Section 3.6 at the end of this chapter. For an alternative existence proof of $\mathbb{R}$ using Dedekind cuts as the elements of $\mathbb{R}$ is covered in Rudin [2, pages 17-21.]. One may also construct the Real numbers using decimal expansions, see T. Gower's notes on real numbers as decimals. We will prove the uniqueness assertion of Theorem 3.3 in Section at the end of this section. From now on we are going to take Theorem 3.3 for granted and derive from this the “familiar” properties of the real numbers.

Observe that $\mathbb{Q}$, $\mathbb{Q}(t)$, $\mathbb{R}(t)$ are not complete and hence are not the real numbers, $\mathbb{R}$. For example $\mathbb{N} \subset \mathbb{Q}(t)$ (or $\mathbb{N} \subset \mathbb{R}(t)$) is bounded by t say but has no least upper bound. However, we do know that $\mathbb{Q} \subset \mathbb{R}$ by Lemma 2.14. We will soon see that $\mathbb{Q}$ is “dense” in $\mathbb{R}$. We now pause to discuss some of the basic properties of $\mathbb{R}$.

Theorem 3.7. Suppose that $\mathbb{R}$ is a complete ordered field which we assume we have already embedded $\mathbb{Q}$ into $\mathbb{R}$ as in Lemma 2.14. Then;

1. For all $x \geq 0$ there exists $n \in \mathbb{N}$ such that $n \geq x$.
2. For all $\varepsilon > 0$ in $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} \leq \varepsilon$.
3. If $\varepsilon \geq 0$ satisfies $\varepsilon \leq 1/n$ for all $n \in \mathbb{N}$ then $\varepsilon = 0$.
4. If $a, b \in \mathbb{R}$ and $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$ or $a \leq b + \varepsilon$ for all $\varepsilon > 0$, then $a \leq b$.

Proof. We take each item in turn.

1. If $n < x$ for all $n \in \mathbb{N}$, then $\mathbb{N}$ is bounded from above and so $a := \sup(\mathbb{N})$ exists in $\mathbb{R}$ by the completeness axiom. As a is the least upper bound for $\mathbb{N}$ there must be an $n \in \mathbb{N}$ such that $n > a - 1$. However this implies $n + 1 > a$ which violates a be an upper bound for $\mathbb{N}$.

¹ Roughly speaking here, you should think of $\alpha = \lim_{n \rightarrow \infty} a_n$ and so $\alpha > 0$ should happen iff $\alpha > \frac{1}{N}$ for some $N \in \mathbb{N}$ which then implies $a_n \geq \frac{1}{N}$ for a.a. n .

2. If $\varepsilon > 0$ in $\mathbb{R}$ and $\frac{1}{n} > \varepsilon$ for all $n \in \mathbb{N}$, then $n < \frac{1}{\varepsilon}$ for all $n \in \mathbb{N}$ which is impossible by item 1.
3. If there exists $\varepsilon > 0$ such that $\varepsilon \leq \frac{1}{n}$ for all n then $n \leq 1/\varepsilon$ for all n which is again impossible by item 1.
4. It suffices to prove the first assertion. We may assume $a \geq b$ for otherwise we are done. If $a \leq b + \frac{1}{n}$ for all n , then $0 \leq a - b \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and hence $a = b$ and in particular $a \leq b$.

■

Proposition 3.8. *If $\mathbb{R}$ is a complete ordered field, then every subset $S \subset \mathbb{R}$ which is bounded from below has a greatest lower bound, $\text{glb}(S) = \inf(S)$. In fact,*

$$\inf(S) = -\sup(-S).$$

Proof. We let $m := -\sup(-S)$. Then we have $-s \leq -m$ for all $s \in S$, i.e. $s \geq m$ for all $s \in S$ so that m is a lower bound for S . Moreover if $\varepsilon > 0$ is given there exists $s_\varepsilon \in S$ such that $-s_\varepsilon \geq -m - \varepsilon$, i.e. $s_\varepsilon \leq m + \varepsilon$. This shows that any lower bound, k of S must satisfy, $k \leq m + \varepsilon$ for all $\varepsilon > 0$, i.e. $k \leq m$. This shows that m is the greatest lower bound for S .

■

Let me sketch one way to construct $\mathbb{R}$ based on Cauchy sequences of rational numbers.

Definition 3.9. *A sequence $\{q_n\}_{n=1}^\infty \subset \mathbb{R}$ **converges to** $0 \in \mathbb{R}$ if for all $\varepsilon > 0$ in $\mathbb{R}$ there exists $N \in \mathbb{N}$ such that $|q_n| \leq \varepsilon$ for all $n \geq N$. Alternatively put, for all $M \in \mathbb{N}$ we have $|q_n| \leq \frac{1}{M}$ for a.a. n .*

Definition 3.10. *A sequence $\{q_n\}_{n=1}^\infty \subset \mathbb{R}$ **converges to** $q \in \mathbb{R}$ if $|q - q_n| \rightarrow 0$ as $n \rightarrow \infty$, i.e. if for all $N \in \mathbb{N}$, $|q - q_n| \leq \frac{1}{N}$ for a.a. n . As usual if $\{q_n\}_{n=1}^\infty$ converges to q we will write $q_n \rightarrow q$ as $n \rightarrow \infty$ or $q = \lim_{n \rightarrow \infty} q_n$.*

Definition 3.11. *A sequence $\{q_n\}_{n=1}^\infty \subset \mathbb{R}$ is **Cauchy** if $|q_n - q_m| \rightarrow 0$ as $m, n \rightarrow \infty$. More precisely we require for each $\varepsilon > 0$ in $\mathbb{R}$ that $|q_m - q_n| \leq \varepsilon$ for a.a. pairs (m, n) , i.e. there should exist $N \in \mathbb{N}$ such that $|q_m - q_n| \leq \varepsilon$ for all $m, n \geq N$.*

The next few results are analogous to what you have already shown in the case $\mathbb{R}$ is replaced by $\mathbb{Q}$. As the proofs are essentially identical to those of Theorem 1.13 and Exercise 1.1 – 1.6.

Proposition 3.12. *If $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ is a convergent sequence then it is Cauchy.² If $\{a_n\}_{n=1}^\infty$ is Cauchy sequence then it is bounded.*

² We are going to show shortly that the converse is true as well!

Theorem 3.13 (Basic Limit Results). *Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $A := \lim_{n \rightarrow \infty} a_n$ and $B := \lim_{n \rightarrow \infty} b_n$ exists in $\mathbb{R}$. Then;*

1. $\lim_{n \rightarrow \infty} |a_n| = |A|$.
2. If $A \neq 0$ then $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{A}$.
3. $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$.
4. $\lim_{n \rightarrow \infty} (a_n b_n) = A \cdot B$.
5. If $a_n \leq b_n$ for all n , then $A \leq B$.
6. If $\{x_n\} \subset \mathbb{R}$ is another sequence such that $a_n \leq x_n \leq b_n$ and $A = B$, then $\lim_{n \rightarrow \infty} x_n = A = B$.

Theorem 3.14. *If $S \subset \mathbb{R}$ is a non-empty set which is bounded from above, then there exists $\{x_n\}_{n=1}^\infty \subset S$ such that $x_n \uparrow \sup S$ as $n \rightarrow \infty$, i.e. $x_n \leq x_{n+1}$ for all n and $\lim_{n \rightarrow \infty} x_n = \sup S$.*

Proof. Let $M := \sup S$. For each $n \in \mathbb{N}$, there exists $y_n \in S$ such that $M \geq y_n \geq M - \frac{1}{n}$. We now let $x_n := \max(\{y_1, \dots, y_n\})$ in which case $M \geq x_n \geq M - \frac{1}{n}$ and x_n is increasing. By the Sandwich theorem it follows that $\lim_{n \rightarrow \infty} x_n = M$.

■

- End of Lecture 5, 10/8/2012

Theorem 3.15. *If $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ is bounded from above and x_n is non-decreasing, then $\lim_{n \rightarrow \infty} x_n = \sup_n x_n$. Similarly if $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ is bounded from below and x_n is non-increasing, then $\lim_{n \rightarrow \infty} x_n = \inf_n x_n$.*

Proof. Let $M := \sup_n x_n$, then for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $M \geq x_{N_\varepsilon} \geq M - \varepsilon$. As x_n is non-decreasing it follows that $M \geq x_n \geq M - \varepsilon$ for all $n \geq N_\varepsilon$, i.e.

$$|M - x_n| \leq \varepsilon \text{ for all } n \geq N_\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we may conclude that $\lim_{n \rightarrow \infty} x_n = M$. If x_n is decreasing instead, then $-x_n \uparrow$ and we have $-\lim_{n \rightarrow \infty} x_n = \sup_n (-x_n) = -\inf_n x_n$.

■

Theorem 3.16. *Suppose that $\mathbb{R}$ is a complete ordered field which we assume we have already embedded $\mathbb{Q}$ into $\mathbb{R}$ as in Lemma 2.14. Then;*

1. For all $m \in \mathbb{R}$, if

$$\alpha_m := \{y \in \mathbb{Q} : y < m\},$$
 then $\sup \alpha_m = m$.
2. If $a, b \in \mathbb{R}$ with $a < b$, then there exists $q \in \mathbb{Q}$ such that $a < q < b$.
3. If $\alpha \subset \mathbb{Q}$ is a cut and $m := \sup \alpha$, then $\alpha = \alpha_m$.

Proof. We take each item in turn.

1. **Proof 1.** Let $\alpha_m := \{y \in \mathbb{Q} : y < m\}$ and $M := \sup \alpha_m \in \mathbb{R}$. Then $M \leq m$. If $M \neq m$ then $M < m$. To see this last case is not possible $\varepsilon := m - M > 0$ and choose $n \in \mathbb{N}$ such that $0 < \frac{1}{n} \leq \varepsilon$. Then choose $y \in \mathbb{Q}$ such that

$$M - \frac{1}{2n} < y < M.$$

From this it follows that

$$M < y + \frac{1}{2n} < M + \frac{1}{2n} < m$$

which shows $y + \frac{1}{2n} \in \alpha_m$ is greater than M violating the assumption that M is an upper bound for α_m .

Proof 2. [Here is a slight rewriting of the above argument.] Choose $y_n \in \alpha_m$ such that $y_n \uparrow M$ as $m \rightarrow \infty$. Choose $n \in \mathbb{N}$ so that $m - M > \frac{1}{n}$. Then $y_m + \frac{1}{n} \uparrow M + \frac{1}{n} < m$ as $m \rightarrow \infty$. So for large m , $y_m + \frac{1}{n} < m$ while $y_m + \frac{1}{n} > M$, i.e. $y_m + \frac{1}{n} \in \alpha_m$ yet $y_m + \frac{1}{n} > M$. This violates the assumption that M is an upper bound for α_m .

2. By item 1. and Theorem 3.14 we can choose $q \in \alpha_b$ to be as close to b as we choose and in particular q can be chosen to be in α_b with $q > a$.
3. You are asked to prove this in Exercise 3.1 below.

■

Exercise 3.1. Suppose that $\alpha \subset \mathbb{Q}$ is a cut as in Definition 2.27. Show α is bounded from above. Then let $m := \sup \alpha$ and show that $\alpha = \alpha_m$, where α_m

$$\alpha_m := \{y \in \mathbb{Q} : y < m\}.$$

Also verify that α_m is a cut for all $m \in \mathbb{R}$. [In this way we see that we may identify $\mathbb{R}$ with the cuts of $\mathbb{Q}$. This should motivate Dedekind's construction of the real numbers as described in Rudin.]

Proposition 3.17 (Rationals are dense in the reals). For all $b \in \mathbb{R}$, there exists $q_n \in \mathbb{Q}$ such that $q_n \uparrow b$. Similarly there exists $p_n \in \mathbb{Q}$ such that $p_n \downarrow b$.

Proof. Given $b \in \mathbb{Q}$ we know that $b = \sup \alpha_b$ by Theorem 3.16. Then by Theorem 3.14 there exists $q_n \in \alpha_b$ such that $q_n \uparrow b$ as $n \rightarrow \infty$. The second assertion can be proved in much the same way as the first. Alternatively, let $q_n \in \mathbb{Q}$ such that $q_n \uparrow -b$ and set $p_n := -q_n \in \mathbb{Q}$. Then $p_n \downarrow b$.

■

Definition 3.18. The real numbers which are not rational are called *irrational*,³ so the irrational numbers are $\mathbb{R} \setminus \mathbb{Q}$.

³ For what it is worth, as dictionary definition of irrational is “not consistent with or using reason.” Lets try to use irrational numbers in a rational way!

Example 3.19 (Euler's number). Let $S_n := \sum_{k=0}^n \frac{1}{k!}$ for all $n \in \mathbb{N}_0$. We define Euler's number to be,

$$e := \lim_{n \rightarrow \infty} S_n = \sup \{S_n : n \in \mathbb{N}_0\} \in \mathbb{R}.$$

From Example 1.17, we have seen that $e \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 3.20 (n^{th} – roots). Let $n \in \mathbb{N}$ and $x > 0$ in $\mathbb{R}$, then there exists a unique $y \in \mathbb{R}_+$ such that $y^n = x$. We of course denote y by $x^{1/n}$ for $\sqrt[n]{x}$. The function $x \rightarrow x^{1/n}$ is increasing. [See Rudin for more properties of $x^{1/n}$ and $x^{m/n}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.]

Proof. Uniqueness. First of $t > s \geq 0$ then $t^n > s^n \geq 0$ as can be proved by induction.⁴ Thus if $x, y \geq 0$ and $x^n = y^n$ then $x = y$ for otherwise $x > y$ or $y > x$ in which case $x^n > y^n$ or $y^n > x^n$ respectively. This shows that there is at most one n^{th} – root if it exists. I also claim that $x^{1/n} < y^{1/n}$ if $x < y$. If not then $x^{1/n} \geq y^{1/n}$ and this would then imply $x = (x^{1/n})^n \geq (y^{1/n})^n = y$ which contradicts $x < y$.

Existence. Let $\Lambda := \{t \in \mathbb{R}_+ : t^n \leq x\}$. If $t = \frac{x}{1+x} \in (0, 1)$, then $t^n \leq t \leq x$ so that $t \in \Lambda$ and $\Lambda \neq \emptyset$. If $t = 1 + x$, then $t^n = (1 + x)^n \geq 1 + nx > x$ and therefore Λ is bounded from above. Hence we may define $y := \sup \Lambda$. We will now show that $y^n = x$.

By Theorem 3.14, there exists $t_k \in \Lambda$ such that $t_k \uparrow y$ as $k \rightarrow \infty$. By definition of Λ , $t_k^n \leq x$ for all k . Passing to the limit as $k \rightarrow \infty$ in this inequality implies $y^n = \lim_{k \rightarrow \infty} t_k^n \leq x$.

If $y^n < x$ then (using the Binomial theorem) and properties of limits,

$$\left(y + \frac{1}{m}\right)^n = y^n + \sum_{k=1}^n \binom{n}{k} y^{n-k} \left(\frac{1}{m}\right)^k \rightarrow y^n < x \text{ as } m \rightarrow \infty.$$

Hence for sufficiently large m we will have $\left(y + \frac{1}{m}\right)^n < x$. But this shows that $y + \frac{1}{m} \in \Lambda$ which violates y being an upper bound for Λ . Therefore we conclude that $y^n = x$.

■

- End of Lecture 6, 10/10/2012.

3.1 Extended real numbers

Notation 3.21 The *extended real numbers* is the set $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, i.e. it is $\mathbb{R}$ with two new points called ∞ and $-\infty$. We use the following conventions,

⁴ The statement holds for $n = 1$ by assumption and if $t^n > s^n$, then $t^{n+1} > ts^n > s^{n+1}$. For the last equality we used $t > s$ implies $ts^n > s \cdot s^n$.

$\pm\infty \cdot a = \pm\infty$ if $a \in \bar{\mathbb{R}}$ with $a > 0$, $\pm\infty \cdot a = \mp\infty$ if $a \in \bar{\mathbb{R}}$ with $a < 0$, $\pm\infty + a = \pm\infty$ for any $a \in \mathbb{R}$, $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$ while the following expressions are **not defined**;

$$\infty - \infty, -\infty + \infty, \infty/\infty, 0 \cdot \infty, \text{ and } \infty \cdot 0.$$

A sequence $a_n \in \bar{\mathbb{R}}$ is said to converge to ∞ ($-\infty$) if for all $M \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $a_n \geq M$ ($a_n \leq M$) for all $n \geq m$. In these case we write $\lim_{n \rightarrow \infty} a_n = \pm\infty$ or $a_n \rightarrow \pm\infty$ as $n \rightarrow \infty$.

For any subset $A \subset \bar{\mathbb{R}}$, let $\sup A$ and $\inf A$ denote the least upper bound and greatest lower bound of A respectively. The convention being that $\sup A = \infty$ if $\infty \in A$ or A is not bounded from above and $\inf A = -\infty$ if $-\infty \in A$ or A is not bounded from below. We will also use the **conventions** that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. The next theorem is a fairly simple but often useful result about computing least upper bounds.

Theorem 3.22 (Sup Sup Theorem). Suppose that A is a subset of $\mathbb{R}$ such that $A = \cup_{\alpha \in I} A_\alpha$ where $A_\alpha \subset A$ and I is some index set. Then

$$\sup A = \sup_{\alpha \in I} \sup A_\alpha.$$

The convention here is that the supremum of a set which is not bounded from above is ∞ and the $\sup \emptyset = -\infty$.

Proof. Let $M := \sup A$ and $M_\alpha := \sup A_\alpha$ for all $\alpha \in I$. As $A_\alpha \subset A$ we have $M_\alpha \leq M$ for all $\alpha \in I$ and therefore $\sup_{\alpha \in I} M_\alpha \leq M$. Conversely, if $\lambda \in A$, then $\lambda \in M_\alpha$ for some $\alpha \in I$ and therefore $\lambda \leq M_\alpha$. From this it follows that $\lambda \leq \sup_{\alpha \in I} M_\alpha$ and as $\lambda \in A$ is arbitrary we may conclude that $M = \sup A \leq \sup_{\alpha \in I} M_\alpha$. ■

The next corollary records a typical way the Sup Sup theorem is used.

Corollary 3.23. Suppose that X and Y are sets and $S : X \times Y \rightarrow \mathbb{R}$ is a function. Then

$$\sup_{x \in X} \sup_{y \in Y} S(x, y) = \sup_{(x, y) \in X \times Y} S(x, y) = \sup_{y \in Y} \sup_{x \in X} S(x, y).$$

In particular, if $S_{m,n} \in \mathbb{R}$ for all $m, n \in \mathbb{N}$, then

$$\sup_m \sup_n S_{m,n} = \sup_{(m,n)} S_{m,n} = \sup_n \sup_m S_{m,n}.$$

Proof. Let $A := \{S(x, y) : (x, y) \in X \times Y\}$, and for $x \in X$ let $A_x := \{S(x, y) : y \in Y\}$. Then $A = \cup_{x \in X} A_x$ and therefore,

$$\sup_{(x, y) \in X \times Y} S(x, y) = \sup A = \sup_{x \in X} \sup_{y \in Y} S(x, y) = \sup_{x \in X} \sup_{y \in Y} S(x, y).$$

The same reasoning also shows,

$$\sup_{(x, y) \in X \times Y} S(x, y) = \sup_{y \in Y} \sup_{x \in X} S(x, y).$$

The next Lemma records some basic limit theorems involving the extended real numbers. ■

Lemma 3.24. Suppose $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are convergent sequences⁵ in $\bar{\mathbb{R}}$, then:

1. If $a_n \leq b_n$ for a.a. n then $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$.
2. If $c \in \mathbb{R}$, $\lim_{n \rightarrow \infty} (ca_n) = c \lim_{n \rightarrow \infty} a_n$.
3. If $\{a_n + b_n\}_{n=1}^\infty$ is convergent and

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \quad (3.1)$$

provided the right side is not of the form $\infty - \infty$.

4. $\{a_n b_n\}_{n=1}^\infty$ is convergent and

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n \quad (3.2)$$

provided the right hand side is not of the form $\pm\infty \cdot 0$ or $0 \cdot (\pm\infty)$.

Before going to the proof consider the simple example where $a_n = n$ and $b_n = -\alpha n + c$ with $\alpha > 0$ and $c \in \mathbb{R}$. Then⁶

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \begin{cases} \infty & \text{if } \alpha < 1 \\ c & \text{if } \alpha = 1 \\ -\infty & \text{if } \alpha > 1 \end{cases}$$

while

$$\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = \infty - \infty.$$

This shows that the requirement that the right side of Eq. (3.1) is not of form $\infty - \infty$ is necessary in Lemma 3.24. Similarly by considering the examples $a_n = n$ and $b_n = n^{-\alpha}$ with $\alpha > 0$ shows the necessity for assuming right hand side of Eq. (3.2) is not of the form $\infty \cdot 0$.

Proof. The proofs of items 1. and 2. are left to the reader.

⁵ The only sequences that do not converge in $\bar{\mathbb{R}}$ are those which oscillate too much.

⁶ This example shows that if you formally arrive at an expression like $\infty - \infty$, then you should work harder to decide what it really means!

Proof of Eq. (3.1). Let $a := \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} b_n$.

Case 1. Suppose $b = \infty$ in which case we must assume $a > -\infty$. In this case, for every $M > 0$, there exists N such that $b_n \geq M$ and $a_n \geq a - 1$ for all $n \geq N$ and this implies

$$a_n + b_n \geq M + a - 1 \text{ for all } n \geq N.$$

Since M is arbitrary it follows that $a_n + b_n \rightarrow \infty$ as $n \rightarrow \infty$. The cases where $b = -\infty$ or $a = \pm\infty$ are handled similarly.

Case 2. If $a, b \in \mathbb{R}$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a - a_n| \leq \varepsilon \text{ and } |b - b_n| \leq \varepsilon \text{ for all } n \geq N.$$

Therefore,

$$|a + b - (a_n + b_n)| = |a - a_n + b - b_n| \leq |a - a_n| + |b - b_n| \leq 2\varepsilon$$

for all $n \geq N$. Since n is arbitrary, it follows that $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$.

Proof of Eq. (3.2). It will be left to the reader to prove the case where $\lim a_n$ and $\lim b_n$ exist in $\mathbb{R}$. I will only consider the case where $a = \lim_{n \rightarrow \infty} a_n \neq 0$ and $\lim_{n \rightarrow \infty} b_n = \infty$ here. Let us also suppose that $a > 0$ (the case $a < 0$ is handled similarly) and let $\alpha := \min(\frac{a}{2}, 1)$. Given any $M < \infty$, there exists $N \in \mathbb{N}$ such that $a_n \geq \alpha$ and $b_n \geq M$ for all $n \geq N$ and for this choice of N , $a_n b_n \geq M\alpha$ for all $n \geq N$. Since $\alpha > 0$ is fixed and M is arbitrary it follows that $\lim_{n \rightarrow \infty} (a_n b_n) = \infty$ as desired. ■

Exercise 3.2. Show $\lim_{n \rightarrow \infty} \alpha^n = \infty$ and $\lim_{n \rightarrow \infty} \frac{1}{\alpha^n} = 0$ whenever $\alpha > 1$.

Exercise 3.3. Suppose $\alpha > 1$ and $k \in \mathbb{N}$, show there is a constant $c = c(\alpha, k) > 0$ such that $\alpha^n \geq cn^k$ for all $n \in \mathbb{N}$. [In words, α^n grows in n faster than any polynomial in n .]

Lemma 3.25. Suppose that $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\lim_{n \rightarrow \infty} a_n = A \in \bar{\mathbb{R}}$. Then every subsequence, $\{b_k := a_{n_k}\}_{k=1}^{\infty}$, also converges to A .

Exercise 3.4. Prove Lemma 3.25.

3.2 Limsups and Liminfs

Notation 3.26 Suppose that $\{x_n\}_{n=1}^{\infty} \subset \bar{\mathbb{R}}$ is a sequence of numbers. Then

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} \text{ and} \quad (3.3)$$

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\}. \quad (3.4)$$

We will also write $\underline{\lim}$ for $\liminf$ and $\overline{\lim}$ for $\limsup$.

Remark 3.27. Notice that if $a_k := \inf\{x_k : k \geq n\}$ and $b_k := \sup\{x_k : k \geq n\}$, then $\{a_k\}$ is an increasing sequence while $\{b_k\}$ is a decreasing sequence. Therefore the limits in Eq. (3.3) and Eq. (3.4) always exist in $\bar{\mathbb{R}}$ (see Theorem 3.15) and

$$\begin{aligned} \liminf_{n \rightarrow \infty} x_n &= \sup_n \inf\{x_k : k \geq n\} \text{ and} \\ \limsup_{n \rightarrow \infty} x_n &= \inf_n \sup\{x_k : k \geq n\}. \end{aligned}$$

Owing to the following exercise, one may reduce properties of the $\liminf$ to those of the $\limsup$.

Exercise 3.5. Show $\liminf_{n \rightarrow \infty} (-a_n) = -\limsup_{n \rightarrow \infty} a_n$.

Exercise 3.6. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence given by,

$$(-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots).$$

Find $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$.

Exercise 3.7. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences such that $a_n \leq b_n$ for a.a. n , then

$$\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n \text{ and } \liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n. \quad (3.5)$$

The following proposition contains some basic properties of liminfs and limsups.

Proposition 3.28. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Then

1. $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$.
2. $\lim_{n \rightarrow \infty} a_n$ exists in $\bar{\mathbb{R}}$ iff

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n \in \bar{\mathbb{R}}.$$

3.

