1. (9/26/03)

1.1. **Introduction.** For our purposes the definition of complex variables is the calculus of **analytic functions**, where a function $F(x, y) = (u(x, y), v(x, y))$ from $\mathbb{R}^2$ to itself is analytic iff it satisfies the Cauchy Riemann equations:

$$u_x = -v_y \text{ and } v_x = u_y.$$

Because this class of functions is so restrictive, the associated calculus has some very beautiful and useful properties which will be explained in this class. The following fact makes the subject useful in applications.

Fact 1.1. Many of the common elementary functions, like x^n, e^x, $\sin x$, $\tan x$, $\ln x$, etc. have **unique** "extensions" to analytic functions. Moreover, the solutions to many ordinary differential equations extend to analytic functions. So the study of analytic functions aids in understanding these class of real valued functions.

1.2. **Book Sections 1-5.**

Definition 1.2 (Complex Numbers)**.** Let $\mathbb{C} = \mathbb{R}^2$ equipped with multiplication rule

$$(a, b)(c, d) \equiv (ac - bd, bc + ad) \tag{1.1}$$

and the usual rule for vector addition. As is standard we will write $0 = (0, 0)$, $1 = (1, 0)$ and $i = (0, 1)$ so that every element z of $\mathbb{C}$ may be written as $z = x1 + yi$ which in the future will be written simply as $z = x + iy$. If $z = x + iy$, let $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

Writing $z = a + ib$ and $w = c + id$, the multiplication rule in Eq. (1.1) becomes

$$(a + ib)(c + id) \equiv (ac - bd) + i(bc + ad) \tag{1.2}$$

and in particular $1^2 = 1$ and $i^2 = -1$.

Proposition 1.3. *The complex numbers $\mathbb{C}$ with the above multiplication rule satisfies the usual definitions of a field. For example $wz = zw$ and $z(w_1 + w_2) = zw_1 + zw_2$, etc. Moreover if $z \neq 0$, z has a multiplicative inverse given by*

$$z^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}. \tag{1.3}$$

Probably the most painful thing to check directly is the associative law, namely

$$u(vw) = (uv)w. \tag{1.4}$$

This can be checked later in polar form easier.

Proof. Suppose $z = a + ib \neq 0$, we wish to find $w = c + id$ such that $zw = 1$ and this happens by Eq. (1.2) iff

$$ac - bd = 1 \text{ and} \tag{1.5}$$

$$bc + ad = 0. \tag{1.6}$$

Now taking $a(1.5) + b(1.6)$ implies $\left(a^2 + b^2\right) c = a$ and so $c = \frac{a}{a^2+b^2}$ and taking $-b(1.5) + a(1.6)$ implies $\left(a^2 + b^2\right) d = -b$ and hence $c = -\frac{b}{a^2+b^2}$ as claimed. ■

Remark 1.4 (Not Done in Class). Here is a way to understand some of the basic properties of $\mathbb{C}$ using our knowledge of linear algebra. Let M_z denote multiplication by $z = a + ib$ then if $w = c + id$ we have

$$M_z w = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

so that $M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI + bJ$ where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. With this notation we have $M_z M_w = M_{zw}$ and since I and J commute it follows that $zw = wz$. Moreover, since matrix multiplication is associative so is complex multiplication, i.e. Eq. (1.4) holds. Also notice that M_z is invertible iff $\det M_z = a^2 + b^2 = |z|^2 \neq 0$ in which case

$$M_z^{-1} = \frac{1}{|z|^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}/|z|^2}$$

as we have already seen above.

Notation 1.5. We will write $1/z$ for z^{-1} and w/z to mean $z^{-1} \cdot w$.