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \quad (3.6)$$

whenever the right side of this equation is not of the form $\infty - \infty$.

4. If $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \limsup_{n \rightarrow \infty} a_n \cdot \limsup_{n \rightarrow \infty} b_n, \quad (3.7)$$

provided the right hand side of (3.7) is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

Proof. Items 1. and 2. will be proved here leaving the remaining items as an exercise to the reader. For item 1. we have

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\} \forall n,$$

and therefore by the Sandwich theorem, $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$.

2. ($\Leftarrow$) Let $A := \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n \in \mathbb{R}$. Since

$$\inf_{k \geq n} a_k \leq a_n \leq \sup_{k \geq n} a_k,$$

if $A \in \mathbb{R}$ then it follows by the sandwich theorem that $\lim_{n \rightarrow \infty} a_n = A$. If $A = \infty$, then for all $M \in \mathbb{N}$ we have $M \leq \inf_{k \geq n} a_k$ for a.a. n . Therefore $a_k \geq M$ for a.a. k and we have shown $\lim_{k \rightarrow \infty} a_k = \infty$. If $A = -\infty$ then for all $M \in \mathbb{N}$ we have $\sup_{k \geq n} a_k \leq -M$ for a.a. n . Therefore $a_k \leq -M$ for a.a. k and we have shown $\lim_{k \rightarrow \infty} a_k = -\infty$.

($\Rightarrow$) Conversely, suppose that $\lim_{n \rightarrow \infty} a_n = A \in \bar{\mathbb{R}}$ exists. If $A \in \mathbb{R}$, then for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|A - a_n| \leq \varepsilon$ for all $n \geq N(\varepsilon)$, i.e.

$$A - \varepsilon \leq a_n \leq A + \varepsilon \text{ for all } n \geq N(\varepsilon).$$

From this we learn that

$$A - \varepsilon \leq \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k \leq A + \varepsilon \text{ for a.a. } n$$

and so passing to the limit as $n \rightarrow \infty$ implies

$$A - \varepsilon \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$A \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq A,$$

i.e. that $A = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$.

If $A = \infty$, then for all $M > 0$ there exists $N = N(M)$ such that $a_n \geq M$ for all $n \geq N$. This shows that $\liminf_{n \rightarrow \infty} a_n \geq M$ and since M is arbitrary it follows that

$$\infty \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n.$$

The proof for the case $A = -\infty$ is analogous to the $A = \infty$ case. ■

Exercise 3.8. Show that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n \quad (3.8)$$

provided that the right side of Eq. (3.8) is well defined, i.e. no $\infty - \infty$ or $-\infty + \infty$ type expressions. (It is OK to have $\infty + \infty = \infty$ or $-\infty - \infty = -\infty$, etc.)

Exercise 3.9. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Show

$$\limsup_{n \rightarrow \infty} (a_n b_n) \leq \limsup_{n \rightarrow \infty} a_n \cdot \limsup_{n \rightarrow \infty} b_n, \quad (3.9)$$

provided the right hand side of (3.9) is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

• End of Lecture 7, 10/12/2012.

Exercise 3.10. If $a_n \geq 0$, then $\lim_{n \rightarrow \infty} a_n = 0$ iff $\limsup_{n \rightarrow \infty} a_n = 0$.

Proposition 3.29. Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of real numbers and let

$$B := \{y \in \mathbb{R} : a_n \geq y \text{ for i.o. } n\}.$$

Then $\sup B = \limsup_{n \rightarrow \infty} a_n$ with the convention that $\sup B = -\infty$ if $B = \emptyset$.

Proof. If $\{a_n\}_{n=1}^\infty$ is not bounded from above, then B is not bounded from above and $\sup B = \infty = \limsup_{n \rightarrow \infty} a_n$. If $B = \emptyset$ so that $\sup B = -\infty$, then for all $y \in \mathbb{R}$ we must have $a_n < y$ for a.a. n . This then implies $\limsup_{n \rightarrow \infty} a_n \leq y$ for all $y \in \mathbb{R}$ from which we conclude that $\limsup_{n \rightarrow \infty} a_n = -\infty$. So let us now assume that $B \neq \emptyset$ and $\{a_n\}_{n=1}^\infty$ is bounded in which case B is bounded from above. Let us set $\beta := \sup B \in \mathbb{R}$ and $a^* := \limsup_{n \rightarrow \infty} a_n$.

If $y > \beta$, then $a_n < y$ for a.a. n from which it follows that $a^* := \limsup_{n \rightarrow \infty} a_n \leq y$. We may now let $y \downarrow \beta$ in order to see that $a^* \leq \beta$.⁷ Now suppose that $y < \beta$, then $a_n \geq y$ for a.a. n and hence $a^* = \limsup_{n \rightarrow \infty} a_n \geq y$. Letting $y \uparrow \beta$ then shows $a^* \geq \beta$. Thus we have shown $a^* = \beta$. ■

Theorem 3.30. There is a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n$. Similarly, there is a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $\lim_{k \rightarrow \infty} a_{n_k} = \liminf_{n \rightarrow \infty} a_n$. Moreover, every convergent subsequence, $\{b_k := a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ satisfies,

$$\liminf_{n \rightarrow \infty} a_n \leq \lim_{k \rightarrow \infty} b_k \leq \limsup_{n \rightarrow \infty} a_n.$$

Proof. Let me prove the last assertion first. Suppose that $b_k := a_{n_k}$ is some convergent subsequence of $\{a_n\}_{n=1}^\infty$. We then have,

$$\inf_{n \geq n_k} a_n \leq b_k \leq \sup_{n \geq n_k} a_n \text{ for all } k \in \mathbb{N}.$$

Passing to the limit in this equation then implies,

$$\liminf_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} \inf_{n \geq n_k} a_n \leq \lim_{k \rightarrow \infty} b_k \leq \lim_{k \rightarrow \infty} \sup_{n \geq n_k} a_n = \limsup_{n \rightarrow \infty} a_n.$$

⁷ This can be done more formally by choosing a sequence $\{y_k\}_{k=1}^\infty$ such that $y_k \downarrow \beta$ so that $a^* \leq y_k$. Now let $k \rightarrow \infty$ to conclude $a^* \leq \lim_{k \rightarrow \infty} y_k = \beta$.

We have used, $\{\inf_{n \geq n_k} a_n\}_{k=1}^\infty$ and $\{\sup_{n \geq n_k} a_n\}_{k=1}^\infty$ are subsequence of the convergent sequences of $\{\inf_{n \geq k} a_n\}_{k=1}^\infty$ and $\{\sup_{n \geq k} a_n\}_{k=1}^\infty$ respectively and therefore converge to the same limits respectively, see Lemma 3.25.

Now let us prove the first assertions. I will cover the limsup case here as the liminf case is similar or can be deduced from the limsup case with the aid of Exercise 3.5. Let $A := \limsup_{n \rightarrow \infty} a_n$. We will need to consider three case, $A \in \mathbb{R}$, $A = \infty$, and $A = -\infty$.

- i) $A \in \mathbb{R}$, then by Proposition 3.29, for all $k \in \mathbb{N}$ we have $A - \frac{1}{k} \leq a_n$ for infinitely many n . In particular we can choose $n_1 < n_2 < n_3 < \dots$ inductively so that $A - \frac{1}{k} \leq a_{n_k}$ for all k . Since

$$A - \frac{1}{k} \leq a_{n_k} \leq \sup_{m \geq n_k} a_m$$

and the limit as $k \rightarrow \infty$ of both extremes of this inequality are A , it follow from the sandwich inequality that $\lim_{k \rightarrow \infty} a_{n_k} = A$.

- ii) If $A = \limsup_{n \rightarrow \infty} a_n = \infty$, then $\sup_{k \geq n} a_k = \infty$ for all $n \in \mathbb{N}$ which implies for all $M < \infty$ that $a_k \geq M$ i.o. k . Working similarly to case i) we can choose $n_1 < n_2 < n_3 < \dots$ so that $a_{n_k} \geq k$ for all k and therefore $\lim_{k \rightarrow \infty} a_{n_k} = \infty$.
- iii) Finally suppose that $A = \limsup_{n \rightarrow \infty} a_n = -\infty$ so that for all $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\sup_{k \geq n} a_k \leq -M$ for all $n \geq N$, i.e. $a_n \leq -M$ for all $n \geq N$. In this case it follows that in fact $\lim_{n \rightarrow \infty} a_n = -\infty$ and we do not have to even choose as subsequence.

■

Corollary 3.31 (Bolzano–Weierstrass Property / Compactness). *Every bounded sequence of real numbers, $\{a_n\}_{n=1}^\infty$, has a convergent in $\mathbb{R}$ subsequence, $\{a_{n_k}\}_{k=1}^\infty$. If we drop the bounded assumption then we may only assert that there is a subsequence which is convergent in $\overline{\mathbb{R}}$.*

Proof. Let $M < \infty$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$, i.e. $-M \leq a_n \leq M$ for all n . We may then conclude from Exercise 3.7 that,

$$-M \leq \limsup_{n \rightarrow \infty} a_n \leq M.$$

It now follows from Theorem 3.30 that there exists a subsequence, $\{a_{n_k}\}_{k=1}^\infty$, of $\{a_n\}_{n=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} a_{n_k} = \limsup_{n \rightarrow \infty} a_n \in [-M, M] \subset \mathbb{R}.$$

■

Theorem 3.32 ($\mathbb{R}$ is Cauchy complete). *If $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ is a Cauchy sequence, then $\lim_{n \rightarrow \infty} a_n$ exists in $\mathbb{R}$ and in fact,*

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. We will give two proofs of this important theorem. Each proof uses the fact that $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ is Cauchy implies $\{a_n\}_{n=1}^\infty$ is bounded. This is proved exactly in the same way as the solution to Exercise 1.2.

First proof. By Corollary 3.31, there is a subsequence, $\{a_{n_k}\}_{k=1}^\infty$, such that $\lim_{k \rightarrow \infty} a_{n_k} = L \in \mathbb{R}$. As in the proof of Exercise 1.7, it follows that $\lim_{n \rightarrow \infty} a_n$ exists and is equal to L .

Second proof. Let $a := \liminf_{n \rightarrow \infty} a_n$ and $b := \limsup_{n \rightarrow \infty} a_n$. It suffices to show $a = b$. As we always know that $a \leq b$ it will suffice to show $b \leq a$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_m - a_n| \leq \varepsilon \text{ for all } m, n \geq N.$$

In particular, for $m, n \geq k \geq N$ we have $a_m \leq a_n + \varepsilon$ and hence

$$b \leq \sup_{m \geq k} a_m \leq a_n + \varepsilon \text{ for all } n \geq k.$$

From this inequality we may further conclude,

$$b \leq \inf_{n \geq k} a_n + \varepsilon \leq a + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have indeed shown $b \leq a$.

■

- End of Lecture 8, 10/15/2012.

Exercise 3.11. Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of real numbers and let

$$A := \{y \in \mathbb{R} : a_n \geq y \text{ for a.a. } n\}.$$

Then $\sup A = \liminf_{n \rightarrow \infty} a_n$ with the convention that $\sup A = -\infty$ if $A = \emptyset$.

Exercise 3.12. Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of real numbers. Show $\limsup_{n \rightarrow \infty} a_n = a^* \in \mathbb{R}$ iff for all $\varepsilon > 0$,

$$\begin{aligned} a_n &\leq a^* + \varepsilon \text{ for a.a. } n. \text{ and} \\ a^* - \varepsilon &\leq a_n \text{ i.o. } n. \end{aligned}$$

Similarly, show $\liminf_{n \rightarrow \infty} a_n = a_* \in \mathbb{R}$ iff for all $\varepsilon > 0$,

$$\begin{aligned} a_n &\leq a_* + \varepsilon \text{ i.o. } n. \text{ and} \\ a_* - \varepsilon &\leq a_n \text{ for a.a. } n. \end{aligned}$$

Notice that this exercise gives another proof of item 2. of Proposition 3.28 in the case all limits are real valued.

Exercise 3.13 (Cauchy Complete $\implies$ L.U.B. Property). Suppose that $\mathbb{R}$ denotes any ordered field which is Cauchy complete. Show $\mathbb{R}$ has the least upper bound property and therefore is the field of real numbers.

3.3 Partitioning the Real Numbers

Notation 3.33 (Intervals) For $a, b \in \mathbb{R}$ with $a < b$ we define,

$$\begin{aligned}(a, b) &:= \{x \in \mathbb{R} : a < x < b\}, \\ [a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, \\ (a, b] &:= \{x \in \mathbb{R} : a < x \leq b\}, \text{ and} \\ [a, b) &:= \{x \in \mathbb{R} : a \leq x < b\}.\end{aligned}$$

We also allow $a = -\infty$ in the *intervals*, (a, b) and $(a, b]$ and allows $b = +\infty$ in the intervals (a, b) and $[a, b)$.

Notation 3.34 (Pairwise disjoint unions) If X is a set and $A_\alpha \subset X$ for $\alpha \in I$, we write $X = \sum_{\alpha \in I} A_\alpha$ to mean; $X = \cup_{\alpha \in I} A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha \neq \beta$.

Exercise 3.14. Suppose that $a, b, c, d \in \mathbb{R}$ such that $a < b \leq c < d$. Show $(a, b] \cap (c, d] = \emptyset$ and $[a, b) \cap [c, d) = \emptyset$.

Lemma 3.35 (Well Ordering II). Suppose that S is a non-empty subset of $\mathbb{Z}$ which is bounded from below, then $\inf(S) \in S$, i.e. S has a (unique) minimizer.

Proof. As S is bounded from below, there exists $k \in \mathbb{Z}$ such that $k \leq s$ for all $s \in S$. Therefore $\tilde{S} := \{s - k + 1 : s \in S\} \subset \mathbb{N}$ and hence by the Well ordering principle, $\min(\tilde{S}) := m \in \mathbb{N}$ exists. That is $m \leq s - k + 1$ for all $s \in S$ and there exists $s_0 \in S$ such that $m = s_0 - k + 1$. These last statements are equivalent to saying,

$$s_0 = m + k - 1 \leq s \text{ for all } s \in S,$$

which is to say $s_0 = \min(S)$. ■

Proposition 3.36. Suppose that $\{S_n\}_{n=-\infty}^{\infty} \subset \mathbb{R}$ such that $S_n < S_{n+1}$ for all $n \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} S_n = \infty$ and $\lim_{n \rightarrow -\infty} S_n = -\infty$. Then

$$\sum_{n \in \mathbb{Z}} (S_{n-1}, S_n] = \mathbb{R} = \sum_{n \in \mathbb{Z}} [S_n, S_{n+1}). \quad (3.10)$$

Proof. The fact that $(S_n, S_{n+1}] \cap (S_m, S_{m+1}] = \emptyset$ follows from Exercise 3.14. For $x \in \mathbb{R}$, let

$$n_0 := \min(\{n \in \mathbb{Z} : x \leq S_n\})$$

which exists since $\{n \in \mathbb{Z} : x \leq S_n\}$ is non-empty as $S_n \rightarrow \infty$ as $n \rightarrow \infty$ and is bounded from below since $S_n \rightarrow -\infty$ as $n \rightarrow -\infty$. It then follows that $x \leq S_{n_0}$ while $x \not\leq S_{n_0-1}$, i.e. $S_{n_0-1} < x \leq S_{n_0}$ and we have shown $x \in (S_{n_0-1}, S_{n_0}]$ which completes the proof of the first equality in Eq. (3.10). The proof of the second equality is similar and so will be omitted. ■

Proposition 3.37. Suppose that $-\infty < a < b < \infty$ and $\{S_n\}_{n=0}^N \subset [a, b]$ such that $a = S_0 < S_1 < \dots < S_{N-1} < S_N = b$, then

$$[a, b) = \sum_{n=1}^N [S_{n-1}, S_n).$$

This result also holds if $N = \infty$ provided we now assume $S_n < S_{n+1}$ for all n , $a = S_0$, and $S_n \uparrow b$ as $n \rightarrow \infty$.

Proof. This proof is very similar to the proof of Proposition 3.36 and so will be omitted. ■

3.4 The Decimal Representation of a Real Number

Lemma 3.38 (Geometric Series). Let $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{Q}$, $m, n \in \mathbb{Z}$ and $S := \sum_{k=n}^m \alpha^k$. Then

$$S = \begin{cases} m - n + 1 & \text{if } \alpha = 1 \\ \frac{\alpha^{m+1} - \alpha^n}{\alpha - 1} & \text{if } \alpha \neq 1 \end{cases}.$$

Proof. When $\alpha = 1$,

$$S = \sum_{k=n}^m 1^k = m - n + 1.$$

If $\alpha \neq 1$, then

$$\alpha S - S = \alpha^{m+1} - \alpha^n.$$

Solving for S gives

$$S = \sum_{k=n}^m \alpha^k = \frac{\alpha^{m+1} - \alpha^n}{\alpha - 1} \text{ if } \alpha \neq 1. \quad (3.11)$$

Taking $\alpha = 10^{-1}$ in Eq. (3.11) implies ■

$$\sum_{k=n}^m 10^{-k} = \frac{10^{-(m+1)} - 10^{-n}}{10^{-1} - 1} = \frac{1}{10^{n-1}} \frac{1 - 10^{-(m-n+1)}}{9}$$

and in particular, for all $M \geq n$,

$$\lim_{m \rightarrow \infty} \sum_{k=n}^m 10^{-k} = \frac{1}{9 \cdot 10^{n-1}} \geq \sum_{k=n}^M 10^{-k}.$$

Definition 3.39 (Decimal Numbers). Let $\mathbb{D}$ denote those sequences $\alpha \in \{0, 1, 2, \dots, 9\}^{\mathbb{Z}}$ with the following properties:

1. there exists $N \in \mathbb{N}$ such that $\alpha_{-n} = 0$ for all $n \geq N$ and
2. $\alpha_n \neq 0$ for some $n \in \mathbb{Z}$.

A **decimal number** is then an expression of the form

$$\alpha_{-N}\alpha_{-N+1} \dots \alpha_0.\alpha_1\alpha_2\alpha_3 \dots$$

For example

$$52 + \sqrt{2} \cong 53.41421356237309504880168872420969807856967187537694807 \dots$$

To every decimal number $\alpha \in \mathbb{D}$ is the sequence $a_n = a_n(\alpha)$ defined for $n \in \mathbb{N}$ by

$$a_n := \sum_{k=-\infty}^n \alpha_k 10^{-k}. \quad (\text{a finite sum}).$$

Since for $m > n$,

$$|a_m - a_n| = \left| \sum_{k=n+1}^m \alpha_k 10^{-k} \right| \leq 9 \sum_{k=n+1}^m 10^{-k} \leq 9 \frac{1}{9 \cdot 10^n} = \frac{1}{10^n},$$

it follows that

$$|a_m - a_n| \leq \frac{1}{10^{\min(m,n)}} \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

which shows $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus to every decimal number we may associate the real number

$$a(\alpha) := \lim_{n \rightarrow \infty} a_n.$$

Theorem 3.40. If $x \geq 0$ is a real number, there exists $\alpha \in \mathbb{D}$ such that $x = a(\alpha)$, i.e. all real numbers can be represented in decimal form.

Proof. If $x = 0$, we can take $\alpha_n = 0$ for all n so that $0 = a(\alpha)$. So suppose that $x > 0$ and let $p := \min(\{n \in \mathbb{N} : x < n\})$. Set $m = p - 1$, then $m \leq x < m + 1$. We then define α_k for $k \leq 0$ so that $m = \alpha_{-N} \dots \alpha_0$. We now construct α_k for $k \geq 1$. For $k = 1$ we write

$$[m, m+1) = \sum_{l=0}^9 [m + \frac{l}{10}, m + \frac{l+1}{10})$$

and then choose $\alpha_1 = l$ if $x \in [m + \frac{l}{10}, m + \frac{l+1}{10})$. We then construct α_2 using,

$$[m + \frac{\alpha_1}{10}, m + \frac{\alpha_1 + 1}{10}) \sum_{l=0}^9 [m + \frac{\alpha_1}{10} + \frac{l}{100}, m + \frac{\alpha_1}{10} + \frac{l+1}{100})$$

and set $\alpha_2 = l$ for $x \in [m + \frac{\alpha_1}{10} + \frac{l}{100}, m + \frac{\alpha_1}{10} + \frac{l+1}{100})$. Continuing this way inductively we construct $\{\alpha_k\}_{k=1}^{\infty}$ such that

$$x \in [m + \sum_{j=1}^k \frac{\alpha_j}{10^j}, m + \sum_{j=1}^{k-1} \frac{\alpha_j}{10^j} + \frac{\alpha_k + 1}{10^k}).$$

It is now easy to see that $x = a(\alpha)$. ■

Remark 3.41. The representation of $x \geq 0$ as a decimal number may not be unique. For example,

$$0.999\underline{9} = \sum_{k=1}^{\infty} \frac{9}{10^k} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 1.000\underline{0}.$$

[Or note that

$$1 - 0.\overbrace{9 \dots 9}^{\text{n-times}} = 0.\overbrace{0 \dots 0}^{\text{n-1 times}} 1 = 10^{-n} \rightarrow 0 \text{ as } n \rightarrow \infty.]$$

On the other hand if we agree to not allow a tail of repeated 9's as an element of $\mathbb{D}$, then the representation would be unique.

3.5 Summary of Key Facts about Real Numbers

1. The real numbers, $\mathbb{R}$, is the unique (up to order preserving field isomorphism) ordered field with the least upper bound property or equivalently which is Cauchy complete.
2. Informally the real numbers are the rational numbers with the (irrational) hole filled in.

3. Monotone bounded sequence always converge in $\mathbb{R}$.
4. A sequence converges in $\mathbb{R}$ iff it is Cauchy.
5. Cauchy sequences are bounded.
6. $\mathbb{N}$ is unbounded from above in $\mathbb{R}$.
7. For all $\varepsilon > 0$ in $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.
8. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. In particular, between any two real numbers $a < b$, there are infinitely many rational and irrational numbers.
9. Decimal numbers map (almost 1-1) into the real numbers by taking the limit of the truncated decimal number.
10. If $a, b, \varepsilon \in \mathbb{R}$, then
 - a) $a \leq b$ by showing that $a \leq b + \varepsilon$ for all $\varepsilon > 0$.
 - b) $a = b$ by proving $a \leq b$ and $b \leq a$ or
 - c) $a = b$ by showing $|b - a| \leq \varepsilon$ for all $\varepsilon > 0$.
11. A number of standard limit theorems hold, see Theorem 3.13.
12. Unlike limits, $\limsup$ and $\liminf$ always exist. Moreover we have; $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$ with equality iff $\lim_{n \rightarrow \infty} a_n$ exists in which case

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n.$$

We may allow the values of $\pm\infty$ in these statements.

13. If $b_k := \{a_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{a_n\}$, then

$$\liminf_{n \rightarrow \infty} a_n \leq \lim_{k \rightarrow \infty} b_k \leq \limsup_{n \rightarrow \infty} a_n$$

and we may choose $\{b_k\}$ so that $\lim_{k \rightarrow \infty} b_k = \limsup_{n \rightarrow \infty} a_n$ or $\lim_{k \rightarrow \infty} b_k = \liminf_{n \rightarrow \infty} a_n$.

14. Bounded sequences of real numbers always have convergence subsequences.
15. If $S \subset \mathbb{R}$ and $A := \sup(S)$, then there exists $\{a_n\}_{n=1}^{\infty} \subset S$ such that $a_n \leq a_{n+1}$ for all n and $\lim_{n \rightarrow \infty} a_n = \sup(S)$.
16. If $S \subset \mathbb{R}$ and $A := \inf(S)$, then there exists $\{a_n\}_{n=1}^{\infty} \subset S$ such that $a_{n+1} \leq a_n$ for all n and $\lim_{n \rightarrow \infty} a_n = \inf(S)$.

3.6 (Optional) Proofs of Theorem 3.6 and Theorem 3.3

In this section, we assume that $\mathbb{R}$ is as describe in Theorem 3.6. The next exercise is relatively straight forward.

Exercise 3.15. Prove the following properties of $\mathbb{R}$.

1. Show addition and multiplication in Theorem 3.6 are well defined.

2. Show $(\mathbb{R}, +, \cdot)$ satisfies the axioms of a field. **Hint:** for constructing multiplicative inverses, make use of Proposition 3.42 below to conclude if $\alpha := [\{a_n\}_{n=1}^{\infty}] \in \mathbb{R}$ and $a \neq 0 = i(0)$, then there exists $N \in \mathbb{N}$ such that $|a_n| \geq \frac{1}{N}$ for a.a. n . By redefining the first few terms of a_n if necessary, you may assume that $|a_n| \geq \frac{1}{N}$ for all n and then take

$$\alpha^{-1} = [\{a_n^{-1}\}_{n=1}^{\infty}].$$

3. Show $i : \mathbb{Q} \rightarrow \mathbb{R}$ is injective homomorphism of fields.

To finish the proof of Theorem 3.6 we must show that P is an ordering on $\mathbb{R}$ with the least upper bound property. This will be carried out in the remainder of this section.

Proposition 3.42. Suppose that $\alpha := [\{a_n\}_{n=1}^{\infty}]$ and $\beta := [\{b_n\}_{n=1}^{\infty}]$ are real numbers. Then precisely one of the following three cases can happen;

1. $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$, i.e. $\alpha = \beta$,
2. there exists $\varepsilon = \frac{1}{N} > 0$ such that $a_n \geq b_n + \varepsilon$ for a.a. n in which case $\alpha > \beta$,
or
3. there exists $\varepsilon = \frac{1}{N} > 0$ such that $b_n \geq a_n + \varepsilon$ for a.a. n in which case $\beta > \alpha$.

Proof. If case 1. does not hold then there exists $\delta > 0$ such that $|a_n - b_n| \geq \delta$ for infinitely many n . There are now two possibilities (which will turn out to me mutually exclusive;

- i) $a_n - b_n \geq \delta$ i.o. n ,
- ii) $b_n - a_n \geq \delta$ i.o. n .

Since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| \geq \delta/3 \text{ and } |b_n - b_m| \geq \delta/3 \text{ for all } m, n \geq N.$$

If case i) holds, we may choose an $m \geq N$ such that $a_m - b_m \geq \delta$ and so for $n \geq N$ we find,

$$\begin{aligned} \delta &\leq a_m - b_m = a_m - a_n + a_n - b_n + b_n - b_m \\ &\leq |a_m - a_n| + a_n - b_n + |b_n - b_m| \\ &= \delta/3 + a_n - b_n + \delta/3 \end{aligned}$$

from which it follows that $a_n - b_n \geq \varepsilon := \delta/3$ for all $n \geq N$ and we are in case 2. Similarly if case ii) holds then we are in fact in case 3. of the proposition. ■

Corollary 3.43. Suppose that $\alpha := [\{a_n\}_{n=1}^{\infty}]$ and $\beta := [\{b_n\}_{n=1}^{\infty}]$ are real numbers, then $\alpha \geq \beta$ iff for all $N \in \mathbb{N}$,

$$a_n - b_n \geq -\frac{1}{N} \text{ for a.a. } n. \quad (3.12)$$

Alternatively put, $\alpha \geq \beta$ iff for all $N \in \mathbb{N}$,

$$b_n \leq a_n + \frac{1}{N} \text{ for a.a. } n.$$

Proof. If $\alpha = \beta$, then $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ and therefore Eq. (3.12) holds. If $\alpha > \beta$, then in fact $a_n - b_n \geq \varepsilon > 0 > -1/N$ for a.a. n .

Conversely, if $\alpha < \beta$, then there exists $\varepsilon > 0$ such that $b_n \geq a_n + \varepsilon$ for a.a. n . Thus if Eq. (3.12) were to also hold we could conclude for each $N \in \mathbb{N}$ that

$$a_n \geq b_n - \frac{1}{N} \geq a_n + \varepsilon - \frac{1}{N} \text{ for a.a. } n.$$

This leads to a contradiction as soon as we choose N so large as to make $1/N < \varepsilon$. Thus if Eq. (3.12) holds we must have $\alpha \geq \beta$. ■

Proposition 3.44. Suppose that $\lambda \in \mathbb{R}$, $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{Q}$, and $\alpha := [\{a_n\}_{n=1}^\infty]$. If $\lambda \leq i(a_k)$ for all k then $\lambda \leq \alpha$. Similarly if $i(a_k) \leq \lambda$ for all k then $\alpha \leq \lambda$.

Proof. Let $\lambda = [\{\lambda_n\}_{n=1}^\infty]$ and suppose that $\lambda \leq i(a_n)$ for all n . For sake of contradiction, suppose that $\lambda > \alpha$, i.e. there exists an $N \in \mathbb{N}$ such that $\lambda_n \geq a_n + \frac{1}{N}$ for a.a. n . The assumption that $\lambda \leq i(a_k)$ implies that $\lambda_n \leq a_k + \frac{1}{3N}$ for a.a. n . Because $\{a_k\}$ is Cauchy, we may conclude there exists $M \in \mathbb{N}$ such that

$$\lambda_n \leq a_k + \frac{1}{2N} \text{ for all } n, k \geq M.$$

By making M even larger if necessary, we may assume that $\lambda_n \geq a_n + \frac{1}{N}$ for all $n \geq M$ as well. From these two inequalities with $k = n \geq M$ we learn

$$a_n + \frac{1}{N} \leq \lambda_n \leq a_n + \frac{1}{2N} \implies \frac{1}{2N} \geq \frac{1}{N}$$

and we have reached the desired contradiction. The fact that $i(a_k) \leq \lambda$ for all k implies $\alpha \leq \lambda$ is proved similarly. Alternatively if $i(a_k) \leq \lambda$ then $-\lambda \leq i(-a_k)$ which implies $-\lambda \leq -\alpha$, i.e. $\alpha \leq \lambda$. ■

With these results in hand, let us now show that $\mathbb{R}$ as defined in Theorem 3.6 has the least upper bound property.

Proof of the least upper bound property. So suppose that $A \subset \mathbb{R}$ is a non empty set which is bounded from above. For each $m \in \mathbb{N}$, let $k_m \in \mathbb{Z}$ be the smallest integer such that $i(a_m) := i(\frac{k_m}{2^m})$ is an upper bound for A . Since, for all $n \geq m$, $a_m - 2^{-m} \leq a_n \leq a_m$, we may conclude that

$$|a_n - a_m| \leq 2^{-\min(n,m)} \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

This shows $\{a_n\}_{n=1}^\infty$ is Cauchy and hence we defined an element $\alpha := [\{a_n\}_{n=1}^\infty] \in \mathbb{R}$. We now will show $\alpha = \sup A$.

If $\lambda \in A$, then $\lambda \leq i(a_n)$ for all n and so by Proposition 3.44 we conclude that $\lambda \leq \alpha$, i.e. α is an upper bound for A . Now suppose that β is another upper bound for A . As $i(a_n - 2^{-n})$ is not an upper bound for A there exists $\lambda \in A$ such that

$$i(a_n - 2^{-n}) < \lambda \leq \beta.$$

So by another application of Proposition 3.44 we learn that

$$\alpha = [\{a_n\}_{n=1}^\infty] = [\{a_n - 2^{-n}\}_{n=1}^\infty] \leq \beta.$$

This shows that α is in fact the least upper bound for A . ■

Theorem 3.45 (Real numbers are unique). Suppose that $\mathbb{F}$ and $\mathbb{G}$ are two complete ordered fields. Then there is a unique order preserving isomorphism, $\varphi : \mathbb{F} \rightarrow \mathbb{G}$.

(Sketch). Suppose that $\varphi : \mathbb{F} \rightarrow \mathbb{G}$ is an order preserving homomorphism. The usual arguments show that any homomorphism, $\varphi : \mathbb{F} \rightarrow \mathbb{G}$ must satisfy $\varphi(q1_{\mathbb{F}}) = q1_{\mathbb{G}}$. We know that $\{q \cdot 1_{\mathbb{F}} : q \in \mathbb{Q}\}$ and $\{q \cdot 1_{\mathbb{G}} : q \in \mathbb{Q}\}$ are dense copies of $\mathbb{Q}$ inside of $\mathbb{F}$ and $\mathbb{G}$ respectively. Now for general $a \in \mathbb{F}$ choose $q_n, p_n \in \mathbb{Q}$ that $q_n 1_{\mathbb{F}} \uparrow a$ and $p_n 1_{\mathbb{F}} \downarrow a$. Since φ is order preserving we must have $q_n 1_{\mathbb{G}} = \varphi(q_n 1_{\mathbb{F}})$ is increasing and $p_n 1_{\mathbb{G}} = \varphi(p_n 1_{\mathbb{F}})$ is decreasing. Moreover, since $p_n - q_n \rightarrow 0$ we must have $\lim_{n \rightarrow \infty} \varphi(q_n 1_{\mathbb{F}}) = \lim_{n \rightarrow \infty} \varphi(p_n 1_{\mathbb{F}})$. Since $\varphi(q_n 1_{\mathbb{F}}) \leq \varphi(a) \leq \varphi(p_n 1_{\mathbb{F}})$ for all n it then follows that $\varphi(a) = \lim_{n \rightarrow \infty} q_n 1_{\mathbb{G}} = \lim_{n \rightarrow \infty} p_n 1_{\mathbb{G}}$ and we have shown φ is uniquely determined.

For the converse, if $q_n \in \mathbb{Q}$ we know that

$$\begin{aligned} |q_n 1_{\mathbb{F}} - q_m 1_{\mathbb{F}}| &= |q_n - q_m| 1_{\mathbb{F}} \text{ and} \\ |q_n 1_{\mathbb{G}} - q_m 1_{\mathbb{G}}| &= |q_n - q_m| 1_{\mathbb{G}}. \end{aligned}$$

Thus if $\{q_n 1_{\mathbb{F}}\}_{n=1}^\infty$ is convergent in $\mathbb{F}$ iff $\{q_n 1_{\mathbb{G}}\}_{n=1}^\infty$ is convergent in $\mathbb{G}$. Thus for any $a \in \mathbb{F}$ we choose $q_n \in \mathbb{Q}$ such that $q_n 1_{\mathbb{F}} \rightarrow a$ and then define $\varphi(a) := \lim_{n \rightarrow \infty} q_n 1_{\mathbb{G}}$. One now checks that this formula is well defined (independent of the choice of $\{q_n\} \subset \mathbb{Q}$ such that $q_n 1_{\mathbb{F}} \rightarrow a$) and defines an order preserving isomorphism. For example, if $a \leq b$ we may choose $\{q_n\} \subset \mathbb{Q}$ and $\{p_n\} \subset \mathbb{Q}$ such that $q_n 1_{\mathbb{F}} \uparrow a$ and $p_n 1_{\mathbb{F}} \downarrow b$. Then $q_n 1_{\mathbb{G}} \leq p_n 1_{\mathbb{G}}$ for all n and letting $n \rightarrow \infty$ shows,

$$\varphi(a) = \lim_{n \rightarrow \infty} q_n 1_{\mathbb{G}} \leq \lim_{n \rightarrow \infty} p_n 1_{\mathbb{G}} = \varphi(b).$$

The other properties of φ are proved similarly. ■

Complex Numbers

Definition 4.1 (Complex Numbers). Let $\mathbb{C} = \mathbb{R}^2$ equipped with multiplication rule

$$(a, b)(c, d) \equiv (ac - bd, bc + ad) \quad (4.1)$$

and the usual rule for vector addition. As is standard we will write $0 = (0, 0)$, $1 = (1, 0)$ and $i = (0, 1)$ so that every element z of $\mathbb{C}$ may be written as $z = x1 + yi$ which in the future will be written simply as $z = x + iy$. If $z = x + iy$, let $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

Writing $z = a + ib$ and $w = c + id$, the multiplication rule in Eq. (4.1) becomes

$$(a + ib)(c + id) \equiv (ac - bd) + i(bc + ad) \quad (4.2)$$

and in particular $1^2 = 1$ and $i^2 = -1$.

Proposition 4.2. The complex numbers $\mathbb{C}$ with the above multiplication rule satisfies the usual definitions of a field – see Definition 2.1. For example $z_1 z_2 = z_2 z_1$, and $z(w_1 + w_2) = zw_1 + zw_2$, etc. Moreover if $z = a + ib \neq 0$, then z has a multiplicative inverse given by

$$z^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}. \quad (4.3)$$

Moreover $\mathbb{C}$ contains $\mathbb{R}$ as sub-field under the identification

$$\mathbb{R} \ni a \rightarrow a\mathbf{1} + 0i = (a, 0) \in \mathbb{C}.$$

Proof. Suppose $z = a + ib \neq 0$, we wish to find $w = c + id$ such that $zw = 1$ and this happens by Eq. (4.2) iff

$$ac - bd = 1 \text{ and} \quad (4.4)$$

$$bc + ad = 0. \quad (4.5)$$

Solving these equations as follows

$$a(4.4) + b(4.5) \implies (a^2 + b^2)c = a \implies \operatorname{Re} w = c = \frac{a}{a^2 + b^2}$$

$$-b(4.4) + a(4.5) \implies (a^2 + b^2)d = -b \implies \operatorname{Im} w = d = -\frac{b}{a^2 + b^2},$$

gives implies the result in Eq. (4.3).

Probably the most painful thing to check directly is the associative law, namely that $[z_1 z_2] z_3 = z_1 [z_2 z_3]$ for all $z_1, z_2, z_3 \in \mathbb{C}$. This is equivalent to showing for all $a, b, u, v, x, y \in \mathbb{R}$ that

$$[(a + ib)(u + iv)](x + iy) = (a + ib)[(u + iv)(x + iy)].$$

We do this by working out both sides as follows;

$$\begin{aligned} LHS &= [(au - bv) + i(av + bu)](x + iy) \\ &= (au - bv)x - (av + bu)y + i[(av + bu)x + (au - bv)y]; \\ RHS &= (a + ib)[(ux - vy) + i(uy + vx)] \\ &= a(ux - vy) - b(uy + vx) + i[b(ux - vy) + a(uy + vx)]. \end{aligned}$$

The reader should now easily see that both of these expressions are in fact equal. The remaining axioms of a field are checked similarly. ■

- End of Lecture 9, 10/17/2012.
- Test 1 took place of lecture 10, 10/22/2012.

Notation 4.3 We will write $1/z$ for z^{-1} and w/z to mean $z^{-1} \cdot w$.

Notation 4.4 (Conjugation and Modulus) If $z = a + ib$ with $a, b \in \mathbb{R}$ let $\bar{z} = a - ib$ and

$$|z|^2 \equiv z\bar{z} = a^2 + b^2.$$

Notice that

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}) \text{ and } \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}). \quad (4.6)$$

Proposition 4.5. Complex conjugation and the modulus operators satisfy;

1. $\bar{\bar{z}} = z$,
2. $\overline{z\bar{w}} = \bar{z}w$ and $\bar{z} + \bar{w} = \overline{z + w}$.
3. $|\bar{z}| = |z|$
4. $|zw| = |z||w|$ and in particular $|z^n| = |z|^n$ for all $n \in \mathbb{N}$.
5. $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$

6. $|z + w| \leq |z| + |w|$.
7. $z = 0$ iff $|z| = 0$.
8. If $z \neq 0$ then

$$z^{-1} := \frac{\bar{z}}{|z|^2}$$

(also written as $\frac{1}{z}$) is the inverse of z .

9. $|z^{-1}| = |z|^{-1}$ and more generally $|z^n| = |z|^n$ for all $n \in \mathbb{Z}$.

Proof. 1. and 3. are geometrically obvious as well as easily verified.

2. Say $z = a + ib$ and $w = c + id$, then $\bar{z}\bar{w}$ is the same as zw with b replaced by $-b$ and d replaced by $-d$, and looking at Eq. (4.2) we see that

$$\bar{z}\bar{w} = (ac - bd) - i(bc + ad) = \overline{zw}.$$

4. $|zw|^2 = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2$ as real numbers and hence $|zw| = |z||w|$.

5. Geometrically obvious or also follows from

$$|z| = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}.$$

6. This is the triangle inequality which may be understood geometrically or by the computation

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) = |z|^2 + |w|^2 + w\bar{z} + \bar{w}z \\ &= |z|^2 + |w|^2 + w\bar{z} + \overline{w\bar{z}} \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \leq |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2. \end{aligned}$$

7. Obvious.

8. Follows from Eq. (4.3). Alternatively if $\rho = \rho + i0 > 0$ is a real number then $\rho^{-1} = \rho^{-1} + i0$ as is easily verified since $\mathbb{R}$ is a sub-field of $\mathbb{C}$. Thus since $\bar{z}z = |z|^2$ we find

$$\frac{1}{|z|^2}\bar{z}z = \frac{1}{|z|^2}|z|^2 = 1 \implies z^{-1} = \frac{1}{|z|^2}\bar{z} = \frac{\operatorname{Re} z}{|z|^2} - i\frac{\operatorname{Im} z}{|z|^2}.$$

$$9. |z^{-1}| = \left| \frac{\bar{z}}{|z|^2} \right| = \frac{|\bar{z}|}{|z|^2} = \frac{1}{|z|}. \quad \blacksquare$$

Corollary 4.6. If $w, z \in \mathbb{C}$, then

$$||z| - |w|| \leq |z - w|.$$

Proof. Just copy the proof of Lemma 1.6. \(\blacksquare\)

Lemma 4.7. For complex numbers $u, v, w, z \in \mathbb{C}$ with $v \neq 0 \neq z$, we have

$$\frac{1}{u} \frac{1}{v} = \frac{1}{uv}, \text{ i.e. } u^{-1}v^{-1} = (uv)^{-1}$$

$$\frac{u}{v} \frac{w}{z} = \frac{uw}{vz} \text{ and}$$

$$\frac{u}{v} + \frac{w}{z} = \frac{uz + vw}{vz}.$$

[These statements hold in any field.]

Proof. For the first item, it suffices to check that

$$(uv)(u^{-1}v^{-1}) = u^{-1}uvv^{-1} = 1 \cdot 1 = 1.$$

The rest follow using

$$\frac{u}{v} \frac{w}{z} = uv^{-1}wz^{-1} = uuv^{-1}z^{-1} = uw(vz)^{-1} = \frac{uw}{vz}.$$

$$\begin{aligned} \frac{u}{v} + \frac{w}{z} &= \frac{zu}{zv} + \frac{vw}{vz} = \frac{zu}{zv} + \frac{vw}{vz} \\ &= (vz)^{-1}(zu + vw) = \frac{uz + vw}{vz}. \end{aligned} \quad \blacksquare$$

Definition 4.8. A sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is **Cauchy** if $|z_n - z_m| \rightarrow 0$ as $m, n \rightarrow \infty$ and is **convergent** to $z \in \mathbb{C}$ if $|z - z_n| \rightarrow 0$ as $n \rightarrow \infty$. As usual if $\{z_n\}_{n=1}^{\infty}$ converges to z we will write $z_n \rightarrow z$ as $n \rightarrow \infty$ or $z = \lim_{n \rightarrow \infty} z_n$.

Theorem 4.9. The complex numbers are complete, i.e. all Cauchy sequences are convergent.

Proof. This follows from the completeness of real numbers and the easily proved observations that if $z_n = a_n + ib_n \in \mathbb{C}$, then

1. $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is Cauchy iff $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ are Cauchy and
2. $z_n \rightarrow z = a + ib$ as $n \rightarrow \infty$ iff $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. \(\blacksquare\)

The complex numbers satisfy all the same limit theorems as the real numbers.

Theorem 4.10. If $\{w_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are convergent sequences of complex numbers, then

1. $\{w_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are Cauchy sequences. \(\blacksquare\)

2. $\lim_{n \rightarrow \infty} (w_n + z_n) = \lim_{n \rightarrow \infty} w_n + \lim_{n \rightarrow \infty} z_n$.
3. $\lim_{n \rightarrow \infty} (w_n \cdot z_n) = \lim_{n \rightarrow \infty} w_n \cdot \lim_{n \rightarrow \infty} z_n$.
4. if we further assume that $\lim_{n \rightarrow \infty} z_n \neq 0$, then

$$\lim_{n \rightarrow \infty} \left(\frac{w_n}{z_n} \right) = \frac{\lim_{n \rightarrow \infty} w_n}{\lim_{n \rightarrow \infty} z_n}.$$

- End of Lecture 11, 10/24/2012.

Lemma 4.11 (Bolzano–Weierstrass property). *Every bounded sequence, $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$, has a convergent subsequence.*

Proof. By assumption there exists $M < \infty$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. Writing $z_n = a_n + ib_n$ with $a_n, b_n \in \mathbb{R}$ we may conclude that $|a_n|, |b_n| \leq M$. According to Corollary 3.31 there exists an increasing function $\mathbb{N} \ni k \rightarrow n_k \in \mathbb{N}$ such that $\lim_{k \rightarrow \infty} a_{n_k} =: A$ exists. Similarly, we can apply Corollary 3.31 again to find an increasing function $\mathbb{N} \ni l \rightarrow k_l \in \mathbb{N}$ such that $\lim_{l \rightarrow \infty} b_{n_{k_l}} =: B$ exists. We now let $w_l := z_{n_{k_l}}$ for $l \in \mathbb{N}$. Then $\{w_l\}_{l=1}^{\infty}$ is a subsequence of $\{z_n\}_{n=1}^{\infty}$ which is convergent to $A + iB \in \mathbb{C}$. Indeed,

$$\begin{aligned} |w_l - (A + iB)| &= |a_{n_{k_l}} - A + i(b_{n_{k_l}} - B)| \\ &\leq |a_{n_{k_l}} - A| + |b_{n_{k_l}} - B| \rightarrow 0 \text{ as } l \rightarrow \infty. \end{aligned}$$

■

Notation 4.12 (Euclidean Spaces) Let $\mathbb{C}^n := \{a := (a_1, \dots, a_n) : a_i \in \mathbb{C}\}$ and $\mathbb{R}^n := \{a := (a_1, \dots, a_n) : a_i \in \mathbb{R}\} \subset \mathbb{C}^n$. For $a, b \in \mathbb{C}^n$ we let $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$,

$$\begin{aligned} a \cdot b &:= a_1 b_1 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i, \text{ and} \\ \|a\| &= \|a\|_2 = \sqrt{a \cdot \bar{a}} = \sqrt{\sum_{i=1}^n |a_i|^2}. \end{aligned} \quad (4.7)$$

Exercise 4.1. If $a, b \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, then $a \cdot b = b \cdot a$,

$$(\lambda a) \cdot b = a \cdot (\lambda b) = \lambda (a \cdot b) \text{ and } \|\lambda a\| = |\lambda| \|a\| = |\lambda| \|\bar{a}\|.$$

If we further assume that $c \in \mathbb{C}^n$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

4.1 A Matrix Perspective (Optional)

Here is a way to understand some of the basic properties of $\mathbb{C}$ using your knowledge of linear algebra. Let $M_z : \mathbb{C} \rightarrow \mathbb{C}$ denote multiplication by $z = a + ib$. We now identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$\mathbb{C} \ni c + id \cong \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2.$$

Using this identification, the product formula

$$zw = (ac - bd) + i(bc + ad),$$

becomes

$$M_z w = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

so that

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI + bJ$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We now have the following simple observations;

1. $J^2 = -I$ and $J^{\text{tr}} = -J$,
2. $M_z M_w = M_w M_z$ because J and I commute,
3. we have

$$M_z M_w = (aI + bJ)(cI + dJ) = (ac - bd)I + (ad + bc)J = M_{zw},$$

4. the associativity of complex multiplication follows from the associativity properties of matrix multiplication,
5. $M_z^{\text{tr}} = aI - bJ = M_{\bar{z}}$ and in particular
6. $M_{\bar{z}\bar{w}} = (M_z M_w)^{\text{tr}} = M_w^{\text{tr}} M_z^{\text{tr}} = M_{\bar{w}} M_{\bar{z}} = M_{\bar{w}\bar{z}}$,
7. $M_z^{\text{tr}} M_z = M_{\bar{z}z} = M_{|z|^2} = \det(M_z)$,
8. $|wz| = \det(M_{wz}) = \det(M_w M_z) = \det(M_w) \det(M_z) = |w| |z|$,
9. M_z is invertible iff $\det(M_z) \neq 0$ which happens iff $|z|^2 \neq 0$ and in this case we know from basic linear algebra that

$$M_z^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{|z|^2} M_z^{\text{tr}} = M_{\frac{1}{|z|^2} \bar{z}},$$

10. With this notation we have $M_z M_w = M_{zw}$ and since I and J commute it follows that $zw = wz$. Moreover, since matrix multiplication is associative so is complex multiplication. Also notice that M_z is invertible iff $\det M_z = a^2 + b^2 = |z|^2 \neq 0$ in which case

$$M_z^{-1} = \frac{1}{|z|^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}/|z|^2}$$

as we have already seen above.

Set Operations, Functions, and Countability

Let $\mathbb{N}$ denote the positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the non-negative integers and $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$ – the positive and negative integers including 0, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. We will also use $\mathbb{F}$ to stand for either of the fields $\mathbb{R}$ or $\mathbb{C}$.

5.1 Set Operations and Functions

Notation 5.1 Given two sets X and Y , let Y^X denote the collection of all functions $f : X \rightarrow Y$. If $X = \mathbb{N}$, we will say that $f \in Y^{\mathbb{N}}$ is a sequence with values in Y and often write f_n for $f(n)$ and express f as $\{f_n\}_{n=1}^{\infty}$. If $X = \{1, 2, \dots, N\}$, we will write Y^N in place of $Y^{\{1, 2, \dots, N\}}$ and denote $f \in Y^N$ by $f = (f_1, f_2, \dots, f_N)$ where $f_n \in Y$.

Notation 5.2 More generally if $\{X_\alpha : \alpha \in A\}$ is a collection of non-empty sets, let $X_A = \prod_{\alpha \in A} X_\alpha$ and $\pi_\alpha : X_A \rightarrow X_\alpha$ be the canonical projection map defined by $\pi_\alpha(x) = x_\alpha$. If $X_\alpha = X$ for some fixed space X , then we will write $\prod_{\alpha \in A} X_\alpha$ as X^A rather than X_A .

Recall that an element $x \in X_A$ is a “**choice function**,” i.e. an assignment $x_\alpha := x(\alpha) \in X_\alpha$ for each $\alpha \in A$. The **axiom of choice** states that $X_A \neq \emptyset$ provided that $X_\alpha \neq \emptyset$ for each $\alpha \in A$.

Notation 5.3 Given a set X , let 2^X denote the **power set** of X – the collection of all subsets of X including the empty set.