Notation 1.6 (Conjugation and Modulous)**.** If $z = a+ib$ with $a, b \in \mathbb{R}$ let $\bar{z} = a-ib$ and

$$|z|^2 \equiv z\bar{z} = a^2 + b^2.$$

Notice that

$$\operatorname{Re} z = \frac{1}{2}(z + \bar{z}) \text{ and } \operatorname{Im} z = \frac{1}{2i}(z - \bar{z}). \tag{1.7}$$

Proposition 1.7. *Complex conjugation and the modulus operators satisfy,*

1. $\bar{\bar{z}} = z$,
2. $\overline{zw} = \bar{z}\bar{w}$ *and* $\bar{z} + \bar{w} = \overline{z + w}$.
3. $|\bar{z}| = |z|$
4. $|zw| = |z|\,|w|$ *and in particular* $|z^n| = |z|^n$ *for all* $n \in \mathbb{N}$.
5. $|\operatorname{Re} z| \le |z|$ *and* $|\operatorname{Im} z| \le |z|$
6. $|z + w| \le |z| + |w|$.
7. $z = 0$ *iff* $|z| = 0$.
8. *If* $z \neq 0$ *then* $z^{-1} := \frac{\bar{z}}{|z|^2}$ *(also written as* $\frac{1}{z}$*) is the inverse of* z.
9. $\left|z^{-1}\right| = |z|^{-1}$ *and more generally* $|z^n| = |z|^n$ *for all* $n \in \mathbb{Z}$.

Proof. 1. and 3. are geometrically obvious.

2. Say $z = a + ib$ and $w = c + id$, then $\bar{z}\bar{w}$ is the same as zw with b replaced by $-b$ and d replaced by $-d$, and looking at Eq. (1.2) we see that

$$\bar{z}\bar{w} = (ac - bd) - i(bc + ad) = \overline{zw}.$$

4. $|zw|^2 = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2\,|w|^2$ as real numbers and hence $|zw| = |z|\,|w|$.

5. Geometrically obvious or also follows from

$$|z| = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}.$$

6. This is the triangle inequality which may be understood geometrically or by the computation

$$\begin{aligned}|z+w|^2 &= (z+w)\left(\overline{z+w}\right) = |z|^2 + |w|^2 + w\bar{z} + \bar{w}z \\ &= |z|^2 + |w|^2 + w\bar{z} + \overline{w\bar{z}} \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \le |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2.\end{aligned}$$

7. Obvious.
8. Follows from Eq. (1.3).
9. $\left|z^{-1}\right| = \left|\frac{\bar{z}}{|z|^2}\right| = \left|\frac{1}{|z|^2}\right| |\bar{z}| = \frac{1}{|z|}$. ■

2. (9/30/03)

2.1. **Left Overs.** Go over Eq. (1.7) and properties 8. and 9. in Proposition 1.7.

Lemma 2.1. *For complex number* $u, v, w, z \in \mathbb{C}$ *with* $v \neq 0 \neq z$*, we have*

$$\frac{1}{u}\frac{1}{v} = \frac{1}{uv}, \text{ i.e. } u^{-1}v^{-1} = (uv)^{-1}$$
$$\frac{u}{v}\frac{w}{z} = \frac{uw}{vz} \text{ and}$$
$$\frac{u}{v} + \frac{w}{z} = \frac{uz+vw}{vz}.$$

Proof. For the first item, it suffices to check that

$$(uv)\left(u^{-1}v^{-1}\right) = u^{-1}uvv^{-1} = 1 \cdot 1 = 1.$$

The rest follow using

$$\frac{u}{v}\frac{w}{z} = uv^{-1}wz^{-1} = uwv^{-1}z^{-1} = uw\,(vz)^{-1} = \frac{uw}{vz}.$$

$$\begin{aligned}\frac{u}{v} + \frac{w}{z} &= \frac{z}{z}\frac{u}{v} + \frac{v}{v}\frac{w}{z} = \frac{zu}{zv} + \frac{vw}{vz} \\ &= (vz)^{-1}(zu+vw) = \frac{uz+vw}{vz}.\end{aligned}$$

■

2.2. **Book Sections 36-37, p. 111-115.** Here we suppose $w(t) = c(t) + id(t)$ and define

$$\dot{w}(t) = \dot{c}(t) + i\dot{d}(t)$$

and

$$\int_\alpha^\beta w(t)\,dt := \int_\alpha^\beta c(t)\,dt + i\int_\alpha^\beta d(t)\,dt$$