The reason for writing the power set of X as 2^X is that if we think of 2 meaning $\{0, 1\}$, then an element of $a \in 2^X = \{0, 1\}^X$ is completely determined by the set

$$A := \{x \in X : a(x) = 1\} \subset X.$$

In this way elements in $\{0, 1\}^X$ are in one to one correspondence with subsets of X .

For $A \in 2^X$ let

$$A^c := X \setminus A = \{x \in X : x \notin A\}$$

and more generally if $A, B \subset X$ let

$$B \setminus A := \{x \in B : x \notin A\} = A \cap B^c.$$

We also define the symmetric difference of A and B by

$$A \Delta B := (B \setminus A) \cup (A \setminus B).$$

As usual if $\{A_\alpha\}_{\alpha \in I}$ is an indexed collection of subsets of X we define the union and the intersection of this collection by

$$\begin{aligned} \cup_{\alpha \in I} A_\alpha &:= \{x \in X : \exists \alpha \in I \ni x \in A_\alpha\} \text{ and} \\ \cap_{\alpha \in I} A_\alpha &:= \{x \in X : x \in A_\alpha \forall \alpha \in I\}. \end{aligned}$$

Example 5.4. Let A, B , and C be subsets of X . Then

$$A \cap (B \cup C) = [A \cap B] \cup [A \cap C].$$

Indeed, $x \in A \cap (B \cup C) \iff x \in A \text{ and } x \in B \cup C \iff x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C \iff x \in A \cap B \text{ or } x \in A \cap C \iff x \in [A \cap B] \cup [A \cap C]$.

Notation 5.5 We will also write $\prod_{\alpha \in I} A_\alpha$ for $\cup_{\alpha \in I} A_\alpha$ in the case that $\{A_\alpha\}_{\alpha \in I}$ are pairwise disjoint, i.e. $A_\alpha \cap A_\beta = \emptyset$ if $\alpha \neq \beta$.

Notice that $\cup$ is closely related to $\exists$ and $\cap$ is closely related to $\forall$. For example let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets from X and define

$$\begin{aligned} \{A_n \text{ i.o.}\} &:= \{x \in X : \#\{n : x \in A_n\} = \infty\} \text{ and} \\ \{A_n \text{ a.a.}\} &:= \{x \in X : x \in A_n \text{ for all } n \text{ sufficiently large}\}. \end{aligned}$$

(One should read $\{A_n \text{ i.o.}\}$ as A_n infinitely often and $\{A_n \text{ a.a.}\}$ as A_n almost always.) Then $x \in \{A_n \text{ i.o.}\}$ iff

$$\forall N \in \mathbb{N} \exists n \geq N \ni x \in A_n$$

and this may be expressed as

$$\{A_n \text{ i.o.}\} = \cap_{N=1}^{\infty} \cup_{n \geq N} A_n.$$

Similarly, $x \in \{A_n \text{ a.a.}\}$ iff

$$\exists N \in \mathbb{N} \ni \forall n \geq N, \quad x \in A_n$$

which may be written as

$$\{A_n \text{ a.a.}\} = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} A_n.$$

We end this section with some notation which will be used frequently in the sequel.

Definition 5.6. If $f : X \rightarrow Y$ is a function and $B \subset Y$, then

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

If $A \subset X$ we also write,

$$f(A) := \{f(x) : x \in A\} \subset Y.$$

Example 5.7. If $f : X \rightarrow Y$ is a function and $B \subset Y$, then $f^{-1}(B^c) = [f^{-1}(B)]^c$ or to be more precise,

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

To prove this notice that

$$x \in f^{-1}(B^c) \iff f(x) \in B^c \iff f(x) \notin B \iff x \notin f^{-1}(B) \iff x \in [f^{-1}(B)]^c.$$

On the other hand, if $A \subset X$ it is **not** necessarily true that $f(A^c) = [f(A)]^c$. For example, suppose that $f : \{1, 2\} \rightarrow \{1, 2\}$ is defined by $f(1) = f(2) = 1$ and $A = \{1\}$. Then $f(A) = f(A^c) = \{1\}$ where $[f(A)]^c = \{1\}^c = \{2\}$.

Notation 5.8 If $f : X \rightarrow Y$ is a function and $\mathcal{E} \subset 2^Y$ let

$$f^{-1}\mathcal{E} := f^{-1}(\mathcal{E}) := \{f^{-1}(E) \mid E \in \mathcal{E}\}.$$

If $\mathcal{G} \subset 2^X$, let

$$f_*\mathcal{G} := \{A \in 2^Y \mid f^{-1}(A) \in \mathcal{G}\}.$$

Definition 5.9. Let $\mathcal{E} \subset 2^X$ be a collection of sets, $A \subset X$, $i_A : A \rightarrow X$ be the **inclusion map** ($i_A(x) = x$ for all $x \in A$) and

$$\mathcal{E}_A = i_A^{-1}(\mathcal{E}) = \{A \cap E : E \in \mathcal{E}\}.$$

5.1.1 Exercises

Let $f : X \rightarrow Y$ be a function and $\{A_i\}_{i \in I}$ be an indexed family of subsets of Y , verify the following assertions.

Exercise 5.1. $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$.

Exercise 5.2. Suppose that $B \subset Y$, show that $B \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (B \setminus A_i)$.

Exercise 5.3. $f^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f^{-1}(A_i)$.

Exercise 5.4. $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$.

Exercise 5.5. Find a counterexample which shows that $f(C \cap D) = f(C) \cap f(D)$ need not hold.

5.2 Counting

In this section, X and Y will denote sets and for $n \in \mathbb{N}$ we let

$$J_n := \{1, 2, \dots, n\} := \{k \in \mathbb{N} : k \leq n\}.$$

Definition 5.10 (Cardinality). We say $\text{card}(X) \leq \text{card}(Y)$ if there exists an injective map, $f : X \rightarrow Y$ and $\text{card}(Y) \geq \text{card}(X)$ if there exists a surjective map $g : Y \rightarrow X$. We say $\text{card}(X) = \text{card}(Y)$ (also denoted as $X \sim Y$) if there exists a bijections, $f : X \rightarrow Y$.

Proposition 5.11. If X and Y are sets, then $\text{card}(X) \leq \text{card}(Y)$ iff $\text{card}(Y) \geq \text{card}(X)$.

Proof. If $f : X \rightarrow Y$ is an injective map, define $g : Y \rightarrow X$ by $g|_{f(X)} = f^{-1}$ and $g|_{Y \setminus f(X)} = x_0 \in X$ chosen arbitrarily. Then $g : Y \rightarrow X$ is surjective.

If $g : Y \rightarrow X$ is a surjective map, then $Y_x := g^{-1}(\{x\}) \neq \emptyset$ for all $x \in X$ and so by the axiom of choice there exists $f \in \prod_{x \in X} Y_x$. Thus $f : X \rightarrow Y$ such that $f(x) \in Y_x$ for all x . As the $\{Y_x\}_{x \in X}$ are pairwise disjoint, it follows that f is injective. ■

Theorem 5.12 (Schröder-Bernstein Theorem). If $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$, then $\text{card}(X) = \text{card}(Y)$. Stated more explicitly; if there exists injective maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$, then there exists a bijective map, $h : X \rightarrow Y$.

Proof. See the Appendix, Theorem B.10. ■

The next theorem summarizes the properties of cardinalities that we will need for this course.

Theorem 5.13 (Cardinality/Counting Summary). *Given a non-empty set X , then one and only one of the following statements holds;*

1. *There exists a unique $n \in \mathbb{N}$ such that $\text{card } X = \text{card } J_n$.*
2. *$\text{card } X = \text{card } \mathbb{N}$,*
3. *$\text{card } X > \text{card } \mathbb{N}$.*
4. *Cases 2. or 3. hold iff $\text{card } J_n \leq \text{card } X$ for all $n \in \mathbb{N}$ which happens iff $\text{card } \mathbb{N} \leq \text{card } X$.*
*If X satisfies case 1. we say X is a **finite set**. If X satisfies case 2 we say X is a **countably infinite set** and if X satisfies case 3. we say X is an **uncountably infinite set**.*
5. *If $S \subset \mathbb{N}$ is an unbounded set, then $\text{card } (S) = \text{card } (\mathbb{N})$.*
6. *If $S \subset \mathbb{N}$ is a bounded set then $\text{card } (S) = \text{card } (J_n)$ for some $n \in \mathbb{N}$.*
7. *If $\{X_k\}_{k=1}^{\infty}$ are sets such that $\text{card } (X_k) \leq \text{card } (\mathbb{N})$, then $\text{card } (\cup_{k=1}^{\infty} X_k) \leq \text{card } (\mathbb{N})$.*
8. *If X and Y are sets such that $\text{card } (X) \leq \text{card } (\mathbb{N})$ and $\text{card } (Y) \leq \text{card } (\mathbb{N})$, then $\text{card } (X \times Y) \leq \text{card } (\mathbb{N})$.*
9. *$\text{card } (\mathbb{Q}) = \text{card } (\mathbb{N})$.*
10. *For any $a < b$ in $\mathbb{R}$, $\text{card } (\mathbb{Q} \cap (a, b)) = \text{card } (\mathbb{N})$ while $\text{card } (\mathbb{Q}^c \cap (a, b)) > \text{card } (\mathbb{N})$.*

Proof. These results will be proved in the remainder of this section. ■

Lemma 5.14. *Suppose $n \in \mathbb{N}$ and $k \in J_{n+1}$, then $\text{card } (J_{n+1} \setminus \{k\}) = \text{card } (J_n)$.*

Proof. If $n = 1$, then $J_2 = \{1, 2\}$ and either $J_2 \setminus \{k\} = J_1$ or $J_2 \setminus \{k\} = \{2\}$, either way $\text{card } (J_2 \setminus \{k\}) = \text{card } (J_1)$. Now suppose that result holds for a given $n \in \mathbb{N}$ and $k \in J_{n+2}$. If $k = \{n+2\}$ we have $J_{n+2} \setminus \{k\} = J_{n+1}$ so $\text{card } (J_{n+2} \setminus \{k\}) = \text{card } (J_{n+1})$ while if $k \in J_{n+1} \subset J_{n+2}$, then $J_{n+2} \setminus \{k\} = (J_{n+1} \setminus \{k\}) \cup \{n+2\} \cong J_n \cup \{n+2\} \cong J_n \cup \{n+1\} = J_{n+1}$.

Alternatively. Let $f : J_{n+1} \rightarrow J_n \setminus \{k\}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{if } x \geq k \end{cases}$$

Then f is the desired bijection. ■

Proposition 5.15. *If $\text{card } (J_n) = \text{card } (J_m)$, then $m = n$. More generally, $\text{card } J_m \leq \text{card } J_n$ iff $m \leq n$ with equality iff $m = n$.*

Proof. We prove this by induction on m . When $m = 1$, $\text{card } (J_n) = \text{card } (J_1)$, then there exists $\varphi : J_1 \rightarrow J_n$ which is a bijection. We must have $\varphi(1) = 1$ as $1 \in \text{Ran } (\varphi)$. If $n > 1$, then $2 \in J_n$ but $2 \notin \text{Ran } (\varphi)$ and hence φ is not bijective. Therefore $n = 1$. Now suppose the statement holds for $m \in \mathbb{N}$. If

$\text{card } (J_{m+1}) = \text{card } (J_n)$, there exists $\varphi : J_{m+1} \rightarrow J_n$ which is a bijection. Let $k = \varphi(m+1) \in J_n$, then $J_n \setminus \{k\} \cong J_{n-1}$ and $\varphi|_{J_m} : J_m \rightarrow J_n \setminus \{k\} \cong J_{n-1}$ is a bijection and therefore by induction $m = n-1$, i.e. $n = m+1$. ■

Proposition 5.16. *We have $\text{card } (J_m) < \text{card } (J_n)$ for all $m < n$. Moreover if $\emptyset \neq X \subsetneq J_n$ then $\text{card } (X) = \text{card } (J_k)$ for some $k < n$.*

Proof. Suppose that $\emptyset \neq X \subsetneq J_n$. Let $X_1 := X$, and $f(1) := \min X$. Then define f and $X_k \subset X$ inductively so that $X_k = X_{k-1} \setminus \{f(k-1)\}$ and $f(k) = \min X_k$. This process can not continue indefinitely for otherwise we would produce an injective map, $f : \mathbb{N} \rightarrow X$ which would imply $\text{card } J_n \geq \text{card } X \geq \text{card } \mathbb{N}$ whereas $\text{card } J_n < \text{card } \mathbb{N}$. So we must conclude there exists $k \in \mathbb{N}$ such that $X_{k+1} = \emptyset$ in which case $f : J_k \rightarrow X$ is a bijection.

By assumption there exists $m \in J_n \setminus X$ so that $X \subset J_n \setminus \{m\} \cong J_{n-1}$ and hence $\text{card } (J_k) = \text{card } (X) \leq \text{card } (J_{n-1})$ from which it follows that $k \leq n-1$, i.e. $k < n$. ■

Example 5.17. More generally if $X \subset Y$ and $\text{card } (X) = m < n = \text{card } (Y)$, then $\text{card } (Y \setminus X) = n - m$ and if X and Y are finite disjoint sets then $\text{card } (X \cup Y) = \text{card } (X) + \text{card } (Y)$. Similarly, $\text{card } (X \times Y) = \text{card } (X) \cdot \text{card } (Y)$.

Proposition 5.18. *If $f : J_n \rightarrow J_n$ is a map, then the following are equivalent,*

1. *f is injective,*
2. *f is surjective,*
3. *f is bijective.*

Proof. If $n = 1$, the only map $f : J_1 \rightarrow J_1$ is $f(1) = 1$. So in this case there is nothing to prove. So now suppose the proposition holds for level n and $f : J_{n+1} \rightarrow J_{n+1}$ is a given map.

If $f : J_{n+1} \rightarrow J_{n+1}$ is an injective map and $f(J_{n+1})$ is a proper subset of J_{n+1} , then $\text{card } (J_{n+1}) < \text{card } (f(J_{n+1})) = \text{card } (J_{n+1})$ which is absurd. Thus f is injective implies f is surjective.

Conversely suppose that $f : J_{n+1} \rightarrow J_{n+1}$ is surjective. Let $g : J_{n+1} \rightarrow J_{n+1}$ be a right inverse, i.e. $f \circ g = \text{id}$, which is necessarily injective, see the proof of Proposition 5.11. By the previous paragraph we know that g is necessarily surjective and therefore $f = g^{-1}$ is a bijection. ■

Definition 5.19. *A set X is said to be **countable** if it is empty or there is an injective function $f : X \rightarrow \mathbb{N}$, otherwise X is said to be **uncountable**.*

Lemma 5.20 (Basic Properties of Countable Sets).

1. *If $A \subset X$ is a subset of a countable set X then A is countable.*

2. Any infinite subset $A \subset \mathbb{N}$ is in one to one correspondence with $\mathbb{N}$.
3. A non-empty set X is countable iff there exists a surjective map, $g : \mathbb{N} \rightarrow X$.
4. If X and Y are countable then $X \times Y$ is countable.
5. Suppose for each $m \in \mathbb{N}$ that A_m is a countable subset of a set X , then $A = \bigcup_{m=1}^{\infty} A_m$ is countable. In short, the countable union of countable sets is still countable.
6. If X is an infinite set and Y is a set with at least two elements, then Y^X is uncountable. In particular 2^X is uncountable for any infinite set X .

Proof. We take each item in turn.

1. If $f : X \rightarrow N$ is an injective map then so is the restriction, $f|_A$, of f to the subset A .
2. Let $f(1) = \min A$ and define f inductively by

$$f(n+1) = \min (A \setminus \{f(1), \dots, f(n)\}).$$

Since A is infinite the process continues indefinitely. The function $f : \mathbb{N} \rightarrow A$ defined this way is a bijection. (This is explained in more detail on Lemma 5.23 below.)

3. If $g : \mathbb{N} \rightarrow X$ is a surjective map, let

$$f(x) = \min g^{-1}(\{x\}) = \min \{n \in \mathbb{N} : f(n) = x\}.$$

Then $f : X \rightarrow \mathbb{N}$ is injective which combined with item 2. (taking $A = f(X)$) shows X is countable. Conversely if $f : X \rightarrow \mathbb{N}$ is injective let $x_0 \in X$ be a fixed point and define $g : \mathbb{N} \rightarrow X$ by $g(n) = f^{-1}(n)$ for $n \in f(X)$ and $g(n) = x_0$ otherwise.

4. Let us first construct a bijection, h , from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. To do this put the elements of $\mathbb{N} \times \mathbb{N}$ into an array of the form

$$\begin{pmatrix} (1,1) & (1,2) & (1,3) & \dots \\ (2,1) & (2,2) & (2,3) & \dots \\ (3,1) & (3,2) & (3,3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and then “count” these elements by counting the sets $\{(i,j) : i+j = k\}$ one at a time. For example let $h(1) = (1,1)$, $h(2) = (2,1)$, $h(3) = (1,2)$, $h(4) = (3,1)$, $h(5) = (2,2)$, $h(6) = (1,3)$ and so on. If $f : \mathbb{N} \rightarrow X$ and $g : \mathbb{N} \rightarrow Y$ are surjective functions, then the function $(f \times g) \circ h : \mathbb{N} \rightarrow X \times Y$ is surjective where $(f \times g)(m,n) := (f(m), g(n))$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$.

5. If $A = \emptyset$ then A is countable by definition so we may assume $A \neq \emptyset$. With out loss of generality we may assume $A_1 \neq \emptyset$ and by replacing A_m by A_1 if necessary we may also assume $A_m \neq \emptyset$ for all m . For each $m \in \mathbb{N}$ let

$a_m : \mathbb{N} \rightarrow A_m$ be a surjective function and then define $f : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{m=1}^{\infty} A_m$ by $f(m,n) := a_m(n)$. The function f is surjective and hence so is the composition, $f \circ h : \mathbb{N} \rightarrow \bigcup_{m=1}^{\infty} A_m$, where $h : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is the bijection defined above.

6. Let us begin by showing $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ is uncountable. For sake of contradiction suppose $f : \mathbb{N} \rightarrow \{0,1\}^{\mathbb{N}}$ is a surjection and write $f(n)$ as $(f_1(n), f_2(n), f_3(n), \dots)$. Now define $a \in \{0,1\}^{\mathbb{N}}$ by $a_n := 1 - f_n(n)$. By construction $f_n(n) \neq a_n$ for all n and so $a \notin f(\mathbb{N})$. This contradicts the assumption that f is surjective and shows $2^{\mathbb{N}}$ is uncountable. For the general case, since $Y_0^X \subset Y^X$ for any subset $Y_0 \subset Y$, if Y_0^X is uncountable then so is Y^X . In this way we may assume Y_0 is a two point set which may as well be $Y_0 = \{0,1\}$. Moreover, since X is an infinite set we may find an injective map $x : \mathbb{N} \rightarrow X$ and use this to set up an injection, $i : 2^{\mathbb{N}} \rightarrow 2^X$ by setting $i(A) := \{x_n : n \in \mathbb{N}\} \subset X$ for all $A \subset \mathbb{N}$. If 2^X were countable we could find a surjective map $f : 2^X \rightarrow \mathbb{N}$ in which case $f \circ i : 2^{\mathbb{N}} \rightarrow \mathbb{N}$ would be surjective as well. However this is impossible since we have already seen that $2^{\mathbb{N}}$ is uncountable. ■

- End of Lecture 14, 10/31/2012.

Corollary 5.21. The set $(0,1) := \{a \in \mathbb{R} : 0 < a < 1\}$ is uncountable while $\mathbb{Q} \cap (0,1)$ is countable.

Proof. From Section 3.4, the set $\{0,1,2,\dots,8\}^{\mathbb{N}}$ can be mapped injectively into $(0,1)$ and therefore it follows from Lemma 5.20 that $(0,1)$ is uncountable. For each $m \in \mathbb{N}$, let $A_m := \{\frac{n}{m} : n \in \mathbb{N} \text{ with } n < m\}$. Since $\mathbb{Q} \cap (0,1) = \bigcup_{m=1}^{\infty} A_m$ and $\#(A_m) < \infty$ for all m , another application of Lemma 5.20 shows $\mathbb{Q} \cap (0,1)$ is countable. ■

Theorem 5.22. Suppose that X is a set. Then $\text{card}(J_n) \leq \text{card}(X)$ for all $n \in \mathbb{N}$ iff $\text{card}(\mathbb{N}) \leq \text{card}(X)$.

Proof. Since $\text{card}(J_n) \leq \text{card}(\mathbb{N})$ for all $n \in \mathbb{N}$ it suffices to prove $\text{card}(J_n) \leq \text{card}(X)$ for all $n \in \mathbb{N}$ implies $\text{card}(\mathbb{N}) \leq \text{card}(X)$. The intuitive idea is as follows.

Suppose we have constructed $f_n : J_n \rightarrow X$ which is injective. If f_n were bijective we would have $\text{card}(J_n) = \text{card}(X)$ and in particular $\text{card}(J_m) > \text{card}(J_n) = \text{card}(X)$ for all $m > n$. Thus there exists $x \in X \setminus f_n(J_n)$ and we then define $f_{n+1} : J_{n+1} \rightarrow X$ so that $f_{n+1}(n+1) = x$ and $f_{n+1}|_{J_n} = f_n$. This process continues indefinitely and so we may construct injective maps $f_n : J_n \rightarrow X$ such that $f_m = f_n|_{J_m}$ for all $m \leq n$. We then define $f(m) := f_n(m)$ where $n \in \mathbb{N}$ is any integer such that $n \geq m$. In this way we construct a function,

$f : \mathbb{N} \rightarrow X$ such that $f|_{J_n} = f_n$ for all n . This function is easily seen to be injective.

Formalities Version 1. Consider the collection of injective maps $f : \mathcal{D}(f) \subset \mathbb{N} \rightarrow X$, where $\mathcal{D}(f)$ is either J_n for some $n \in \mathbb{N}$ or is $\mathbb{N}$. We say $f \leq g$ if $\mathcal{D}(f) \subset \mathcal{D}(g)$ and $f = g|_{\mathcal{D}(f)}$. It is easy to see that every linearly ordered collection of such maps has an upper bound and so by Zorn's lemma (see Theorem B.6), there exists a maximal element, f . If $\mathcal{D}(f) \neq \mathbb{N}$ then $\mathcal{D}(f) = J_n$ for some n . By the last paragraph we could extend f to injective map on J_{n+1} violating the maximality of f . Thus $\mathcal{D}(f) = \mathbb{N}$ and we have found an injective map from $\mathbb{N}$ to X .

Formalities Version 2. (This argument will avoid the use of Zorn's Lemma.) By assumption, for each $n \in \mathbb{N}$ there exists an injective map, $f_n : J_n \rightarrow X$. We now let $Y := \cup_{n \in \mathbb{N}} f_n(J_n) \subset X$. We may construct a surjective map (but not necessarily injective map) $F : \mathbb{N} \rightarrow Y$. From this map we then define $\psi : Y \rightarrow \mathbb{N}$ by $\psi(y) := \min F^{-1}(\{y\})$ so that $\psi : Y \rightarrow \mathbb{N}$ is now injective. Suppose for the sake of contradiction that $\psi(Y) \subset J_N$ for some $N \in \mathbb{N}$, i.e. $\psi(Y)$ is a bounded set. Then using our above arguments, we know that $\text{card}(\psi(Y)) = \text{card}(J_k)$ for some $k \leq N$. On the other hand, $f_n : J_n \rightarrow Y$ being injective implies $\text{card}(\psi(Y)) \geq \text{card}(J_n)$ for all $n \in \mathbb{N}$. As both of these statements can not be correct at the same time we conclude that $\psi(Y)$ is unbounded. We may now apply Lemma 5.23 in order to see that $\text{card}(Y) = \text{card}(\psi(Y)) = \text{card}(\mathbb{N})$. From this it follows that $\text{card}(\mathbb{N}) \leq \text{card}(X)$. ■

Lemma 5.23. *If $S \subset \mathbb{N}$ is an unbounded set, then $\text{card}(S) = \text{card}(\mathbb{N})$.*

Proof. Idea: Define $f : \mathbb{N} \rightarrow S$ via, let

$$\begin{aligned} S_1 &:= S \text{ and } f(1) := \min S_1, \\ S_2 &:= S_1 \setminus \{f(1)\} \text{ and } f(2) := \min S_2, \\ S_3 &:= S_2 \setminus \{f(2)\} \text{ and } f(3) := \min S_3 \\ &\vdots \end{aligned}$$

In more detail, let T denote those $n \in \mathbb{N}$ such that there exists $f : J_n \rightarrow S$ and $\{S_k \subset S\}_{k=1}^n$ satisfying, $S_1 = S$, $f(k) = \min S_k$ and $S_{k+1} = S_k \setminus \{f(k)\}$ for $1 \leq k < n$. If $n \in T$, we may define $S_{n+1} := S_n \setminus \{f(n)\}$ and $f(n+1) := \min S_{n+1}$ in order to show $n+1 \in T$. Thus $T = \mathbb{N}$ and we have constructed an injective map, $f : \mathbb{N} \rightarrow S$. Moreover $\cap_{k \in \mathbb{N}} S_k \subset \mathbb{N} \setminus J_n$ for all n and therefore $\cap_{k \in \mathbb{N}} S_k = \emptyset$. Thus it follows that f is a bijection. ■

Definition 5.24. *We say a set X is infinite if $\text{card}(X) \geq \text{card}(\mathbb{N})$ or equivalently, $\text{card}(X) \geq n$ for all $n \in \mathbb{N}$. One could also say X is an infinite set if $X \neq \emptyset$ and $\text{card}(X \setminus \{x\}) = \text{card}(X)$ for some (or all) $x \in X$.*

5.2.1 Exercises

Exercise 5.6. Show that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$.

Exercise 5.7. Let $\mathbb{Q}[t]$ be the set of polynomial functions, p , such that p has rational coefficients. That is $p \in \mathbb{Q}[t]$ iff there exists $n \in \mathbb{N}_0$ and $a_k \in \mathbb{Q}$ for $0 \leq k \leq n$ such that

$$p(t) = \sum_{k=0}^n a_k t^k \text{ for all } t \in \mathbb{R}. \quad (5.1)$$

Show $\mathbb{Q}[t]$ is a countable set.

Definition 5.25 (Algebraic Numbers). *A real number, $x \in \mathbb{R}$, is called **algebraic number**, if there is a non-zero polynomial $p \in \mathbb{Q}[t]$ such that $p(x) = 0$. [That is to say, $x \in \mathbb{R}$ is algebraic if it is the root of a non-zero polynomial with coefficients from $\mathbb{Q}$.]*

Note that for all $q \in \mathbb{Q}$, $p(t) := t - q$ satisfies $p(q) = 0$. Hence all rational numbers are algebraic. But there are many more algebraic numbers, for example $y^{1/n}$ is algebraic for all $y \geq 0$ and $n \in \mathbb{N}$.

Exercise 5.8. Show that the set of algebraic numbers is countable. [Hint: any polynomial of degree n has at most n – real roots.] In particular, “most” irrational numbers are not algebraic numbers, i.e. there is still any uncountable number of non-algebraic numbers.

Metric Spaces

Metric Spaces

Definition 6.1. A function $d : X \times X \rightarrow [0, \infty)$ is called a metric if

1. (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$,
2. (Non-degenerate) $d(x, y) = 0$ if and only if $x = y \in X$, and
3. (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 6.2. Here are a few immediate examples of metric spaces;

1. Let X be any set and then define,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

2. Let $X = \mathbb{R}$ with $d(x, y) := |y - x|$. Notice that

$$\begin{aligned} d(x, z) &= |x - y + y - z| \\ &\leq |x - y| + |y - z| = d(x, y) + d(y, z). \end{aligned}$$

3. Let X be any subset of $\mathbb{C}$ and define $d(w, z) := |z - w|$.

In general our typical example of a metric space will often be a generalization of the last example above, see Example 6.4.

6.1 Normed Spaces [Linear Algebra Meets Analysis]

Definition 6.3. A **norm** on a vector space Z is a function $\|\cdot\| : Z \rightarrow [0, \infty)$ such that

1. (Homogeneity) $\|\lambda f\| = |\lambda| \|f\|$ for all $\lambda \in \mathbb{F}$ and $f \in Z$.
2. (Triangle inequality) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in Z$.
3. (Positive definite) $\|f\| = 0$ implies $f = 0$.

A pair $(Z, \|\cdot\|)$ where Z is a vector space and $\|\cdot\|$ is a norm on Z is called a **normed vector space** or **normed space** for short.

Example 6.4. If $(Z, \|\cdot\|)$ is a normed space, then $d(x, y) := \|x - y\|$ is a metric on Z and restricts to a metric on any subset of Z .

Example 6.5 (Normed Spaces). The following are normed spaces;

1. $Z = \mathbb{R}$ with $\|x\| = |x|$.
2. $Z = \mathbb{C}$ with $\|z\| = |z|$.
3. $Z = \mathbb{C}^n$ with

$$\|z\|_1 := \sum_{i=1}^n |z_i| \text{ for } z = (z_1, \dots, z_n) \in Z.$$

The triangle inequality is easily verified here since,

$$\begin{aligned} \|z + w\|_1 &= \sum_{i=1}^n |z_i + w_i| \\ &\leq \sum_{i=1}^n [|z_i| + |w_i|] \\ &= \sum_{i=1}^n |z_i| + \sum_{i=1}^n |w_i| = \|z\|_1 + \|w\|_1. \end{aligned}$$

4. Let X be a set and for any function $f : X \rightarrow \mathbb{C}$, let

$$\|f\|_u := \sup_{x \in X} |f(x)|.$$

Then $Z := \{f : X \rightarrow \mathbb{C} : \|f\|_u < \infty\}$ is a vector space and $\|\cdot\|_u$ is a norm on Z .

Exercise 6.1. Verify the last item of Example 6.5. That is let X be a set and for any function $f : X \rightarrow \mathbb{C}$, let

$$\|f\|_u := \sup_{x \in X} |f(x)|.$$

Show $Z := \{f : X \rightarrow \mathbb{C} : \|f\|_u < \infty\}$ is a vector space and $\|\cdot\|_u$ is a norm on Z .

Our next goal is to show that $\|\cdot\|_2$ defined in Eq. (4.7) defines a norm on $\mathbb{R}^n$ and $\mathbb{C}^n$. We will begin by proving the important Cauchy-Schwarz inequality.

Lemma 6.6. If $x, y \geq 0$ and $\rho > 0$, then

$$xy \leq \frac{1}{2} \left(\rho x^2 + \frac{1}{\rho} y^2 \right) \quad (6.1)$$

with equality when $\rho = y/x$ in the case $x > 0$.

Proof. Since

$$0 \leq \left(\sqrt{\rho}x - \frac{y}{\sqrt{\rho}} \right)^2 = \rho x^2 + \frac{1}{\rho} y^2 - 2xy,$$

with equality iff $\sqrt{\rho}x = \frac{y}{\sqrt{\rho}}$, i.e. iff $\rho = y/x$, we see that

$$xy \leq \frac{1}{2} \left(\rho x^2 + \frac{1}{\rho} y^2 \right)$$

with equality iff $\rho = y/x$. ■

Theorem 6.7 (Cauchy-Schwarz Inequality). For $a, b \in \mathbb{C}^n$, $|a \cdot b| \leq \|a\|_2 \cdot \|b\|_2$.

Proof. The inequality holds true if $a = 0$ so we may now assume $a \neq 0$. Using Lemma 6.6 with $x = |a_i|$ and $y = |b_i|$ we find for any $\rho > 0$ that

$$\begin{aligned} |a \cdot b| &\leq \left| \sum_{i=1}^n a_i b_i \right| \leq \sum_{i=1}^n |a_i b_i| = \sum_{i=1}^n |a_i| \cdot |b_i| \\ &\leq \sum_{i=1}^n \frac{1}{2} \left(\rho |a_i|^2 + \frac{1}{\rho} |b_i|^2 \right) = \frac{1}{2} \left[\rho \|a\|_2^2 + \frac{1}{\rho} \|b\|_2^2 \right]. \end{aligned}$$

Taking $\rho = \|b\|_2 / \|a\|_2$ then completes the proof. ■

Theorem 6.8 (Triangle Inequality). For $a, b \in \mathbb{C}^n$,

$$\|a + b\|_2 \leq \|a\|_2 + \|b\|_2.$$

Proof. We have

$$\begin{aligned} \|a + b\|_2^2 &= (a + b) \cdot (\overline{a + b}) = (a + b) \cdot (\bar{a} + \bar{b}) \\ &= (a + b) \cdot \bar{a} + (a + b) \cdot \bar{b} = \|a\|_2^2 + b \cdot \bar{a} + a \cdot \bar{b} + \|b\|_2^2 \\ &= \|a\|_2^2 + \|b\|_2^2 + 2 \operatorname{Re}(a \cdot \bar{b}) \leq \|a\|_2^2 + \|b\|_2^2 + 2 |\operatorname{Re}(a \cdot \bar{b})| \\ &\leq \|a\|_2^2 + \|b\|_2^2 + 2 |(a \cdot \bar{b})| \\ &\leq \|a\|_2^2 + \|b\|_2^2 + 2 \|a\|_2 \cdot \|\bar{b}\|_2 \quad (\text{Theorem 6.7}) \\ &= \|a\|_2^2 + \|b\|_2^2 + 2 \|a\|_2 \cdot \|b\|_2 = (\|a\|_2 + \|b\|_2)^2. \end{aligned}$$

■

- End of Lecture 12, 10/26/2012.

For the next two exercise you will be using some concepts from calculus which we will develop in detail next quarter. For now, I assume you know what the Riemann integral is for continuous functions on $[0, 1]$ with values in $\mathbb{R}$. Let Z denote the continuous functions¹ on $[0, 1]$ with values in $\mathbb{R}$. The only properties that you need to know about the Riemann integral are;

1. The integral is linear, namely for all $f, g \in Z$ and $\lambda \in \mathbb{R}$,

$$\int_0^1 (f(t) + \lambda g(t)) dt = \int_0^1 f(t) dt + \lambda \int_0^1 g(t) dt.$$

2. If $f, g \in Z$ and $f(t) \leq g(t)$ for all $t \in [0, 1]$, then

$$\int_0^1 f(t) dt \leq \int_0^1 g(t) dt.$$

3. For all $f \in Z$,

$$\left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt.$$

In fact this item follows from items 1. and 2. Indeed, since $\pm f(t) \leq |f(t)|$ for all t , we find

$$\pm \int_0^1 f(t) dt = \int_0^1 \pm f(t) dt \leq \int_0^1 |f(t)| dt \iff \left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt.$$

Exercise 6.2. Let Z denote the continuous function on $[0, 1]$ with values in $\mathbb{R}$ and for $f \in Z$ let

$$\|f\|_1 := \int_0^1 |f(t)| dt.$$

Show $\|\cdot\|_1$ satisfies,

1. (Homogeneity) $\|\lambda f\| = |\lambda| \|f\|$ for all $\lambda \in \mathbb{R}$ and $f \in Z$.
2. (Triangle inequality) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in Z$.

Remark 6.9 (An interpretation of $\|f\|_1$). If we interpret $f(t)$ as the speed of a particle on the real line at time t , then $\|f\|_1$ represents the total distance (including retracing of its path) the particle travels over the time interval $[0, 1]$.

Exercise 6.3. Let Z denote the continuous function on $[0, 1]$ with values in $\mathbb{R}$ and for $f \in Z$ let

$$\|f\|_2 := \sqrt{\int_0^1 |f(t)|^2 dt}.$$

Show;

¹ The notion of continuity will be formally developed shortly.

1. for $f, g \in Z$ that

$$\left| \int_0^1 f(t) g(t) dt \right| \leq \|f\|_2 \cdot \|g\|_2,$$

2. Homogeneity) $\|\lambda f\|_2 = |\lambda| \|f\|_2$ for all $\lambda \in \mathbb{R}$ and $f \in Z$, and
 3. (Triangle inequality) $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$ for all $f, g \in Z$.

Remark 6.10 (An interpretation of $\|f\|_2$). Let us suppose that $f(t)$ is the voltage across a 1 Ohm resistor. By Ohms law the current through this resistor is $f(t)/1 = f(t)$ and the power dissipated by the resistor at time t is (Voltage·Current) is $f(t)^2$. The work done over the time interval, $[0, 1]$ is then

$$\int_0^1 \text{Power}(t) dt = \int_0^1 f^2(t) dt = \|f\|_2^2.$$

On the other hand if we had a constant voltage of $\|f\|_2$ across the resistor over the time interval $[0, 1]$, the work done over this period would again be $\|f\|_2^2$. Thus $\|f\|_2$ is often referred to as the RMS voltage (root mean squared voltage) and represents the equivalent DC voltage necessary to produce the same amount of work over the time interval $[0, 1]$.

- End of Lecture 12, 10/26/2012. [We started Section 5.1 above as well this day.]

6.2 Sequences in Metric Spaces

Definition 6.11. A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X, d) is said to be **convergent** if there exists a point $x \in X$ such that $\lim_{n \rightarrow \infty} d(x, x_n) = 0$. In this case we write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Exercise 6.4. Show that x in Definition 6.11 is necessarily unique.

Definition 6.12 (Cauchy sequences). A sequence $\{x_n\}_{n=1}^\infty$ in a metric space (X, d) is **Cauchy** provided that $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$, i.e. for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq \varepsilon \text{ when } m, n \geq N(\varepsilon).$$

Exercise 6.5. Show that convergent sequences in metric spaces are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x, y) = |x - y|$. Choose a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^\infty$ is $(\mathbb{Q}, d)$ – Cauchy but not $(\mathbb{Q}, d)$ – convergent. Of course the sequence is convergent in $\mathbb{R}$.

Exercise 6.6. If $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in a metric space (X, d) , $\lim_{n \rightarrow \infty} d(x_n, y)$ exists in $\mathbb{R}$ for all $y \in X$. In particular, $\{d(x_n, y)\}_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}$ for all $y \in X$.

Definition 6.13. A metric space (X, d) (or normed space $(X, \|\cdot\|)$) is **complete** if all Cauchy sequences are convergent sequences. A complete normed space is called a **Banach space**.

Theorem 6.14 (Completeness of $\mathbb{C}^n$). Let $n \in \mathbb{N}$ and $\|\cdot\|$ denote any one of the norms, $\|\cdot\|_1$, $\|\cdot\|_\infty$, or $\|\cdot\|_2$ on $\mathbb{C}^n$. Then $(\mathbb{C}^n, \|\cdot\|)$ is complete.

Proof. If $\{z(k) := (z_1(k), \dots, z_n(k))\}_{k=1}^\infty$ is a sequence in $\mathbb{C}^n$, then $z(k) \rightarrow z = (z_1, \dots, z_n) \in \mathbb{C}^n$ as $k \rightarrow \infty$ iff $\lim_{k \rightarrow \infty} z_i(k) = z_i$ for $1 \leq i \leq n$. Similarly, $\{z(k)\}_{k=1}^\infty$ is Cauchy iff $\{z_i(k)\}_{k=1}^\infty$ is Cauchy for all $1 \leq i \leq n$. From these remarks we see that the completeness of $(\mathbb{C}^n, \|\cdot\|)$ follows directly from the completeness of $\mathbb{C}$ in Theorem 4.9. ■

Lemma 6.15. Let X be a set and for any function $f : X \rightarrow \mathbb{C}$, let

$$\|f\|_u := \sup_{x \in X} |f(x)|.$$

Then $Z := \{f : X \rightarrow \mathbb{C} : \|f\|_u < \infty\}$ is a Banach space, i.e. $(Z, \|\cdot\|_u)$ is a complete normed space.

Proof. Let $\{f_n\}_{n=1}^\infty \subset Z$ be a Cauchy sequence. Since for any $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \quad (6.2)$$

which shows that $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence of complex numbers. Because $\mathbb{C}$ is complete, $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists for all $x \in X$. Passing to the limit $n \rightarrow \infty$ in Eq. (6.2) implies

$$|f(x) - f_m(x)| \leq \liminf_{n \rightarrow \infty} \|f_n - f_m\|_\infty$$

and taking the supremum over $x \in X$ of this inequality implies

$$\|f - f_m\|_\infty \leq \liminf_{n \rightarrow \infty} \|f_n - f_m\|_\infty \rightarrow 0 \text{ as } m \rightarrow \infty$$

showing $f_m \rightarrow f$ in Z . ■

Exercise 6.7. Let X be a set and (Y, ρ) be a complete metric space. Suppose that $f_n : X \rightarrow Y$ are functions such that

$$\delta_{m,n} := \sup_{x \in X} d(f_n(x), f_m(x)) \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Show there exists a (unique) functions, $f : X \rightarrow Y$ such that

$$\lim_{n \rightarrow \infty} \sup_{x \in X} d(f_n(x), f(x)) = 0.$$

Hint: mimic the proof of Lemma 6.15.

6.3 Continuity

Suppose now that (X, ρ) and (Y, d) are two metric spaces and $f : X \rightarrow Y$ is a function.

Definition 6.16 (Continuity). A function $f : X \rightarrow Y$ is **continuous at** $x \in X$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(f(x), f(x')) < \varepsilon \text{ provided that } \rho(x, x') < \delta. \quad (6.3)$$

The function f is said to be **continuous** if f is continuous at all points $x \in X$.

Definition 6.17 (Sequential Continuity). A function $f : X \rightarrow Y$ is **continuous at** $x \in X$ if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x) \text{ for all } \{x_n\}_{n=1}^{\infty} \subset X \text{ with } \lim_{n \rightarrow \infty} x_n = x.$$

We say f is **sequentially continuous** on X if it is continuous at all points in X .

Theorem 6.18. Continuity and sequential continuity are the same notions.

Proof. Suppose that $f : X \rightarrow Y$ is continuous at $x \in X$ and $\{x_n\}_{n=1}^{\infty} \subset X$ such that $\lim_{n \rightarrow \infty} x_n = x$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ provided that $\rho(x, x') < \delta$. However since $x_n \rightarrow x$, we know that $\rho(x, x_n) < \delta$ a.a. n and therefore $d(f(x), f(x_n)) < \varepsilon$ for a.a. n . This shows $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$.

For the converse direction, suppose that f is not continuous at x . Then there exists $\varepsilon > 0$ such that for any $\delta = \frac{1}{n} > 0$ there exists $x_n \in X$ such that $d(f(x), f(x_n)) \geq \varepsilon$ while $\rho(x, x_n) < \frac{1}{n}$. We then have $\lim_{n \rightarrow \infty} x_n = x$ while $\lim_{n \rightarrow \infty} f(x_n) \neq f(x)$. ■

Example 6.19. The functions $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ defined by $f(z) = 1/z$ is continuous. Indeed, if $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ and $\lim_{n \rightarrow \infty} z_n = z \in \mathbb{C} \setminus \{0\}$, then

$$\lim_{n \rightarrow \infty} f(z_n) = \lim_{n \rightarrow \infty} \frac{1}{z_n} = \frac{1}{z} = f(z).$$

Exercise 6.8. Suppose that (X, d) is a metric space and $f, g : X \rightarrow \mathbb{C}$ are two continuous functions on X . Show;

1. $f + g$ is continuous,
2. $f \cdot g$ is continuous,
3. f/g is continuous provided $g(x) \neq 0$ for all $x \in X$.

Exercise 6.9. Show the following functions from $\mathbb{C}$ to $\mathbb{C}$ are continuous.

1. $f(z) = c$ for all $z \in \mathbb{C}$ where $c \in \mathbb{C}$ is a constant.

2. $f(z) = |z|$.
3. $f(z) = z$ and $f(z) = \bar{z}$.
4. $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$.
5. $f(z) = \sum_{m,n=0}^N a_{m,n} z^m \bar{z}^n$ where $a_{m,n} \in \mathbb{C}$.

Exercise 6.10. Suppose now that (X, ρ) , (Y, d) , and (Z, δ) are three metric spaces and $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Let $x \in X$ and $y = f(x) \in Y$, show $g \circ f : X \rightarrow Z$ is continuous at x if f is continuous at x and g is continuous at y . Recall that $(g \circ f)(x) := g(f(x))$ for all $x \in X$. In particular this implies that if f is continuous on X and g is continuous on Y then $f \circ g$ is continuous on X .

Theorem 6.20 (A monotone continuity criteria). Suppose that (Y, d) is a metric space, $X = (a, b) \subset \mathbb{R}$, and $f : X \rightarrow Y$ is a function. Then f is continuous at $x \in X$ iff $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ whenever $\{x_n\}_{n=1}^{\infty} \subset X$ converges monotonically to x as $n \rightarrow \infty$. In other words, it is sufficient to check sequential continuity along sequences which are either increasing or decreasing.

Proof. ($\Leftarrow$) This direction follows from Theorem 6.18.

($\Rightarrow$) Suppose there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x_\delta \in X$ $\ni |x_\delta - x| \leq \delta$ with $d(f(x_\delta), f(x)) \geq \varepsilon$. Then either $x_\delta > x$ i.o. δ or $x_\delta < x$ i.o. δ . If $x_\delta > x$ i.o. δ , we choose $\delta_n \downarrow 0$ such that $y_n := x_{\delta_n} > x$ for all n . Then take $x_n = \min(y_1, \dots, y_n)$. Then $x_n \downarrow x$ while $|f(x_n) - f(x)| \geq \varepsilon$, i.e., $\lim_{n \rightarrow \infty} f(x_n) \neq f(x)$. Similarly if $x_\delta < x$ i.o. δ , we choose $\delta_n \downarrow 0$ such that $y_n := x_{\delta_n} < x$ for all n . Then take $x_n = \max(y_1, \dots, y_n)$. Then $x_n \uparrow x$ while $|f(x_n) - f(x)| \geq \varepsilon$, i.e., $\lim_{n \rightarrow \infty} f(x_n) \neq f(x)$. ■

Exercise 6.11 (Continuity of $x^{1/m}$). Show for each $m \in \mathbb{N}$ that the function $f(x) := x^{1/m}$ is continuous on $[0, \infty)$.

Exercise 6.12 (Differentiability of $x^{1/m}$). Show for each $m \in \mathbb{N}$ that the function $f(x) := x^{1/m}$ is differentiable on $(0, \infty)$ and that

$$\frac{d}{dx} x^{1/m} := \lim_{y \rightarrow x} \frac{y^{1/m} - x^{1/m}}{y - x} = \frac{1}{m} x^{1/m-1}.$$

Exercise 6.13 (Intermediate value theorem). Suppose that $-\infty < a < b < \infty$ and $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function such that $f(a) \leq f(b)$. Show for any $y \in [f(a), f(b)]$, there exists a $c \in [a, b]$ such that $f(c) = y$.² **Hint:** Let $S := \{t \in [a, b] : f(t) \leq y\}$ and let $c := \sup(S)$.

² The same result holds for $y \in [f(b), f(a)]$ if $f(b) \leq f(a)$ – just replace f by $-f$ in this case.

Exercise 6.14. Let $f : [a, b] \rightarrow [c, d]$ be a strictly increasing (i.e. $f(x_1) < f(x_2)$ whenever $x_1 < x_2$) continuous function such that $f(a) = c$ and $f(b) = d$. Then f is bijective and the inverse function, $g := f^{-1} : [c, d] \rightarrow [a, b]$, is strictly increasing and is continuous.

Definition 6.21 (Pointwise Convergence). Let (X, d) and (Y, ρ) be metric spaces and $f_n : X \rightarrow Y$ be functions for each $n \in \mathbb{N}$. We say that f_n **converges pointwise to** $f : X \rightarrow Y$ provided $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in X$, i.e. provided

$$\lim_{n \rightarrow \infty} d(f(x), f_n(x)) = 0 \text{ for each } x \in X.$$

Definition 6.22 (Uniform Convergence). Let (X, d) and (Y, ρ) be metric spaces and $f_n : X \rightarrow Y$ be functions for each $n \in \mathbb{N}$. We say that f_n **converges uniformly to** $f : X \rightarrow Y$ provided

$$\delta_n := \sup_{x \in X} d(f(x), f_n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hopefully it is clear to the reader the uniform convergence implies pointwise convergence. The next theorem is a basic fact about uniform convergence which does not hold in general for pointwise convergence. The proof of this theorem is a bit subtle but well worth mastering as the method will arise over and over again.

Theorem 6.23 (Uniform Convergence Preserves Continuity). Suppose that $\{f_n\}_{n=1}^{\infty}$ are continuous functions from X to Y and f_n converges uniformly to $f : X \rightarrow Y$. If f_n is continuous at $x \in X$ for all n then f is continuous at x as well. In particular if f_n is continuous on X for all n then f is continuous on X as well.

Proof. We must show $\lim_{k \rightarrow \infty} f(x_k) = f(x)$ whenever $\{x_k\}_{k=1}^{\infty} \subset X$ is a convergent sequence such that $x := \lim_{k \rightarrow \infty} x_k \in X$. So assume we are given such a sequence $\{x_k\}_{k=1}^{\infty}$. Then for any $n \in \mathbb{N}$ we have,

$$\begin{aligned} d(f(x), f(x_k)) &\leq d(f(x), f_n(x)) + d(f_n(x), f(x_k)) \\ &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_k)) + d(f_n(x_k), f(x_k)) \\ &\leq \delta_n + d(f_n(x), f_n(x_k)) + \delta_n. \end{aligned}$$

Therefore,

$$\limsup_{k \rightarrow \infty} d(f(x), f(x_k)) \leq \limsup_{k \rightarrow \infty} d(f_n(x), f_n(x_k)) + 2\delta_n = 2\delta_n,$$

wherein we have used the continuity of f_n for the last equality. Thus we have shown

$$\limsup_{k \rightarrow \infty} d(f(x), f(x_k)) \leq 2\delta_n$$

which upon passing to the limit as $n \rightarrow \infty$ shows $\limsup_{k \rightarrow \infty} d(f(x), f(x_k)) = 0$. This suffices to show $\lim_{k \rightarrow \infty} f(x_k) = f(x)$. ■

Exercise 6.15. Let $f_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $f_n(x) = x^n$ for $x \in [0, 1]$. Show $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ exists for all $x \in [0, 1]$ and find f explicitly. Show that f_n does **not** converge to f uniformly.

Example 6.24. Let Z denote the vector space of continuous functions on $[0, 1]$ with values in $\mathbb{R}$ and for $f \in Z$ let

$$\|f\|_1 := \int_0^1 |f(t)| dt.$$

Show $(Z, \|\cdot\|_1)$ is **not** complete. Hint: For the last assertion find a sequence of $\{f_n\}_{n=1}^{\infty} \subset Z$ which is “trying” to converge to the function $f = 1_{[\frac{1}{2}, 1]} \notin Z$.