Example 2.2.

$$\int_0^{\pi/2}\left(e^t + i\sin t\right)dt = e^{\frac{1}{2}\pi} - 1 + i.$$

Theorem 2.3. *If* $z(t) = a(t) + ib(t)$ *and* $w(t) = c(t) + id(t)$ *and* $\lambda = u + iv \in \mathbb{C}$ *then*

(1) $\frac{d}{dt}(w(t) + z(t)) = \dot{w}(t) + \dot{z}(t)$
(2) $\frac{d}{dt}[w(t)z(t)] = w\dot{z} + \dot{w}z$
(3) $\int_\alpha^\beta [w(t) + \lambda z(t)]\,dt = \int_\alpha^\beta w(t)\,dt + \lambda\int_\alpha^\beta z(t)\,dt$
(4) $\int_\alpha^\beta \dot{w}(t)dt = w(\beta) - w(\alpha)$ *In particular if* $\dot{w} = 0$ *then* w *is constant.*
(5)
$$\int_\alpha^\beta \dot{w}(t)z(t)dt = -\int_\alpha^\beta w(t)\dot{z}(t)dt + w(t)\,z(t)\,|_\alpha^\beta.$$
(6)
$$\left|\int_\alpha^\beta w(t)\,dt\right| \leq \int_\alpha^\beta |w(t)|\,dt.$$

Proof. 1. and 4. are easy.

2.

$$\begin{aligned}\frac{d}{dt}[wz] &= \frac{d}{dt}(ac-bd)+i\frac{d}{dt}(bc+ad)\\ &= (\dot{a}c-\dot{b}d)+i(\dot{b}c+\dot{a}d)\\ &+(a\dot{c}-b\dot{d})+i(b\dot{c}+a\dot{d})\\ &= \dot{w}z+w\dot{z}.\end{aligned}$$

3. The only interesting thing to check is that

$$\int_\alpha^\beta \lambda z(t)\,dt = \lambda\int_\alpha^\beta z(t)\,dt.$$

Again we simply write out the real and imaginary parts:

$$\begin{aligned}\int_\alpha^\beta \lambda z(t)\,dt &= \int_\alpha^\beta (u+iv)(a(t)+ib(t))\,dt\\ &= \int_\alpha^\beta (ua(t)-vb(t)+i[ub(t)+va(t)])\,dt\\ &= \int_\alpha^\beta (ua(t)-vb(t))\,dt + i\int_\alpha^\beta [ub(t)+va(t)]\,dt\end{aligned}$$

while

$$\begin{aligned}\int_\alpha^\beta \lambda z(t)\,dt &= (u+iv)\int_\alpha^\beta [a(t)+ib(t)]\,dt\\ &= (u+iv)\left(\int_\alpha^\beta a(t)\,dt + i\int_\alpha^\beta b(t)\,dt\right)\\ &= \int_\alpha^\beta (ua(t)-vb(t))\,dt + i\int_\alpha^\beta [ub(t)+va(t)]\,dt.\end{aligned}$$

Shorter Alternative: Just check it for $\lambda = i$, this is the only new thing over the real variable theory.

5.

$$w(t)z(t)|_\alpha^\beta = \int_\alpha^\beta \frac{d}{dt}[w(t)z(t)]\,dt = \int_\alpha^\beta \dot{w}(t)z(t)dt + \int_\alpha^\beta w(t)\dot{z}(t)dt.$$

6. Let $\rho \geq 0$ and $\theta \in \mathbb{R}$ be chosen so that

$$\int_\alpha^\beta w(t)\,dt = \rho e^{i\theta},$$

then

$$\begin{aligned}\left|\int_\alpha^\beta w(t)\,dt\right| &= \rho = e^{-i\theta}\int_\alpha^\beta w(t)\,dt = \int_\alpha^\beta e^{-i\theta}w(t)\,dt\\ &= \int_\alpha^\beta \operatorname{Re}\left[e^{-i\theta}w(t)\right]dt \leq \int_\alpha^\beta \left|\operatorname{Re}\left[e^{-i\theta}w(t)\right]\right|dt\\ &\leq \int_\alpha^\beta \left|e^{-i\theta}w(t)\right|dt = \int_\alpha^\beta |w(t)|\,dt.\end{aligned}$$

■

2.3. **Application.** We would like to use the above ideas to find a "natural" extension of the function e^x to a function e^z with $z \in \mathbb{C}$. The idea is that since

$$\frac{d}{dt}e^{tx} = xe^{tx} \text{ with } e^{0x} = 1$$

we might try to define e^z so that

$$\frac{d}{dt}e^{tz} = ze^{tz} \text{ with } e^{0z} = 1. \tag{2.1}$$

Proposition 2.4. *If there is a function e^z such that Eq. (2.1) holds, then it satisfies:*

(1) $e^{-z} = \frac{1}{e^z}$ *and*
(2) $e^{w+z} = e^w e^z$.

Proof. 1. By the product rule,

$$\frac{d}{dt}\left[e^{-tz}e^{tz}\right] = -ze^{-tz}e^{tz} + e^{-tz}ze^{tz} = 0$$

and therefore, $e^{-tz}e^{tz} = e^{-0z}e^{0z} = 1$. Taking $t = 1$ proves 1.

2. Again by the product rule,

$$\frac{d}{dt}\left[e^{-t(w+z)}e^{tw}e^{tz}\right] = 0$$

and so $e^{-t(w+z)}e^{tw}e^{tz} = e^{-t(w+z)}e^{tw}e^{tz}|_{t=0} = 1$. Taking $t = 1$ then shows $e^{-(w+z)}e^w e^z = 1$ and then using Item 1. proves Item 2. ■

According to Proposition 2.4, to find the desired function e^z it suffices to find e^{iy}. So let us write

$$e^{it} = x(t) + iy(t)$$

then by assumption $\frac{d}{dt}e^{it} = ie^{it}$ with $e^{i0} = 1$ implies

$$\dot{x} + i\dot{y} = i(x + iy) = -y + ix \text{ with } x(0) = 1 \text{ and } y(0) = 0$$

or equivalently that

$$\dot{x} = -y,\ \dot{y} = x \text{ with } x(0) = 1 \text{ and } y(0) = 0.$$

This equation implies

$$\ddot{x}(t) = -\dot{y}(t) = -x(t) \text{ with } x(0) = 1 \text{ and } \dot{x}(0) = 0$$

which has the unique solution $x(t) = \cos t$ in which case $y(t) = -\frac{d}{dt}\cos t = \sin t$. This leads to the following definition.

Definition 2.5 (Euler's Formula)**.** For $\theta \in \mathbb{R}$ let $e^{i\theta} := \cos\theta + i\sin\theta$ and for $z = x + iy$ let

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y). \tag{2.2}$$

Quickly review e^z and its properties, in particular Euler's formula.

Theorem 2.6. *The function e^z defined by Eq. (2.2) satisfies Eq. (2.1) and hence the results of Proposition 2.4. Also notice that $\overline{e^z} = e^{\bar{z}}$.*

Proof. This is proved on p. 112 of the book and the proof goes as follows,

$$\frac{d}{dt}e^{tz} = \frac{d}{dt}\left[e^{tx}e^{ity}\right] = xe^{tx}e^{ity} + e^{tx}iye^{ity} = ze^{tx}e^{ity} = ze^{tz}.$$

The last equality follows from

$$\begin{aligned}\overline{e^{z}} &= \overline{e^{x}\left(\cos y + i\sin y\right)} = \overline{e^{x}}\,\overline{\left(\cos y + i\sin y\right)} = e^{x}\left(\cos y - i\sin y\right)\\ &= e^{x}\left(\cos\left(-y\right) + i\sin\left(-y\right)\right) = e^{\bar{z}}.\end{aligned}$$

■

Corollary 2.7 (Addition formulas)**.** *For* $\alpha, \beta \in \mathbb{R}$ *we have*

$$\begin{aligned}\cos\left(\alpha+\beta\right) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \sin\left(\alpha+\beta\right) &= \cos\alpha\sin\beta + \cos\beta\sin\alpha.\end{aligned}$$