Solution. Let

$$f_n(t) = (0 \vee \left[n(t - \frac{1}{2})\right]) \wedge 1 = \begin{cases} 0 & \text{if } t \leq 1/2 \\ n(t - \frac{1}{2}) & \text{if } 1/2 \leq t \leq 1/2 + n^{-1} \\ 1 & \text{if } 1/2 + n^{-1} \leq t. \end{cases}$$

The pointwise limit of these functions is

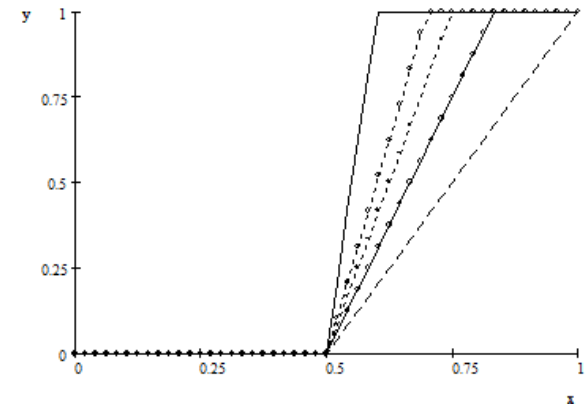


Fig. 6.1. Plot of some of the functions f_n .

$$g(t) := \lim_{n \rightarrow \infty} f_n(t) = \begin{cases} 0 & \text{if } t \leq 1/2 \\ 1 & \text{if } t > 1/2 \end{cases}$$

and it is easily seen that $\{f_n\}_{n=1}^\infty$ is Cauchy in L^1 - norm. Suppose, for the sake of contradiction, that there exists a continuous function f such that $\|f - f_n\|_1 \rightarrow 0$. Clearly $f(t) = 0$ for $t \leq 1/2$ and $f(t) = 1$ for $t > 1/2$ for otherwise $\|f - f_n\|_1 \geq \varepsilon > 0$ for some $\varepsilon > 0$ and all n sufficiently large. But this shows that f would have to be the discontinuous function g above. So we must conclude there exists no such continuous function f .

The following continuity facts of the metric d will be used frequently in the remainder of these notes.

Lemma 6.25 (Distance to a Set). *For any non empty subset $A \subset X$, let*

$$d_A(x) := \inf\{d(x, a) | a \in A\},$$

then

$$|d_A(x) - d_A(y)| \leq d(x, y) \quad \forall x, y \in X. \quad (6.4)$$

In particular, $d_A : X \rightarrow [0, \infty)$ is continuous.

Proof. Let $a \in A$ and $x, y \in X$, then

$$d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a).$$

Take the infimum over a in the above equation shows that

$$d_A(x) \leq d(x, y) + d_A(y) \quad \forall x, y \in X.$$

Therefore, $d_A(x) - d_A(y) \leq d(x, y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \leq d(x, y)$ which implies Eq. (6.4). ■

Corollary 6.26. *The function d satisfies,*

$$|d(x, y) - d(x', y')| \leq d(y, y') + d(x, x').$$

In particular $d : X \times X \rightarrow [0, \infty)$ is continuous in the sense that $d(x, y)$ is close to $d(x', y')$ if x is close to x' and y is close to y' .

Proof. First Proof. By Lemma 6.25 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned} |d(x, y) - d(x', y')| &\leq |d(x, y) - d(x, y')| + |d(x, y') - d(x', y')| \\ &\leq d(y, y') + d(x, x'). \end{aligned}$$

Second Proof. By the triangle inequality,

$$d(x, y) \leq d(x, x') + d(x', y) \leq d(x, x') + d(x', y') + d(y', y)$$

from which it follows that

$$d(x, y) - d(x', y') \leq d(x, x') + d(y', y).$$

Interchanging x with x' and y with y' in this inequality shows,

$$d(x', y') - d(x, y) \leq d(x, x') + d(y', y)$$

and the result follows from the last two inequalities. ■

6.4 Closed and Open Sets

Let (X, d) be a metric space.

Definition 6.27. *Let (X, d) be a metric space. The **open ball** $B(x, \delta) \subset X$ centered at $x \in X$ with radius $\delta > 0$ is the set*

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

*We will often also write $B(x, \delta)$ as $B_x(\delta)$. We also define the **closed ball** centered at $x \in X$ with radius $\delta > 0$ as the set $C_x(\delta) := \{y \in X : d(x, y) \leq \delta\}$.*

Definition 6.28. *A set $E \subset X$ is **bounded** if $E \subset B(x, R)$ for some $x \in X$ and $R < \infty$. A set $F \subset X$ is **closed** iff every convergent sequence $\{x_n\}_{n=1}^\infty$ which is contained in F has its limit back in F . A set $V \subset X$ is **open** iff V^c is closed. We will write $F \sqsubset X$ to indicate F is a closed subset of X and $V \subset_o X$ to indicate the V is an open subset of X . We also let τ_d denote the collection of open subsets of X relative to the metric d .*

Lemma 6.29. *If $f : X \rightarrow \mathbb{R}$ is a continuous function and $k \in \mathbb{R}$, then the following sets are closed,*

$$A := \{x \in X : f(x) \leq k\}, \quad B := \{x \in X : f(x) = k\}, \quad \text{and} \quad C := \{x \in X : f(x) \geq k\}.$$

Proof. The proof that A , B , and C are closed all go the same way so let me just check that A is closed. To this end, suppose that $\{x_n\}_{n=1}^\infty$ is a sequence in A such that $x := \lim_{n \rightarrow \infty} x_n$ exists in X . Since $x_n \in A$, $f(x_n) \leq k$ and therefore,

$$k \geq \lim_{n \rightarrow \infty} f(x_n) = f(x)$$

wherein the last equality we have used the definition of f being continuous. By definition of A it then follows that $x \in A$ and so we have checked that A is closed. ■

Example 6.30 (Closed Balls). Closed balls are closed. Indeed, we have seen $f(y) := d(x, y)$ is continuous and therefore

$$C_x(\delta) := \{y \in X : d(x, y) \leq \delta\} = \{y \in X : f(y) \leq \delta\}$$

is a closed set. Notice that $\{x\} = C_x(0)$ is a closed set for all $x \in X$.

Example 6.31. The following subsets of $\mathbb{C}$ are closed;

1. $\{z \in \mathbb{C} : a \leq \operatorname{Im} z \leq b\}$ for all $a \leq b$ in $\mathbb{R}$.
2. $\{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b\}$ for all $a \leq b$ in $\mathbb{R}$.
3. $\{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } a \leq \operatorname{Re} z \leq b\}$ for all $a \leq b$ in $\mathbb{R}$.

Example 6.32 (Open Balls). Open balls in metric spaces are open sets. Indeed let $f(y) := d(x, y)$, then

$$B_x(\delta)^c := X \setminus B_x(\delta) = \{y \in X : d(x, y) \geq \delta\} = \{y \in X : f(y) \geq \delta\}$$

which is closed since f is continuous. Thus $B_x(\delta)$ is open.

Theorem 6.33. *The closed subsets of $\mathbb{C}$ have the following properties;*

1. X and $\emptyset$ are closed.
2. If $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed subsets of X , then $\cap_{\alpha \in I} C_\alpha$ is closed in X .
3. If A and B are closed sets then $A \cup B$ is closed.

Proof. 3. Let $\{z_n\}_{n=1}^\infty \subset A \cup B$ such that $\lim_{n \rightarrow \infty} z_n =: z$ exists. Then $z_n \in A$ i.o. or $z_n \in B$ i.o. For sake of definiteness say $z_n \in A$ i.o. in which case we may choose a subsequence, $w_k := z_{n_k} \in A$ for all k . Since $\lim_{k \rightarrow \infty} w_k = z$ and A is closed it follows that $z \in A$ and hence $z \in A \cup B$. Thus we have shown $A \cup B$ is closed. ■

Exercise 6.16. Prove item 2. of Theorem 6.33. If $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed subsets of X , then $\cap_{\alpha \in I} C_\alpha$ is closed in X .

Exercise 6.17. Given an example of a collection of closed subsets, $\{A_n\}_{n=1}^\infty$, of $\mathbb{C}$ such that $\cup_{n=1}^\infty A_n$ is not closed.

Corollary 6.34. *Let (X, d) be a metric space. Then the collection of open subsets, τ_d , of X satisfy;*

1. X and $\emptyset$ are in τ_d .
2. τ_d is closed under taking arbitrary unions. i.e. if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets then $\cup_{\alpha \in I} U_\alpha$ is open..
3. τ_d is closed under taking finite intersections, i.e. if U and V are open sets then $U \cap V$ is open as well.

Exercise 6.18. Prove Corollary 6.34.

Exercise 6.19. Let U be a subset of $\mathbb{C}$. Show the following are equivalent;

1. U is open,
2. for all $z \in U$ there exists $\rho > 0$ such that $B_z(\rho) \subset U$.
3. U can be written as a union of open balls.

Exercise 6.20. Show that $V \subset X$ is open iff for every $x \in V$ there is a $\delta > 0$ such that $B_x(\delta) \subset V$. In particular show $B_x(\delta)$ is open for all $x \in X$ and $\delta > 0$.

Hint: by definition V is not open iff V^c is not closed.

Exercise 6.21. Show $U := \mathbb{C} \setminus \{z_0\}$ is open for any $z_0 \in \mathbb{C}$. More generally, show that $U := \mathbb{C} \setminus S$ is open for all whenever S is a finite subset of $\mathbb{C}$.

Exercise 6.22. Let (X, d) be a complete metric space. Let $A \subset X$ be a subset of X viewed as a metric space using $d|_{A \times A}$. Show that $(A, d|_{A \times A})$ is complete iff A is a closed subset of X .

Lemma 6.35. *For any non empty subset $A \subset X$, let $d_A(x) := \inf\{d(x, a) | a \in A\}$, then*

$$|d_A(x) - d_A(y)| \leq d(x, y) \quad \forall x, y \in X \quad (6.5)$$

and in particular if $x_n \rightarrow x$ in X then $d_A(x_n) \rightarrow d_A(x)$ as $n \rightarrow \infty$. Moreover the set $F_\varepsilon := \{x \in X | d_A(x) \geq \varepsilon\}$ is closed in X .

Proof. Let $a \in A$ and $x, y \in X$, then

$$d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a).$$

Take the infimum over a in the above equation shows that

$$d_A(x) \leq d(x, y) + d_A(y) \quad \forall x, y \in X.$$

Therefore, $d_A(x) - d_A(y) \leq d(x, y)$ and by interchanging x and y we also have that $d_A(y) - d_A(x) \leq d(x, y)$ which implies Eq. (6.5). If $x_n \rightarrow x \in X$, then by Eq. (6.5),

$$|d_A(x) - d_A(x_n)| \leq d(x, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $\lim_{n \rightarrow \infty} d_A(x_n) = d_A(x)$. Now suppose that $\{x_n\}_{n=1}^\infty \subset F_\varepsilon$ and $x_n \rightarrow x$ in X , then

$$d_A(x) = \lim_{n \rightarrow \infty} d_A(x_n) \geq \varepsilon$$

since $d_A(x_n) \geq \varepsilon$ for all n . This shows that $x \in F_\varepsilon$ and hence F_ε is closed. ■

Definition 6.36. *A subset $A \subset X$ is a **neighborhood** of x if there exists an open set $V \subset_o X$ such that $x \in V \subset A$. We will say that $A \subset X$ is an **open neighborhood** of x if A is open and $x \in A$.*

Example 6.37. Let $x \in X$ and $\delta > 0$, then $C_x(\delta)$ and $B_x(\delta)^c$ are closed subsets of X . For example if $\{y_n\}_{n=1}^\infty \subset C_x(\delta)$ and $y_n \rightarrow y \in X$, then $d(y_n, x) \leq \delta$ for all n and using Corollary 6.26 it follows $d(y, x) \leq \delta$, i.e. $y \in C_x(\delta)$. A similar proof shows $B_x(\delta)^c$ is closed, see Exercise 6.20.

Lemma 6.38 (Approximating open sets from the inside by closed sets). *Let A be a closed subset of X and $F_\varepsilon := \{x \in X | d_A(x) \geq \varepsilon\} \sqsubset X$ be as in Lemma 6.35. Then $F_\varepsilon \uparrow A^c$ as $\varepsilon \downarrow 0$.*

Proof. It is clear that $d_A(x) = 0$ for $x \in A$ so that $F_\varepsilon \subset A^c$ for each $\varepsilon > 0$ and hence $\cup_{\varepsilon > 0} F_\varepsilon \subset A^c$. Now suppose that $x \in A^c \subset_o X$. By Exercise 6.20 there exists an $\varepsilon > 0$ such that $B_x(\varepsilon) \subset A^c$, i.e. $d(x, y) \geq \varepsilon$ for all $y \in A$. Hence $x \in F_\varepsilon$ and we have shown that $A^c \subset \cup_{\varepsilon > 0} F_\varepsilon$. Finally it is clear that $F_\varepsilon \subset F_{\varepsilon'}$ whenever $\varepsilon' \leq \varepsilon$. ■

Definition 6.39. Given a set A contained in a metric space X , let $\bar{A} \subset X$ be the **closure of A** defined by

$$\bar{A} := \{x \in X : \exists \{x_n\} \subset A \ni x = \lim_{n \rightarrow \infty} x_n\}.$$

That is to say $\bar{A}$ contains all **limit points** of A . We say A is **dense in X** if $\bar{A} = X$, i.e. every element $x \in X$ is a limit of a sequence of elements from A . A metric space is said to be **separable** if it contains a countable dense subset, D .

Exercise 6.23. Given $A \subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A} = \cap \{F : A \subset F \subset X \text{ with } F \text{ closed}\}. \quad (6.6)$$

That is to say $\bar{A}$ is the smallest closed set containing A .

Exercise 6.24. If D is a dense subset of a metric space (X, d) and $E \subset X$ is a subset such that to every point $x \in D$ there exists $\{x_n\}_{n=1}^\infty \subset E$ with $x = \lim_{n \rightarrow \infty} x_n$, then E is also a dense subset of X . If points in E well approximate every point in D and the points in D well approximate the points in X , then the points in E also well approximate all points in X .

Exercise 6.25. Suppose (X, d) is a metric space which contains an uncountable subset $A \subset X$ with the property that there exists $\varepsilon > 0$ such that $d(a, b) \geq \varepsilon$ for all $a, b \in A$ with $a \neq b$. Show that (X, d) is **not** separable.

Exercise 6.26. Let $Y = BC(\mathbb{R}, \mathbb{C})$ be the Banach space of continuous bounded complex valued functions on $\mathbb{R}$ equipped with the uniform norm, $\|f\|_u := \sup_{x \in \mathbb{R}} |f(x)|$. Further let $C_0(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ such that vanish at infinity, i.e. $\lim_{x \rightarrow \pm\infty} f(x) = 0$. Also let $C_c(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ with compact support, i.e. there exists $N < \infty$ such that $f(x) = 0$ if $|x| \geq N$. Show $C_0(\mathbb{R}, \mathbb{C})$ is a closed subspace of Y and that

$$\overline{C_c(\mathbb{R}, \mathbb{C})} = C_0(\mathbb{R}, \mathbb{C}).$$

6.5 Exercises

Exercise 6.27. Let (X, d) be a metric space. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ is a sequence and set $\varepsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \varepsilon_k \leq \sum_{k=n}^{\infty} \varepsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^{\infty} \varepsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^\infty$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^\infty$ is a convergent sequence and $x = \lim_{n \rightarrow \infty} x_n$ then

$$d(x, x_n) \leq \sum_{k=n}^{\infty} \varepsilon_k.$$

Exercise 6.28. Show that (X, d) is a complete metric space iff every sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ is a convergent sequence in X . You may find it useful to prove the following statements in the course of the proof.

1. If $\{x_n\}$ is Cauchy sequence, then there is a subsequence $y_j := x_{n_j}$ such that $\sum_{j=1}^\infty d(y_{j+1}, y_j) < \infty$.
2. If $\{x_n\}_{n=1}^\infty$ is Cauchy and there exists a subsequence $y_j := x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j \rightarrow \infty} y_j$ exists, then $\lim_{n \rightarrow \infty} x_n$ also exists and is equal to x .

Exercise 6.29. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is a C^2 - function such that $f(0) = 0$, $f' > 0$ and $f'' \leq 0$ and (X, ρ) is a metric space. Show that $d(x, y) = f(\rho(x, y))$ is a metric on X . In particular show that

$$d(x, y) := \frac{\rho(x, y)}{1 + \rho(x, y)}$$

is a metric on X . (Hint: use calculus to verify that $f(a+b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$.)

Exercise 6.30. Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a sequence of metric spaces, $X := \prod_{n=1}^\infty X_n$, and for $x = (x(n))_{n=1}^\infty$ and $y = (y(n))_{n=1}^\infty$ in X let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}. \quad (6.7)$$

Show:

1. (X, d) is a metric space,
2. a sequence $\{x_k\}_{k=1}^\infty \subset X$ converges to $x \in X$ iff $x_k(n) \rightarrow x(n) \in X_n$ as $k \rightarrow \infty$ for each $n \in \mathbb{N}$ and

3. X is complete if X_n is complete for all n .

Exercise 6.31. Suppose (X, ρ) and (Y, d) are metric spaces and A is a dense subset of X .

1. Show that if $F : X \rightarrow Y$ and $G : X \rightarrow Y$ are two continuous functions such that $F = G$ on A then $F = G$ on X . **Hint:** consider the set $C := \{x \in X : F(x) = G(x)\}$.
2. Suppose $f : A \rightarrow Y$ is a function which is uniformly continuous, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(f(a), f(b)) < \varepsilon \text{ for all } a, b \in A \text{ with } \rho(a, b) < \delta.$$

Show there is a unique continuous function $F : X \rightarrow Y$ such that $F = f$ on A . **Hint:** each point $x \in X$ is a limit of a sequence consisting of elements from A .

3. Let $X = \mathbb{R} = Y$ and $A = \mathbb{Q} \subset X$, find a function $f : \mathbb{Q} \rightarrow \mathbb{R}$ which is continuous on $\mathbb{Q}$ but does **not** extend to a continuous function on $\mathbb{R}$.

6.6 Sequential Compactness

Suppose that (X, d) and (Y, ρ) are metric spaces.

Definition 6.40. As subset $K \subset X$ is **(sequentially) compact** if every sequence $\{z_n\}_{n=1}^\infty \subset K$ has a convergent subsequence, $\{w_k := z_{n_k}\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} w_k \in K$.

Example 6.41. Suppose that $F \subset X$ is an unbounded set, i.e. for all $n \in \mathbb{N}$ there exists $z_n \in F$ such that $d(x, z_n) \geq n$. The sequence $\{z_n\}_{n=1}^\infty$ and all of its subsequences are unbounded and therefore not Cauchy in X and hence not convergent in X . This shows that compact sets must be bounded.

Example 6.42. Suppose that $F \subset X$ is not closed. Then there exists $\{z_n\}_{n=1}^\infty \subset F$ such that $z := \lim_{n \rightarrow \infty} z_n \notin F$. Moreover, although every subsequence of $\{z_n\}_{n=1}^\infty$ is convergent, they all still converge to $z \notin F$. This shows that a compact set must be closed.

Lemma 6.43 (Bolzano–Weierstrass property for $\mathbb{C}^D$). Let $D \in \mathbb{N}$. Every bounded sequence, $\{z(n)\}_{n=1}^\infty \subset \mathbb{C}^D$, has a convergent subsequence.

Proof. By assumption there exists $M < \infty$ such that $\|z(n)\| = d(z(n), 0) \leq M$ for all $n \in \mathbb{N}$. Writing $z(n) = (z_1(n), \dots, z_D(n)) \in \mathbb{C}^D$. Since $|z_i(n)| \leq \|z(n)\|$ it follows that $\{z_i(n)\}_{n=1}^\infty$ is a bounded sequence in $\mathbb{C}$. Hence by the Bolzano–Weierstrass property for $\mathbb{C}$ we replace $z(n)$ by a subsequence

$z(n_k)$ such that $\lim_{k \rightarrow \infty} z(n_k) = z_1$ exists. We may now replace the original z by this new subsequence and then find a further subsequence $z(n_k)$ such that $\lim_{k \rightarrow \infty} z_i(n_k) = z_i$ exists for $i = 1, 2$. We may continue this way inductively to find a subsequence such that $\lim_{k \rightarrow \infty} z_i(n_k) = z_i$ exists for all $1 \leq i \leq D$. It then follows that $\lim_{k \rightarrow \infty} \|z - z(n_k)\| = 0$ as desired where $z := (z_1, \dots, z_D)$. ■

Theorem 6.44 (Bolzano–Weierstrass / Heine–Borel theorem). As subset $K \subset \mathbb{C}^D$ is **compact** iff it is closed and bounded.

Proof. In light of Examples 6.41 and 6.42 we are left to show that closed and bounded sets are compact. So let $K \subset \mathbb{C}^D$ be a closed and bounded set and $\{z_n\}_{n=1}^\infty$ be any sequence in K . According to Lemma 6.43, $\{z_n\}_{n=1}^\infty$ has a convergent subsequence, $\{w_k := z_{n_k}\}_{k=1}^\infty$. Since $w_k \in K$ for all k and K is closed it necessarily follows that $\lim_{k \rightarrow \infty} w_k \in K$ which shows K is compact. ■

Example 6.45 (Warning!). It is **not** true that a closed and bounded subset of an arbitrary metric space (X, d) is necessarily compact. For example let Z denote the vector space of continuous functions on $[0, 1]$ with values in $\mathbb{R}$ and for $f \in Z$ let $\|f\| = \sup_{t \in [0, 1]} |f(t)|$. Then the set $C := \{f_n\}_{n=0}^\infty$ where

$$f_n(t) = 2^{-(n+2)} \begin{cases} 0 & \text{if } t \in [0, 2^{-(n+1)}] \cup [2^{-n}, 1] \\ t - 2^{-(n+1)} & \text{if } 2^{-(n+1)} \leq t \leq 3 \cdot 2^{-(n+2)} \\ 2^{-n} - t & \text{if } 3 \cdot 2^{-(n+2)} \leq t \leq 2^{-n}. \end{cases}$$

[So $f_n(t)$ is a shark tooth over the interval $[2^{-(n+1)}, 2^{-n}]$.] Notice that $\|f_n\| = 1$ for all n so that C is bounded. Moreover $\|f_n - f_m\| = 1$ for all $m \neq n$, therefore there are no convergent subsequence of C . The reader should use this fact to see that C is closed and bounded but not compact!

Exercise 6.32 (Extreme value theorem). Let K be compact subset of X and $f : K \rightarrow \mathbb{R}$ be a continuous function. Show $-\infty < \inf_{x \in K} f(x) \leq \sup_{x \in K} f(x) < \infty$ and there exists $a, b \in K$ such that $f(a) = \inf_{x \in K} f(x)$ and $f(b) = \sup_{x \in K} f(x)$. **Hint:** first argue that there exists $\{z_n\}_{n=1}^\infty \subset K$ such that $f(z_n) \uparrow \sup_{x \in K} f(x)$ as $n \rightarrow \infty$.

Exercise 6.33 (Uniform Continuity). Let K be compact subset of X and $f : K \rightarrow \mathbb{C}$ be a continuous function. Show that f is uniformly continuous, i.e. if $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ if $w, z \in K$ with $|w - z| < \delta$. **Hint:** prove the contrapositive.

Exercise 6.34. If (X, d) is a metric space and $K \subset X$ is compact. Show subset, $C \subset K$, which is closed is compact as well.

Exercise 6.35. If $K \subset \mathbb{R}$ is compact then $\sup(K) \in K$, i.e. $\sup(K) = \max(K)$.

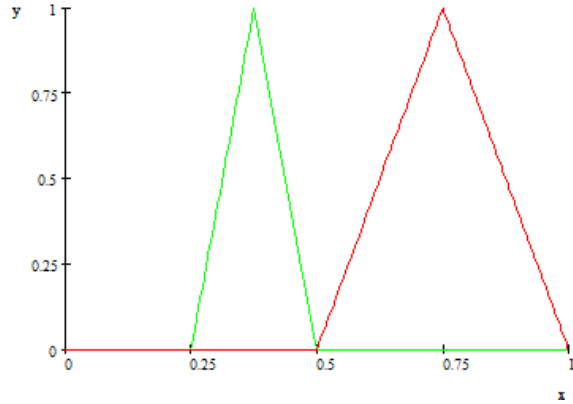


Fig. 6.2. Here are the plots of f_0 and f_1 .

Exercise 6.36. Let (X, d) and (Y, ρ) be metric spaces, $K \subset X$ be a compact set, and $f : K \rightarrow Y$ be a continuous function. Show $f(K)$ is compact in Y . In particular, for $C \subset K$ closed, we have $f(C)$ is closed and in fact compact in Y .

Exercise 6.37. Let $f : [a, b] \rightarrow [c, d]$ be a strictly increasing continuous function such that $f(a) = c$ and $f(b) = d$ and $g := f^{-1} : [c, d] \rightarrow [a, b]$ as in Exercise 6.14. Give one or better yet two alternative proofs that g is continuous based on compactness arguments.

Definition 6.46. Let Z be a vector space. We say that two norms, $|\cdot|$ and $\|\cdot\|$, on Z are equivalent if there exists constants $\alpha, \beta \in (0, \infty)$ such that

$$\|f\| \leq \alpha |f| \quad \text{and} \quad |f| \leq \beta \|f\| \quad \text{for all } f \in Z.$$

Theorem 6.47. Let Z be a finite dimensional vector space. Then any two norms $|\cdot|$ and $\|\cdot\|$ on Z are equivalent. (This is typically not true for norms on infinite dimensional spaces.)