Proof. These follow by comparing the real and imaginary parts of the identity

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = \cos\left(\alpha+\beta\right) + i\sin\left(\alpha+\beta\right)$$

while

$$\begin{aligned}e^{i\alpha}e^{i\beta} &= \left(\cos\alpha + i\sin\alpha\right)\cdot\left(\cos\beta + i\sin\beta\right)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i\left(\cos\alpha\sin\beta + \cos\beta\sin\alpha\right).\end{aligned}$$

■

3. (10/1/03)

Exercise 3.1. Suppose $a, b \in \mathbb{R}$, show

$$\int_0^T e^{at}e^{ibt}dt = \int_0^T e^{(a+ib)t}dt = \int_0^T \frac{1}{a+ib}\frac{d}{dt}e^{(a+ib)t}dt = \frac{1}{a+ib}\left[e^{aT}e^{ibT} - 1\right].$$

By comparing the real and imaginary parts of both sides of this integral find explicit formulas for the two real integrals

$$\int_0^T e^{at}\cos(bt)\,dt \text{ and}$$
$$\int_0^T e^{at}\sin(bt)\,dt.$$

3.1. **Polar/Exponential Form of Complex Numbers: Sections $6-9$.** Bruce: Give the geometric interpretation of each of the following properties.

(1) $z = re^{i\theta} = |z|\,e^{i\theta}$.
(2) $\bar{z} = |z|\,e^{-i\theta}$ and $z^{-1} = \bar{z}/|z|^2 = |z|^{-1}\,e^{-i\theta}$
(3) If $w = |w|\,e^{i\alpha}$ then

$$zw = |z|\,|w|\,e^{i(\theta+\alpha)} \text{ and}$$
$$z/w = zw^{-1} = |z|\,e^{i\theta}\cdot|w|^{-1}\,e^{-i\alpha} = |z|\,|w|^{-1}\,e^{i(\theta-\alpha)}.$$

In particular

$$z^n = |z|^n\,e^{in\theta} \text{ for } n \in \mathbb{Z}.$$

Notation 3.2. If $z \neq 0$ we let $\theta = \operatorname{Arg}(z)$ if $-\pi < \theta \leq \pi$ and $z = |z|\,e^{i\theta}$ while we define

$$\arg(z) = \left\{\theta \in \mathbb{R} : z = |z|\,e^{i\theta}\right\}.$$

Notice that

$$\arg(z) = \operatorname{Arg}(z) + 2\pi\mathbb{Z}.$$

Similarly we define $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and

$$\log(z) = \ln|z| + i\arg(z) = \ln|z| + i\operatorname{Arg}(z) + 2\pi i\mathbb{Z}.$$

Example 3.3.

(1) Work out $(1+i)\left(\sqrt{3}+i\right)$ in polar form.

$$(1+i)\left(\sqrt{3}+i\right) = \sqrt{2}e^{i\pi/4}\cdot 2e^{i\pi/6} = 2\sqrt{2}e^{i5\pi/12}.$$

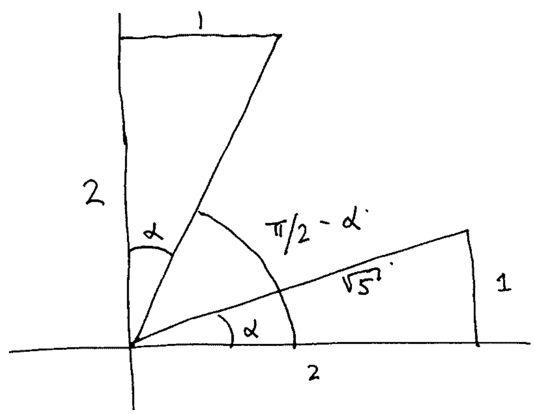
Note here that

$$\arg(1+i) = \pi/4 + 2\pi\mathbb{Z} \text{ and } \arg\left(\sqrt{3}+i\right) = \pi/6 + 2\pi\mathbb{Z}$$
$$\operatorname{Arg}(1+i) = \pi/4 \text{ and } \operatorname{Arg}\left(\sqrt{3}+i\right) = \pi/6$$

(2) Let $\alpha = \tan^{-1}(1/2)$ then

$$\frac{5i}{2+i} = \frac{5e^{i\pi/2}}{\sqrt{5}e^{i\tan^{-1}(1/2)}} = \sqrt{5}e^{i\left(\pi/2-\tan^{-1}(1/2)\right)} = 1+2i$$

by drawing the triangles.



(3) General theory of finding n^{th} – roots if a number $z = \rho e^{i\alpha}$. Let $w = re^{i\theta}$ then $z = \rho e^{i\alpha} = w^n = r^n e^{in\theta}$ happens iff

$$\rho = |z| = |w^n| = |w|^n = r^n \text{ or } r = \rho^{1/n} \text{ and}$$
$$e^{i\alpha} = e^{in\theta} \text{ i.e. } e^{i(n\theta-\alpha)} = 1, \text{ i.e. } n\theta - \alpha \in 2\pi\mathbb{Z}.$$

Therefore

$$\begin{aligned} z^{1/n} &= |z|^{1/n} e^{i\frac{1}{n}(\alpha+2\pi\mathbb{Z})} = |z|^{1/n} e^{i\frac{1}{n}\arg(z)} \\ &= \left\{ |z|^{1/n} e^{i\frac{1}{n}(\alpha+2\pi k)} : k = 0, 1, 2, \dots, n-1 \right\}. \end{aligned}$$

(4) Find all fourth roots of $(1+i)$.

$$(1+i) = \sqrt{2} e^{i(\pi/4+2\pi\mathbb{Z})}$$

and so

$$(1+i)^{1/4} = 2^{1/8} e^{i\left(\pi/16+\frac{1}{8}\pi\mathbb{Z}\right)} = \left\{ 2^{1/8} e^{i\left(\pi/16+\frac{1}{2}\pi k\right)} : k = 0, 1, 2, 3 \right\}.$$

4. (10/03/2003)

4.1. **More on Roots and multi-valued arithmetic.**

Notation 4.1. Suppose $A \subset \mathbb{C}$ and $B \subset \mathbb{C}$, the we let

$$A \cdot B := \{ab : a \in A \text{ and } b \in B\} \text{ and}$$
$$A \pm B := \{a \pm b : a \in A \text{ and } b \in B\}.$$

Proposition 4.2. $\arg(zw) = \arg(z) + \arg(w)$ *while it is not in general true the* $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$.

Proof. Suppose $z = |z| e^{i\theta}$ and $w = |w| e^{i\alpha}$, then

$$\arg(zw) = (\theta + \alpha + 2\pi\mathbb{Z})$$

while

$$\arg(z) + \arg(w) = (\theta + 2\pi\mathbb{Z}) + (\alpha + 2\pi\mathbb{Z}) = (\theta + \alpha + 2\pi\mathbb{Z}).$$

Example: Let $z = i$ and $w = -1$, then $\operatorname{Arg}(i) = \pi/2$ and $\operatorname{Arg}(-1) = \pi$ so that

$$\operatorname{Arg}(i) + \operatorname{Arg}(-1) = \frac{3\pi}{2}$$

while

$$\operatorname{Arg}(i \cdot (-1)) = -\pi/2.$$

■

The following proposition summarizes item 3. of Example 3.3 above and gives an application of Proposition 4.2.

Proposition 4.3. *Suppose that $w \in \mathbb{C}$, then the set of n^{th} – roots, $w^{1/n}$ of w is*

$$w^{1/n} = \sqrt[n]{|w|} e^{i \frac{1}{n} \arg(w)}.$$

Moreover if $z \in \mathbb{C}$ then

$$(wz)^{1/n} = w^{1/n} \cdot z^{1/n} \tag{4.1}$$

In particular this implies if w_0 is an n^{th}– root of w, then

$$w^{1/n} = \left\{ w_0 e^{i \frac{k}{n} 2\pi} : k = 0, 1, \ldots, n-1 \right\}.$$

DRAW picture of the placement of the roots on the circle of radius $\sqrt[n]{|w|}$.