Proof. Let $\{f_i\}_{i=1}^n$ be a basis for Z and define a new norm on Z by

$$\left\| \sum_{i=1}^n a_i f_i \right\|_2 := \sqrt{\sum_{i=1}^n |a_i|^2} \quad \text{for } a_i \in \mathbb{F}.$$

By the triangle inequality for the norm $|\cdot|$, we find

$$\left| \sum_{i=1}^n a_i f_i \right| \leq \sum_{i=1}^n |a_i| |f_i| \leq \sqrt{\sum_{i=1}^n |f_i|^2} \sqrt{\sum_{i=1}^n |a_i|^2} \leq M \left\| \sum_{i=1}^n a_i f_i \right\|_2$$

where $M = \sqrt{\sum_{i=1}^n |f_i|^2}$. Thus we have $|f| \leq M \|f\|_2$ for all $f \in Z$ and this inequality shows that $|\cdot|$ is continuous relative to $\|\cdot\|_2$. Since the normed space $(Z, \|\cdot\|_2)$ is homeomorphic and isomorphic to $\mathbb{F}^n$ with the standard euclidean norm, the closed bounded set, $S := \{f \in Z : \|f\|_2 = 1\} \subset Z$, is a compact subset of Z relative to $\|\cdot\|_2$. Therefore by Exercise 6.32 there exists $f_0 \in S$ such that

$$m = \inf \{|f| : f \in S\} = |f_0| > 0.$$

Hence given $0 \neq f \in Z$, then $\frac{f}{\|f\|_2} \in S$ so that

$$m \leq \left| \frac{f}{\|f\|_2} \right| = |f| \frac{1}{\|f\|_2}$$

or equivalently

$$\|f\|_2 \leq \frac{1}{m} |f|.$$

This shows that $|\cdot|$ and $\|\cdot\|_2$ are equivalent norms. Similarly one shows that $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent and hence so are $|\cdot|$ and $\|\cdot\|$. ■

Corollary 6.48. If $(Z, \|\cdot\|)$ is a finite dimensional normed space, then $A \subset Z$ is compact iff A is closed and bounded relative to the given norm, $\|\cdot\|$.

Corollary 6.49. Every finite dimensional normed vector space $(Z, \|\cdot\|)$ is complete. In particular any finite dimensional subspace of a normed vector space is automatically closed.

Proof. If $\{f_n\}_{n=1}^\infty \subset Z$ is a Cauchy sequence, then $\{f_n\}_{n=1}^\infty$ is bounded and hence has a convergent subsequence, $g_k = f_{n_k}$, by Corollary 6.48. It is now routine to show $\lim_{n \rightarrow \infty} f_n = f := \lim_{k \rightarrow \infty} g_k$. ■

6.7 Connectedness

Definition 6.50. (X, τ) is **disconnected** if there exist non-empty open sets U and V of X such that $U \cap V = \emptyset$ and $X = U \cup V$. We say $\{U, V\}$ is a **disconnection** of X . The topological space (X, τ) is called **connected** if it is not disconnected, i.e. if there is no disconnection of X . If $A \subset X$ we say A is connected iff (A, τ_A) is connected where τ_A is the relative topology on A . Explicitly, A is disconnected in (X, τ) iff there exists $U, V \in \tau$ such that $U \cap A \neq \emptyset$, $U \cap A \neq \emptyset$, $A \cap U \cap V = \emptyset$ and $A \subset U \cup V$.

The reader should check that the following statement is an equivalent definition of connectivity. A topological space (X, τ) is connected iff the only sets $A \subset X$ which are both open and closed are the sets X and $\emptyset$. This version of the definition is often used in practice.

Remark 6.51. Let $A \subset Y \subset X$. Then A is connected in X iff A is connected in Y .

Proof. Since

$$\tau_A := \{V \cap A : V \subset X\} = \{V \cap A \cap Y : V \subset X\} = \{U \cap A : U \subset_o Y\},$$

the relative topology on A inherited from X is the same as the relative topology on A inherited from Y . Since connectivity is a statement about the relative topologies on A , A is connected in X iff A is connected in Y . ■

Theorem 6.52 (The Connected Subsets of $\mathbb{R}$). *The connected subsets of $\mathbb{R}$ are intervals.*

Proof. Suppose that $A \subset \mathbb{R}$ is a connected subset and that $a, b \in A$ with $a < b$. If there exists $c \in (a, b)$ such that $c \notin A$, then $U := (-\infty, c) \cap A$ and $V := (c, \infty) \cap A$ would form a disconnection of A . Hence $(a, b) \subset A$. Let $\alpha := \inf(A)$ and $\beta := \sup(A)$ and choose $\alpha_n, \beta_n \in A$ such that $\alpha_n < \beta_n$ and $\alpha_n \downarrow \alpha$ and $\beta_n \uparrow \beta$ as $n \rightarrow \infty$. By what we have just shown, $(\alpha_n, \beta_n) \subset A$ for all n and hence $(\alpha, \beta) = \cup_{n=1}^{\infty} (\alpha_n, \beta_n) \subset A$. From this it follows that $A = (\alpha, \beta)$, $[\alpha, \beta)$, $(\alpha, \beta]$ or $[\alpha, \beta]$, i.e. A is an interval.

Conversely suppose that A is a sub-interval of $\mathbb{R}$. For the sake of contradiction, suppose that $\{U, V\}$ is a disconnection of A with $a \in U$, $b \in V$. After relabelling U and V if necessary we may assume that $a < b$. Since A is an interval $[a, b] \subset A$. Let $p = \sup([a, b] \cap U)$, then because U and V are open, $a < p < b$. Now p can not be in U for otherwise $\sup([a, b] \cap U) > p$ and p can not be in V for otherwise $p < \sup([a, b] \cap U)$. From this it follows that $p \notin U \cup V$ and hence $A \neq U \cup V$ contradicting the assumption that $\{U, V\}$ is a disconnection of A .

Alternative proof of the converse. Because of Theorem 6.55 below, it suffices to assume that A is an open interval. For the sake of contradiction, suppose that $\{U, V\}$ is a disconnection of A with $a \in U$, $b \in V$. After relabelling U and V if necessary we may assume that $a < b$. Let $J_a = (\alpha, \beta)$ be the maximal open interval in U which contains a . (See Exercise ?? or the footnote to Proposition ?? for the structure of open subsets of $\mathbb{R}$.) If $\beta \in U$ we could extend J_a to the right and still be in U violating the definition of β . Moreover we can not have $\beta \in V$ because in this case J_a would not be in U . Therefore $\beta \notin U \cup V = A$ while on the other hand $a < \beta < b$ and so $\beta \in A$ as A is an interval and we have reached the desired contradiction. ■

For $x \in V$, let $a_x := \inf\{a : (a, x] \subset V\}$ and $b_x := \sup\{b : [x, b) \subset V\}$. Since V is open, $a_x < x < b_x$ and it is easily seen that $J_x := (a_x, b_x) \subset V$. Moreover if $y \in V$ and $J_x \cap J_y \neq \emptyset$, then $J_x = J_y$. The collection, $\{J_x : x \in V\}$, is at most countable since we may label each $J \in \{J_x : x \in V\}$ by choosing a rational number $r \in J$. Letting $\{J_n : n < N\}$, with $N = \infty$ allowed, be an enumeration of $\{J_x : x \in V\}$, we have $V = \coprod_{n < N} J_n$ as desired.

The following elementary but important lemma is left as an exercise to the reader.

Lemma 6.53. *Suppose that $f : X \rightarrow Y$ is a continuous map between topological spaces. Then $f(X) \subset Y$ is connected if X is connected.*

Here is a typical way these connectedness ideas are used.

Example 6.54. Suppose that $f : X \rightarrow Y$ is a continuous map between two topological spaces, the space X is connected and the space Y is “ T_1 ,” i.e. $\{y\}$ is a closed set for all $y \in Y$ as in Definition ?? below. Further assume f is locally constant, i.e. for all $x \in X$ there exists an open neighborhood V of x in X such that $f|_V$ is constant. Then f is constant, i.e. $f(X) = \{y_0\}$ for some $y_0 \in Y$. To prove this, let $y_0 \in f(X)$ and let $W := f^{-1}(\{y_0\})$. Since $\{y_0\} \subset Y$ is a closed set and since f is continuous $W \subset X$ is also closed. Since f is locally constant, W is open as well and since X is connected it follows that $W = X$, i.e. $f(X) = \{y_0\}$.

As a concrete application of this result, suppose that X is a connected open subset of $\mathbb{R}^d$ and $f : X \rightarrow \mathbb{R}$ is a C^1 – function such that $\nabla f \equiv 0$. If $x \in X$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X$, we have, for any $|v| < \varepsilon$ and $t \in [-1, 1]$, that

$$\frac{d}{dt} f(x + tv) = \nabla f(x + tv) \cdot v = 0.$$

Therefore $f(x + v) = f(x)$ for all $|v| < \varepsilon$ and this shows f is locally constant. Hence, by what we have just proved, f is constant on X .

Theorem 6.55 (Properties of Connected Sets). *Let (X, τ) be a topological space.*

1. *If $B \subset X$ is a connected set and X is the disjoint union of two open sets U and V , then either $B \subset U$ or $B \subset V$.*
2. *If $A \subset X$ is connected,*
 - a) *then $\bar{A}$ is connected.*
 - b) *More generally, if A is connected and $B \subset \text{acc}(A)$ or $B \subset \text{bd}(A)$, then $A \cup B$ is connected as well. (Recall that $\text{acc}(A)$ – the set of accumulation points of A was defined in Definition ?? above. Moreover by Exercise ??, we know that $\text{acc}(A) \setminus A = \text{bd}(A) \setminus A$. What we are really showing here is that for any B such that $A \subset B \subset \bar{A}$, then B is connected.)*
3. *If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected sets such that $E_\alpha \cap E_\beta \neq \emptyset$ for all $\alpha, \beta \in A$,³ then $Y := \bigcup_{\alpha \in A} E_\alpha$ is connected as well.*

³ One may assume much less here. What we really need is for any $\alpha, \beta \in A$ there exists $\{\alpha_i\}_{i=0}^n$ in A such that $\alpha_0 = \alpha$, $\alpha_n = \beta$, and $E_{\alpha_i} \cap E_{\alpha_{i+1}} \neq \emptyset$ for all $0 \leq i < n$. Moreover if we make use of item 4. it suffices to assume that

4. Suppose $A, B \subset X$ are non-empty connected subsets of X such that $\bar{A} \cap B \neq \emptyset$, then $A \cup B$ is connected in X .
5. Every point $x \in X$ is contained in a unique maximal connected subset C_x of X and this subset is closed. The set C_x is called the **connected component** of x .

Proof.

1. Since B is the disjoint union of the relatively open sets $B \cap U$ and $B \cap V$, we must have $B \cap U = B$ or $B \cap V = B$ for otherwise $\{B \cap U, B \cap V\}$ would be a disconnection of B .
2. a) Let $Y = \bar{A}$ be equipped with the relative topology from X . Suppose that $U, V \subset_o Y$ form a disconnection of $Y = \bar{A}$. Then by 1. either $A \subset U$ or $A \subset V$. Say that $A \subset U$. Since U is both open and closed in Y , it follows that $Y = \bar{A} \subset U$. Therefore $V = \emptyset$ and we have a contradiction to the assumption that $\{U, V\}$ is a disconnection of $Y = \bar{A}$. Hence we must conclude that $Y = \bar{A}$ is connected as well.
b) Now let $Y = A \cup B$ with $B \subset \text{acc}(A)$, then

$$\bar{A}^Y = \bar{A} \cap Y = (A \cup \text{acc}(A)) \cap Y = A \cup B.$$

Because A is connected in Y , by (2a) $Y = A \cup B = \bar{A}^Y$ is also connected.

3. Let $Y := \bigcup_{\alpha \in A} E_\alpha$. By Remark 6.51, we know that E_α is connected in Y for each $\alpha \in A$. If $\{U, V\}$ were a disconnection of Y , by item (1), either $E_\alpha \subset U$ or $E_\alpha \subset V$ for all α . Let $\Lambda = \{\alpha \in A : E_\alpha \subset U\}$ then $U = \bigcup_{\alpha \in \Lambda} E_\alpha$ and $V = \bigcup_{\alpha \in A \setminus \Lambda} E_\alpha$. (Notice that neither Λ or $A \setminus \Lambda$ can be empty since U and V are not empty.) Since

$$\emptyset = U \cap V = \bigcup_{\alpha \in \Lambda, \beta \in \Lambda^c} (E_\alpha \cap E_\beta) \neq \emptyset.$$

we have reached a contradiction and hence no such disconnection exists.

4. (A good example to keep in mind here is $X = \mathbb{R}$, $A = (0, 1)$ and $B = [1, 2)$.) For sake of contradiction suppose that $\{U, V\}$ were a disconnection of $Y = A \cup B$. By item (1) either $A \subset U$ or $A \subset V$, say $A \subset U$ in which case $B \subset V$. Since $Y = A \cup B$ we must have $A = U$ and $B = V$ and so we may conclude: A and B are disjoint subsets of Y which are both open and closed. This implies

$$A = \bar{A}^Y = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = A \cup (\bar{A} \cap B)$$

and therefore

$$\emptyset = A \cap B = [A \cup (\bar{A} \cap B)] \cap B = \bar{A} \cap B \neq \emptyset$$

$$\bar{E}_{\alpha_i} \cap E_{\alpha_{i+1}} \cup E_{\alpha_i} \cap \bar{E}_{\alpha_{i+1}} \neq \emptyset \text{ for all } 0 \leq i < n.$$

which gives us the desired contradiction.

Alternative proof. Let $A' := A \cup [B \cap \bar{A}]$ so that $A \cup B = A' \cup B$. By item 2b. we know A' is still connected and since $A' \cap B \neq \emptyset$ we may now apply item 3. to finish the proof.

5. Let $\mathcal{C}$ denote the collection of connected subsets $C \subset X$ such that $x \in C$. Then by item 3., the set $C_x := \bigcup \mathcal{C}$ is also a connected subset of X which contains x and clearly this is the unique maximal connected set containing x . Since $\bar{C}_x$ is also connected by item (2) and C_x is maximal, $C_x = \bar{C}_x$, i.e. C_x is closed.

Theorem 6.56 (Intermediate Value Theorem). Suppose that (X, τ) is a connected topological space and $f : X \rightarrow \mathbb{R}$ is a continuous map. Then f satisfies the intermediate value property. Namely, for every pair $x, y \in X$ such that $f(x) < f(y)$ and $c \in (f(x), f(y))$, there exists $z \in X$ such that $f(z) = c$.

Proof. By Lemma 6.53, $f(X)$ is a connected subset of $\mathbb{R}$. So by Theorem 6.52, $f(X)$ is a subinterval of $\mathbb{R}$ and this completes the proof. ■

Definition 6.57. A topological space X is **path connected** if to every pair of points $\{x_0, x_1\} \subset X$ there exists a continuous **path**, $\sigma \in C([0, 1], X)$, such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The space X is said to be **locally path connected** if for each $x \in X$, there is an open neighborhood $V \subset X$ of x which is path connected.

Proposition 6.58. Let X be a topological space.

1. If X is path connected then X is connected.
2. If X is connected and locally path connected, then X is path connected.
3. If X is any connected open subset of $\mathbb{R}^n$, then X is path connected.

Proof. The reader is asked to prove this proposition in Exercises ?? – ?? below. ■

Proposition 6.59 (Stability of Connectedness Under Products). Let (X_α, τ_α) be connected topological spaces. Then the product space $X_A = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology is connected.

Proof. Let us begin with the case of two factors, namely assume that X and Y are connected topological spaces, then we will show that $X \times Y$ is connected as well. Given $x \in X$, let $f_x : Y \rightarrow X \times Y$ be the map $f_x(y) = (x, y)$ and notice that f_x is continuous since $\pi_X \circ f_x(y) = x$ and $\pi_Y \circ f_x(y) = y$ are continuous maps. From this we conclude that $\{x\} \times Y = f_x(Y)$ is connected by Lemma 6.53. A similar argument shows $X \times \{y\}$ is connected for all $y \in Y$.

Let $p = (x_0, y_0) \in X \times Y$ and C_p denote the connected component of p . Since $\{x_0\} \times Y$ is connected and $p \in \{x_0\} \times Y$ it follows that $\{x_0\} \times Y \subset C_p$ and hence C_p is also the connected component (x_0, y) for all $y \in Y$. Similarly, $X \times \{y\} \subset C_{(x_0, y)} = C_p$ is connected, and therefore $X \times \{y\} \subset C_p$. So we have shown $(x, y) \in C_p$ for all $x \in X$ and $y \in Y$, see Figure 6.3. By induction the theorem holds whenever A is a finite set, i.e. for products of a finite number of connected spaces.

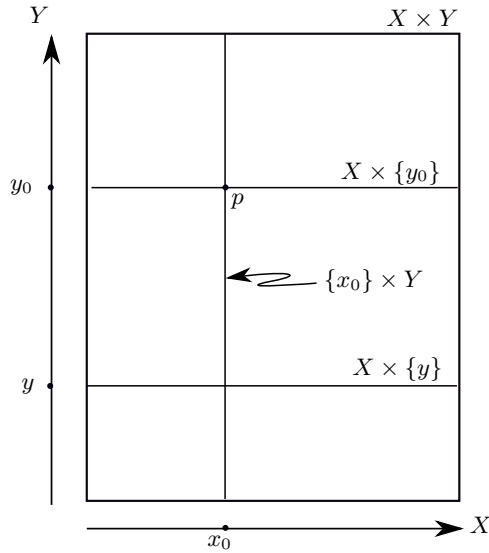


Fig. 6.3. This picture illustrates why the connected component of p in $X \times Y$ must contain all points of $X \times Y$.

For the general case, again choose a point $p \in X_A = X^A$ and again let $C = C_p$ be the connected component of p . Recall that C_p is closed and therefore if C_p is a proper subset of X_A , then $X_A \setminus C_p$ is a non-empty open set. By the definition of the product topology, this would imply that $X_A \setminus C_p$ contains a non-empty open set of the form

$$V := \cap_{\alpha \in \Lambda} \pi_\alpha^{-1}(V_\alpha) = V_\Lambda \times X_{A \setminus \Lambda} \subset X_A \setminus C_p \quad (6.8)$$

where $\Lambda \subset \subset A$ and $V_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda$.

On the other hand, let $\varphi : X_A \rightarrow X_A$ by $\varphi(y) = x$ where

$$x_\alpha = \begin{cases} y_\alpha & \text{if } \alpha \in \Lambda \\ p_\alpha & \text{if } \alpha \notin \Lambda. \end{cases}$$

If $\alpha \in \Lambda$, $\pi_\alpha \circ \varphi(y) = y_\alpha = \pi_\alpha(y)$ and if $\alpha \in A \setminus \Lambda$ then $\pi_\alpha \circ \varphi(y) = p_\alpha$ so that in every case $\pi_\alpha \circ \varphi : X_A \rightarrow X_\alpha$ is continuous and therefore φ is continuous. Since

X_A is a product of a finite number of connected spaces and so is connected and thus so is the continuous image, $\varphi(X_A) = X_A \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda} \subset X_A$. Since $p \in \varphi(X_A)$ we must have

$$X_A \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda} \subset C_p. \quad (6.9)$$

Hence it follows from Eqs. (6.8) and (6.9) that

$$V_\Lambda \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda} = \left(X_A \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda} \right) \cap V \subset C_p \cap [X_A \setminus C_p] = \emptyset$$

which is a contradiction since $V_\Lambda \times \{p_\alpha\}_{\alpha \in A \setminus \Lambda} \neq \emptyset$. ■

A

Appendix: Notation and Logic

The following abbreviations along with their negations are used throughout these notes.

Symbol	Meaning	Negation
$\forall$	for all	$\exists$
$\exists$	there exists	$\forall$
, or “space”	then	$\ni$
$\ni$	such that	, or “space”
a.a.	almost always	i.o.
i.o.	infinitely often	a.a.
$=$	equals	$\neq$
$\neq$	not equals	$=$
$\leq$	less than or equal	$>$
$>$	greater than	$\leq$

Here are some examples.

1. $a_n = b_n$ i.o. $n \iff \#(\{n : a_n = b_n\}) = \infty$. The negation of $\#(\{n : a_n = b_n\}) = \infty$ is

$$\#(\{n : a_n = b_n\}) < \infty \iff a_n \neq b_n \text{ for a.a. } n.$$

2. $\lim_{n \rightarrow \infty} a_n = L$ is by definition the statement;

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \ni \forall n \geq N, |L - a_n| \leq \varepsilon.$$

This may also be written as

$$\forall \varepsilon > 0, |L - a_n| \leq \varepsilon \text{ for a.a. } n.$$

3. The negation of the previous statement is $\lim_{n \rightarrow \infty} a_n \neq L$ which translates to

$$\exists \varepsilon > 0 \ni \forall N \in \mathbb{N}, \exists n \geq N \ni |L - a_n| > \varepsilon.$$

This last statement is also equivalent to;

$$\exists \varepsilon > 0 \ni |L - a_n| > \varepsilon \text{ i.o. } n.$$

It is sometimes useful to reformulate this last statement as; there exists $\varepsilon > 0$ and an increasing function $\mathbb{N} \ni k \rightarrow n_k \in \mathbb{N}$ such that

$$|L - a_{n_k}| > \varepsilon \text{ for all } k \in \mathbb{N}.$$

B

Appendix: More Set Theoretic Properties (highly optional)

B.1 Appendix: Zorn's Lemma and the Hausdorff Maximal Principle (optional)

Definition B.1. A partial order $\leq$ on X is a relation with following properties;

1. If $x \leq y$ and $y \leq z$ then $x \leq z$.
2. If $x \leq y$ and $y \leq x$ then $x = y$.
3. $x \leq x$ for all $x \in X$.

Example B.2. Let Y be a set and $X = 2^Y$. There are two natural partial orders on X .

1. Ordered by inclusion, $A \leq B$ is $A \subset B$ and
2. Ordered by reverse inclusion, $A \leq B$ if $B \subset A$.

Definition B.3. Let $(X, \leq)$ be a partially ordered set we say X is **linearly** or **totally** ordered if for all $x, y \in X$ either $x \leq y$ or $y \leq x$. The real numbers $\mathbb{R}$ with the usual order $\leq$ is a typical example.

Definition B.4. Let $(X, \leq)$ be a partial ordered set. We say $x \in X$ is a **maximal** element if for all $y \in X$ such that $y \geq x$ implies $y = x$, i.e. there is no element larger than x . An **upper bound** for a subset E of X is an element $x \in X$ such that $x \geq y$ for all $y \in E$.

Example B.5. Let

$$X = \{a = \{1\} \ b = \{1, 2\} \ c = \{3\} \ d = \{2, 4\} \ e = \{2\} \}$$

ordered by set inclusion. Then b and d are maximal elements despite that fact that $b \not\leq d$ and $d \not\leq b$. We also have;

1. If $E = \{a, c, e\}$, then E has **no** upper bound.
2. If $E = \{a, e\}$, then b is an upper bound.
3. If $E = \{e\}$, then b and d are upper bounds.

Theorem B.6. The following are equivalent.

1. **The axiom of choice:** to each collection, $\{X_\alpha\}_{\alpha \in A}$, of non-empty sets there exists a “choice function,” $x : A \rightarrow \prod_{\alpha \in A} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for all $\alpha \in A$, i.e. $\prod_{\alpha \in A} X_\alpha \neq \emptyset$.

2. **The Hausdorff Maximal Principle:** Every partially ordered set has a **maximal** (relative to the inclusion order) linearly ordered subset.

3. **Zorn's Lemma:** If X is partially ordered set such that every linearly ordered subset of X has an upper bound, then X has a maximal element.¹

Proof. ($2 \Rightarrow 3$) Let X be a partially ordered subset as in 3 and let $\mathcal{F} = \{E \subset X : E \text{ is linearly ordered}\}$ which we equip with the inclusion partial ordering. By 2. there exist a maximal element $E \in \mathcal{F}$. By assumption, the linearly ordered set E has an upper bound $x \in X$. The element x is maximal, for if $y \in Y$ and $y \geq x$, then $E \cup \{y\}$ is still an linearly ordered set containing E . So by maximality of E , $E = E \cup \{y\}$, i.e. $y \in E$ and therefore $y \leq x$ showing which combined with $y \geq x$ implies that $y = x$.²

($3 \Rightarrow 1$) Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of non-empty sets, we must show $\prod_{\alpha \in A} X_\alpha$ is not empty. Let $\mathcal{G}$ denote the collection of functions $g : D(g) \rightarrow \prod_{\alpha \in A} X_\alpha$ such that $D(g)$ is a subset of A , and for all $\alpha \in D(g)$, $g(\alpha) \in X_\alpha$. Notice that $\mathcal{G}$ is not empty, for we may let $\alpha_0 \in A$ and $x_0 \in X_{\alpha_0}$ and then set $D(g) = \{\alpha_0\}$ and $g(\alpha_0) = x_0$ to construct an element of $\mathcal{G}$. We now put a partial order on $\mathcal{G}$ as follows. We say that $f \leq g$ for $f, g \in \mathcal{G}$ provided that $D(f) \subset D(g)$ and $f = g|_{D(f)}$. If $\Phi \subset \mathcal{G}$ is a linearly ordered set, let $D(h) = \bigcup_{g \in \Phi} D(g)$ and for $\alpha \in D(g)$ let $h(\alpha) = g(\alpha)$. Then $h \in \mathcal{G}$ is an upper bound for Φ . So by Zorn's

¹ If X is a countable set we may prove Zorn's Lemma by induction. Let $\{x_n\}_{n=1}^\infty$ be an enumeration of X , and define $E_n \subset X$ inductively as follows. For $n = 1$ let $E_1 = \{x_1\}$, and if E_n have been chosen, let $E_{n+1} = E_n \cup \{x_{n+1}\}$ if x_{n+1} is an upper bound for E_n otherwise let $E_{n+1} = E_n$. The set $E = \bigcup_{n=1}^\infty E_n$ is a linearly ordered (you check) subset of X and hence by assumption E has an upper bound, $x \in X$. I claim that his element is maximal, for if there exists $y = x_m \in X$ such that $y \geq x$, then x_m would be an upper bound for E_{m-1} and therefore $y = x_m \in E_m \subset E$. That is to say if $y \geq x$, then $y \in E$ and hence $y \leq x$, so $y = x$. (Hence we may view Zorn's lemma as a “jazzed” up version of induction.)

² Similarly one may show that $3 \Rightarrow 2$. Let $\mathcal{F} = \{E \subset X : E \text{ is linearly ordered}\}$ and order $\mathcal{F}$ by inclusion. If $\mathcal{M} \subset \mathcal{F}$ is linearly ordered, let $E = \bigcup_{A \in \mathcal{M}} A$. If $x, y \in E$ then $x \in A$ and $y \in B$ for some $A, B \in \mathcal{M}$. Now $\mathcal{M}$ is linearly ordered by set inclusion so $A \subset B$ or $B \subset A$ i.e. $x, y \in A$ or $x, y \in B$. Since A and B are linearly order we must have either $x \leq y$ or $y \leq x$, that is to say E is linearly ordered. Hence by 3. there exists a maximal element $E \in \mathcal{F}$ which is the assertion in 2.

Lemma there exists a maximal element $h \in \mathcal{G}$. To finish the proof we need only show that $D(h) = A$. If this were not the case, then let $\alpha_0 \in A \setminus D(h)$ and $x_0 \in X_{\alpha_0}$. We may now define $D(\tilde{h}) = D(h) \cup \{\alpha_0\}$ and

$$\tilde{h}(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in D(h) \\ x_0 & \text{if } \alpha = \alpha_0. \end{cases}$$

Then $h \leq \tilde{h}$ while $h \neq \tilde{h}$ violating the fact that h was a maximal element.

(1 $\Rightarrow$ 2) Let $(X, \leq)$ be a partially ordered set. Let $\mathcal{F}$ be the collection of linearly ordered subsets of X which we order by set inclusion. Given $x_0 \in X$, $\{x_0\} \in \mathcal{F}$ is linearly ordered set so that $\mathcal{F} \neq \emptyset$. Fix an element $P_0 \in \mathcal{F}$. If P_0 is not maximal there exists $P_1 \in \mathcal{F}$ such that $P_0 \subsetneq P_1$. In particular we may choose $x \notin P_0$ such that $P_0 \cup \{x\} \in \mathcal{F}$. The idea now is to keep repeating this process of adding points $x \in X$ until we construct a maximal element P of $\mathcal{F}$. We now have to take care of some details. We may assume with out loss of generality that $\tilde{\mathcal{F}} = \{P \in \mathcal{F} : P \text{ is not maximal}\}$ is a non-empty set. For $P \in \tilde{\mathcal{F}}$, let $P^* = \{x \in X : P \cup \{x\} \in \mathcal{F}\}$. As the above argument shows, $P^* \neq \emptyset$ for all $P \in \tilde{\mathcal{F}}$. Using the axiom of choice, there exists $f \in \prod_{P \in \tilde{\mathcal{F}}} P^*$. We now define $g : \mathcal{F} \rightarrow \mathcal{F}$ by

$$g(P) = \begin{cases} P & \text{if } P \text{ is maximal} \\ P \cup \{f(x)\} & \text{if } P \text{ is not maximal.} \end{cases} \quad (\text{B.1})$$

The proof is completed by Lemma B.7 below which shows that g must have a fixed point $P \in \mathcal{F}$. This fixed point is maximal by construction of g . ■

Lemma B.7. *The function $g : \mathcal{F} \rightarrow \mathcal{F}$ defined in Eq. (B.1) has a fixed point.*³

Proof. The **idea of the proof** is as follows. Let $P_0 \in \mathcal{F}$ be chosen arbitrarily. Notice that $\Phi = \{g^{(n)}(P_0)\}_{n=0}^{\infty} \subset \mathcal{F}$ is a linearly ordered set and it is therefore easily verified that $P_1 = \bigcup_{n=0}^{\infty} g^{(n)}(P_0) \in \mathcal{F}$. Similarly we may repeat the process to construct $P_2 = \bigcup_{n=0}^{\infty} g^{(n)}(P_1) \in \mathcal{F}$ and $P_3 = \bigcup_{n=0}^{\infty} g^{(n)}(P_2) \in \mathcal{F}$, etc. etc. Then take $P_{\infty} = \bigcup_{n=0}^{\infty} P_n$ and start again with P_0 replaced by P_{∞} . Then keep going this way until eventually the sets stop increasing in size, in which case we have found our fixed point. The problem with this strategy is that we may never win. (This is very reminiscent of constructing measurable sets and the way out is to use measure theoretic like arguments.)

Let us now start the **formal proof**. Again let $P_0 \in \mathcal{F}$ and let $\mathcal{F}_1 = \{P \in \mathcal{F} : P_0 \subset P\}$. Notice that $\mathcal{F}_1$ has the following properties:

1. $P_0 \in \mathcal{F}_1$.
2. If $\Phi \subset \mathcal{F}_1$ is a totally ordered (by set inclusion) subset then $\bigcup \Phi \in \mathcal{F}_1$.
3. If $P \in \mathcal{F}_1$ then $g(P) \in \mathcal{F}_1$.

Let us call a general subset $\mathcal{F}' \subset \mathcal{F}$ satisfying these three conditions a tower and let

$$\mathcal{F}_0 = \bigcap \{\mathcal{F}' : \mathcal{F}' \text{ is a tower}\}.$$

Standard arguments show that $\mathcal{F}_0$ is still a tower and clearly is the smallest tower containing P_0 . (Morally speaking $\mathcal{F}_0$ consists of all of the sets we were trying to constructed in the “idea section” of the proof.) We now claim that $\mathcal{F}_0$ is a linearly ordered subset of $\mathcal{F}$. To prove this let $\Gamma \subset \mathcal{F}_0$ be the linearly ordered set

$$\Gamma = \{C \in \mathcal{F}_0 : \text{for all } A \in \mathcal{F}_0 \text{ either } A \subset C \text{ or } C \subset A\}.$$

Shortly we will show that $\Gamma \subset \mathcal{F}_0$ is a tower and hence that $\mathcal{F}_0 = \Gamma$. That is to say $\mathcal{F}_0$ is linearly ordered. Assuming this for the moment let us finish the proof.

Let $P \equiv \bigcup \mathcal{F}_0$ which is in $\mathcal{F}_0$ by property 2 and is clearly the largest element in $\mathcal{F}_0$. By 3. it now follows that $P \subset g(P) \in \mathcal{F}_0$ and by maximality of P , we have $g(P) = P$, the desired fixed point. So to finish the proof, we must show that Γ is a tower. First off it is clear that $P_0 \in \Gamma$ so in particular Γ is not empty. For each $C \in \Gamma$ let

$$\Phi_C := \{A \in \mathcal{F}_0 : \text{either } A \subset C \text{ or } g(C) \subset A\}.$$

We will begin by showing that $\Phi_C \subset \mathcal{F}_0$ is a tower and therefore that $\Phi_C = \mathcal{F}_0$. 1. $P_0 \in \Phi_C$ since $P_0 \subset C$ for all $C \in \Gamma \subset \mathcal{F}_0$. 2. If $\Phi \subset \Phi_C \subset \mathcal{F}_0$ is totally ordered by set inclusion, then $A_{\Phi} := \bigcup \Phi \in \mathcal{F}_0$. We must show $A_{\Phi} \in \Phi_C$, that is that $A_{\Phi} \subset C$ or $C \subset A_{\Phi}$. Now if $A \subset C$ for all $A \in \Phi$, then $A_{\Phi} \subset C$ and hence $A_{\Phi} \in \Phi_C$. On the other hand if there is some $A \in \Phi$ such that $g(C) \subset A$ then clearly $g(C) \subset A_{\Phi}$ and again $A_{\Phi} \in \Phi_C$. 3. Given $A \in \Phi_C$ we must show $g(A) \in \Phi_C$, i.e. that

$$g(A) \subset C \text{ or } g(C) \subset g(A). \quad (\text{B.2})$$

There are three cases to consider: either $A \subsetneq C$, $A = C$, or $g(C) \subset A$. In the case $A = C$, $g(C) = g(A) \subset g(A)$ and if $g(C) \subset A$ then $g(C) \subset A \subset g(A)$ and Eq. (B.2) holds in either of these cases. So assume that $A \subsetneq C$. Since $C \in \Gamma$, either $g(A) \subset C$ (in which case we are done) or $C \subset g(A)$. Hence we may assume that

$$A \subsetneq C \subset g(A).$$

Now if C were a proper subset of $g(A)$ it would then follow that $g(A) \setminus A$ would consist of at least two points which contradicts the definition of g . Hence we

³ Here is an easy proof if the elements of $\mathcal{F}$ happened to all be finite sets and there existed a set $P \in \mathcal{F}$ with a maximal number of elements. In this case the condition that $P \subset g(P)$ would imply that $P = g(P)$, otherwise $g(P)$ would have more elements than P .

must have $g(A) = C \subset C$ and again Eq. (B.2) holds, so Φ_C is a tower. It is now easy to show Γ is a tower. It is again clear that $P_0 \in \Gamma$ and Property 2. may be checked for Γ in the same way as it was done for Φ_C above. For Property 3., if $C \in \Gamma$ we may use $\Phi_C = \mathcal{F}_0$ to conclude for all $A \in \mathcal{F}_0$, either $A \subset C \subset g(C)$ or $g(C) \subset A$, i.e. $g(C) \in \Gamma$. Thus Γ is a tower and we are done. ■

B.2 Cardinality

In mathematics, the essence of counting a set and finding a result n , is that it establishes a one to one correspondence (or bijection) of the set with the set of numbers $\{1, 2, \dots, n\}$. A fundamental fact, which can be proved by mathematical induction, is that no bijection can exist between $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\}$ unless $n = m$; this fact (together with the fact that two bijections can be composed to give another bijection) ensures that counting the same set in different ways can never result in different numbers (unless an error is made). This is the fundamental mathematical theorem that gives counting its purpose; however you count a (finite) set, the answer is the same. In a broader context, the theorem is an example of a theorem in the mathematical field of (finite) combinatorics—hence (finite) combinatorics is sometimes referred to as “the mathematics of counting.”

Many sets that arise in mathematics do not allow a bijection to be established with $\{1, 2, \dots, n\}$ for any natural number n ; these are called infinite sets, while those sets for which such a bijection does exist (for some n) are called finite sets. Infinite sets cannot be counted in the usual sense; for one thing, the mathematical theorems which underlie this usual sense for finite sets are false for infinite sets. Furthermore, different definitions of the concepts in terms of which these theorems are stated, while equivalent for finite sets, are inequivalent in the context of infinite sets.

The notion of counting may be extended to them in the sense of establishing (the existence of) a bijection with some well understood set. For instance, if a set can be brought into bijection with the set of all natural numbers, then it is called “countably infinite.” This kind of counting differs in a fundamental way from counting of finite sets, in that adding new elements to a set does not necessarily increase its size, because the possibility of a bijection with the original set is not excluded. For instance, the set of all integers (including negative numbers) can be brought into bijection with the set of natural numbers, and even seemingly much larger sets like that of all finite sequences of rational numbers are still (only) countably infinite. Nevertheless there are sets, such as the set of real numbers, that can be shown to be “too large” to admit a bijection with the natural numbers, and these sets are called “uncountable.” Sets for which there exists a bijection between them are said to have the same cardinality, and in the most general sense counting a set can be taken to mean determining its

cardinality. Beyond the cardinalities given by each of the natural numbers, there is an infinite hierarchy of infinite cardinalities, although only very few such cardinalities occur in ordinary mathematics (that is, outside set theory that explicitly studies possible cardinalities).

Counting, mostly of finite sets, has various applications in mathematics. One important principle is that if two sets X and Y have the same finite number of elements, and a function $f : X \rightarrow Y$ is known to be injective, then it is also surjective, and vice versa. A related fact is known as the pigeonhole principle, which states that if two sets X and Y have finite numbers of elements n and m with $n > m$, then any map $f : X \rightarrow Y$ is not injective (so there exist two distinct elements of X that f sends to the same element of Y); this follows from the former principle, since if f were injective, then so would its restriction to a strict subset S of X with m elements, which restriction would then be surjective, contradicting the fact that for x in X outside S , $f(x)$ cannot be in the image of the restriction. Similar counting arguments can prove the existence of certain objects without explicitly providing an example. In the case of infinite sets this can even apply in situations where it is impossible to give an example; for instance there must exist real numbers that are not computable numbers, because the latter set is only countably infinite, but by definition a non-computable number cannot be precisely specified.

The domain of enumerative combinatorics deals with computing the number of elements of finite sets, without actually counting them; the latter usually being impossible because infinite families of finite sets are considered at once, such as the set of permutations of $\{1, 2, \dots, n\}$ for any natural number n .

B.3 Formalities of Counting

Definition B.8. We say $\text{card}(X) \leq \text{card}(Y)$ if there exists an injective map, $f : X \rightarrow Y$ and $\text{card}(Y) \geq \text{card}(X)$ if there exists a surjective map $g : Y \rightarrow X$. We say $\text{card}(X) = \text{card}(Y)$ if there exists a bijections, $f : X \rightarrow Y$.

Proposition B.9. We have $\text{card}(X) \leq \text{card}(Y)$ iff $\text{card}(Y) \geq \text{card}(X)$.

Proof. If $f : X \rightarrow Y$ is an injective map, define $g : Y \rightarrow X$ by $g|_{f(X)} = f^{-1}$ and $g|_{Y \setminus f(X)} = x_0 \in X$ chosen arbitrarily. Then $g : Y \rightarrow X$ is surjective.

If $g : Y \rightarrow X$ is a surjective map, then $Y_x := g^{-1}(\{x\}) \neq \emptyset$ for all $x \in X$ and so by the axiom of choice there exists $f \in \prod_{x \in X} Y_x$. Thus $f : X \rightarrow Y$ such that $f(x) \in Y_x$ for all x . As the $\{Y_x\}_{x \in X}$ are pairwise disjoint, it follows that f is injective. ■

Theorem B.10 (Schröder-Bernstein Theorem). If $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$, then $\text{card}(X) = \text{card}(Y)$. Stated more explicitly; if there exists injective maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$, then there exists a bijective map, $h : X \rightarrow Y$.

Proof. Starting with an $x \in X$ we may form the sequence of “ancestors” of x , namely $\text{ancestor}(x) := (x, y_1, x_1, y_2, \dots)$ where $y_1 = g^{-1}(x)$, $x_1 = f^{-1}(y_1)$, $y_2 = g^{-1}(x_1), \dots$, i.e.

$$x \xrightarrow{g^{-1}} y_1 \xrightarrow{f^{-1}} x_1 \xrightarrow{g^{-1}} y_2 \xrightarrow{f^{-1}} \dots$$

We continue this process of inverse iterates as long as it is possible, i.e. we can construct y_{n+1} if $x_n \in g(Y)$ and x_{n+1} if $y_{n+1} \in f(X)$. There are now three possibilities;

1. $\text{ancestor}(x)$ has infinite length so the process never gets stuck in which case we say $x \in X_\infty$, read as start in X and end never get stuck.
2. $\text{ancestor}(x)$ is finite and the last term in the sequence is in X , in which case we say $x \in X_X$ (read as start in X and end in (get stuck in) X).
3. $\text{ancestor}(x)$ is finite and the last term in the sequence is in Y , in which case we say $x \in X_Y$ (read as start in X and end in (get stuck in) Y).

In this way we partition X into three disjoint sets, X_∞ , X_X , and X_Y . Similarly we may partition Y into Y_∞ , Y_Y , and Y_X . Let us now observe that,

1. $f(X_\infty) = Y_\infty$. Indeed if $x \in X_\infty$ then $\text{ancestor}(f(x)) = (x, \text{ancestor}(x))$ is an infinite sequence, i.e. $f(x) \in Y_\infty$. Moreover if $y \in Y_\infty$, then $\text{ancestor}(y) = (y, \text{ancestor}(x))$ where $f(x) = y$ so that $x \in X_\infty$ and $y \in f(X_\infty)$. Thus we have shown $f : X_\infty \rightarrow Y_\infty$ is a bijection, i.e. $\text{card}(X_\infty) = \text{card}(Y_\infty)$,
2. $f(X_X) = Y_X$. Indeed if $x \in X_X$ then again $\text{ancestor}(f(x)) = (x, \text{ancestor}(x))$ which ends in X so that $f(x) \in Y_X$. Moreover if $y \in Y_X$, then $\text{ancestor}(y) = (y, \text{ancestor}(x))$ where $f(x) = y$ so that $\text{ancestor}(x)$ ends in X , i.e. $y \in f(X_X)$. Thus we have shown $f : X_X \rightarrow Y_X$ is a bijection, i.e. $\text{card}(X_X) = \text{card}(Y_X)$.
3. By the same argument as in item 2. it follow that $g : Y_Y \rightarrow X_Y$ is a bijection, i.e. $\text{card}(X_Y) = \text{card}(Y_Y)$.

The last three statement implies $\text{card}(X) = \text{card}(Y)$. We may in fact define a bijection, $h : X \rightarrow Y$, by

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_\infty \cup X_X \\ g|_{Y_Y}^{-1}(x) & \text{if } x \in X_Y \end{cases}.$$

■

Definition B.11. We say $\text{card}(X) < \text{card}(Y)$ if $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(X) \neq \text{card}(Y)$, i.e. $\text{card}(X) < \text{card}(Y)$ if there exists an injective map, $f : X \rightarrow Y$, but not bijective map exists. Similarly we say $\text{card}(Y) > \text{card}(X)$ if $\text{card}(Y) \geq \text{card}(X)$ and $\text{card}(Y) \neq \text{card}(X)$, i.e. $\text{card}(Y) > \text{card}(X)$ if there exists a surjective map $g : Y \rightarrow X$ but no bijective map exists.

Proposition B.12. For any non-empty set X , $\text{card}(X) < \text{card}(2^X)$.

Proof. Define $f : X \rightarrow 2^X$ by $f(x) = \{x\}$. Then f is an injective map and hence $\text{card}(X) \leq \text{card}(2^X)$. Now suppose that $g : X \rightarrow 2^X$ is any map. Let $X_0 = \{x \in X : x \notin g(x)\} \subset X$. I claim that $X_0 \notin g(X)$.

Indeed suppose there exists $x_0 \in X$ such that $g(x_0) = X_0$. If $x_0 \in X_0$, then $x_0 \notin g(x_0) = X_0$ which is impossible. Similarly if $x_0 \notin X_0 = g(x_0)$ then $x_0 \in X_0$ and again we have reached a contradiction. Thus we must conclude that $X_0 \notin g(X)$. Thus there are no surjective maps, $g : X \rightarrow 2^X$ so that $\text{card}(X) \neq \text{card}(2^X)$. ■

Proposition B.13. If $\text{card}(X) < \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(Z)$, then $\text{card}(X) < \text{card}(Z)$.

Proof. If there exists an injective map, $f : Z \rightarrow X$ then composing this with an injective map, $g : X \rightarrow Y$ gives an injective map, $g \circ f : Z \rightarrow Y$ and there for $\text{card}(Z) \leq \text{card}(Y)$. But this would imply that $\text{card}(X) = \text{card}(Z)$. ■

Definition B.14. Let $A_n := \{1, 2, \dots, n\}$ for all $n \in \mathbb{N}$ and write n for $\text{card}(A_n)$.

Proposition B.15. We have $\text{card}(A_m) < \text{card}(A_n)$ for all $m < n$. Moreover if $\emptyset \neq X \subsetneq A_n$ then $\text{card}(X) = \text{card}(A_k)$ for some $k < n$.

Proof. If $f : A_1 \rightarrow A_2$, then either $f(1) = 1$ or $f(1) = 2$. In either case f is injective but not bijective so that $\text{card}(A_2) < \text{card}(A_1)$. Let S_n be the statement that $\text{card}(A_k) < \text{card}(A_l)$ for all $1 \leq k < l \leq n$ and for any proper subset $X \subset A_n$ we have $\text{card}(X) = \text{card}(A_m)$ for some $m < n$. Then we have just shown that S_2 is true. So suppose that S_n is now true. As $f : A_k \rightarrow A_l$ defined by $f(m) = m$ for all $m \in A_k$ is a injection when $k < l$ we always have $\text{card}(A_k) \leq \text{card}(A_l)$. Now suppose that $\text{card}(A_k) = \text{card}(A_{n+1})$ for some $k \leq n$. Then there exists a bijection, $f : A_{n+1} \rightarrow A_k$. In this case $f(A_n)$ is a proper subset of A_k and therefore $\text{card}(f(A_n)) < \text{card}(A_k)$ but on the other hand $\text{card}(f(A_n)) = \text{card}(A_n) \geq \text{card}(A_k)$ which is a contradiction. So no such bijection can exists and we have shown $\text{card}(A_k) < \text{card}(A_{n+1})$ for all $k \leq n$. Finally suppose that $X \subset A_{n+1}$ is proper subset. If $X \subset A_n$ then $\text{card}(X) = \text{card}(A_k)$ for some $k \leq n$ by the induction hypothesis. On the other hand if $n+1 \in X$, let $X' := X \setminus \{n+1\} \subsetneq A_n$. Therefore by the induction hypothesis $\text{card}(X') = \text{card}(A_k)$ for some $k < n$. It is then clear that $\text{card}(X) = \text{card}(A_{k+1})$ where $k+1 < n$, indeed we map $X := X' \cup \{n+1\} \rightarrow A_k \cup \{k+1\} = A_{k+1}$. ■

Example B.16. $\text{card}(A_n \setminus \{k\}) = n - 1$ for $k \in A_n$. Indeed, let $f : A_{n-1} \rightarrow A_n \setminus \{k\}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x < k \\ x+1 & \text{if } x \geq k \end{cases}.$$

Then f is the desired bijection. More generally if $X \subset Y$ and $\text{card}(X) = m < n = \text{card}(Y)$, then $\text{card}(Y \setminus X) = n - m$ and if X and Y are finite disjoint sets then $\text{card}(X \cup Y) = \text{card}(X) + \text{card}(Y)$. Similarly, $\text{card}(X \times Y) = \text{card}(X) \cdot \text{card}(Y)$.

Proposition B.17. *If $f : A_n \rightarrow A_n$ is a map, then the following are equivalent,*

1. f is injective,
2. f is surjective,
3. f is bijective.

Moreover $\text{card}(\text{Bijec}(A_n)) = n!$.

Proof. If $n = 1$, the only map $f : A_1 \rightarrow A_1$ is $f(1) = 1$. So in this case there is nothing to prove. So now suppose the proposition holds for level n and $f : A_{n+1} \rightarrow A_{n+1}$ is a given map.

If $f : A_{n+1} \rightarrow A_{n+1}$ is an injective map and $f(A_{n+1})$ is a proper subset of A_{n+1} , then $\text{card}(A_{n+1}) < \text{card}(f(A_{n+1})) = \text{card}(A_{n+1})$ which is absurd. Thus f is injective implies f is surjective.

Conversely suppose that $f : A_{n+1} \rightarrow A_{n+1}$ is surjective. Let $g : A_{n+1} \rightarrow A_{n+1}$ be a right inverse, i.e. $f \circ g = \text{id}$, which is necessarily injective, see the proof of Proposition B.9. By the pervious paragraph we know that g is necessarily surjective and therefore $f = g^{-1}$ is a bijection.

It now only remains to prove $\text{card}(\text{Bijec}(A_n)) = n!$ which we again do by induction. For $n = 1$ the result is clear. So suppose it holds at level n . If $f : A_{n+1} \rightarrow A_{n+1}$ is a bijection. Given $1 \leq k \leq n+1$ let

$$\text{Bij}_k(A_{n+1}) := \{f \in \text{Bij}(A_{n+1}) : f(n+1) = k\}.$$

For $f \in \text{Bij}_k(A_{n+1})$, we have $f : A_n \rightarrow A_{n+1} \setminus \{k\} \cong A_n$ is a bijection. Thus $\text{Bij}_k(A_{n+1}) \cong \text{Bij}(A_n)$ and

$$\text{Bij}(A_{n+1}) = \sum_{k=1}^{n+1} \text{Bij}_k(A_{n+1})$$

we have

$$\begin{aligned} \text{card}(\text{Bij}(A_{n+1})) &= \sum_{k=1}^{n+1} \text{card}(\text{Bij}_k(A_{n+1})) \\ &= \sum_{k=1}^{n+1} \text{card}(\text{Bij}(A_n)) = \sum_{k=1}^{n+1} n! \\ &= (n+1)n! = (n+1)!. \end{aligned}$$

■

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