Proof. It only remains to prove Eq. (4.1) and this is done using

$$\begin{aligned} w^{1/n} \cdot z^{1/n} &= \sqrt[n]{|w|} e^{i \frac{1}{n} \arg(w)} \sqrt[n]{|z|} e^{i \frac{1}{n} \arg(z)} \\ &= \sqrt[n]{|w|\,|z|} e^{i \frac{1}{n} [\arg(w) + \arg(z)]} = \sqrt[n]{|wz|} e^{i \frac{1}{n} \arg(wz)} \\ &= (wz)^{1/n}. \end{aligned}$$

■

Theorem 4.4 (Quadratic Formula)**.** *Suppose $a, b, c \in \mathbb{C}$ with $a \neq 0$ then the general solution to the equation*

$$az^2 + bz + c = 0$$

is

$$z = \frac{-b \pm \left(b^2 - 4ac\right)^{1/2}}{2a}.$$

Proof. The proof goes as in the real case by observing

$$0 = az^2 + bz + c = a\left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

and so

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots of this equation then shows

$$z + \frac{b}{2a} = \frac{\left(b^2 - 4ac\right)^{1/2}}{2a}$$

which is the quadratic formula. ■

4.2. **Regions and Domains:**

(1) Regions in the plain. **Definition:** a domain is a **connected open** subset of $\mathbb{C}$. Examples:
   (a) $\{z : |z - 1 + 2i| < 4\}$.
   (b) $\{z : |z - 1 + 2i| \leq 4\}$.
   (c) $\{z : |z - 1 + 2i| = 4\}$
   (d) $\left\{z : z = re^{i\theta} \text{ with } r > 0 \text{ and } -\pi < \theta < \pi\right\}$
   (e) $\left\{z : z = re^{i\theta} \text{ with } r \geq 0 \text{ and } -\pi < \theta \leq \pi\right\}$.

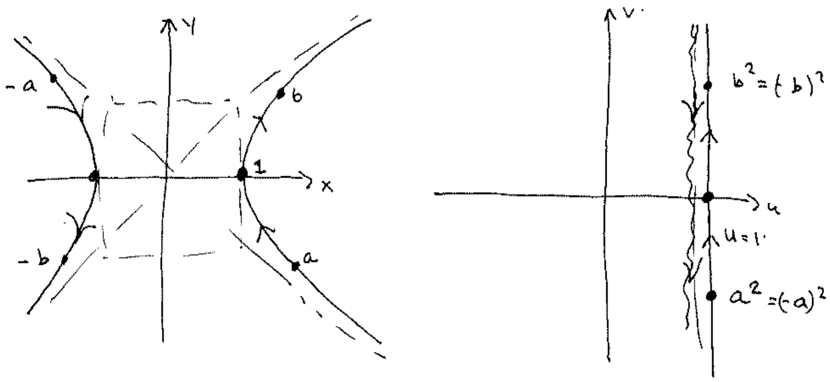
5. (10/06/2003)

5.1. **Functions from $\mathbb{C}$ to $\mathbb{C}$..**

(1) Complex functions, $f : D \to \mathbb{C}$. Point out that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy \in D$. Examples: (Mention domains)
  (a) $f(z) = z$, $u = x$, $v = y$
  (b) $f(z) = z^2$, $u = x^2 - y^2$, $v = 2xy$ also look at it as $f\left(re^{i\theta}\right) = r^2 e^{i2\theta}$.
    (i) So rays through the origin go to rays through the origin.
    (ii) Also arcs of circles centered at 0 go over to arcs of circles centered at 0.
    (iii) Also notice that if we hold x constant, then $y = v/2x$ and so $u = x^2 - \frac{v^2}{4x^2}$ which is the graph of a parabola.
    (iv) Bruce !!: Do the examples where $\operatorname{Re} f(z) = 1$ and $\operatorname{Im} f(z) = 1$ to get pre-images which are two hyperbolas. Explain the orientation traversed. See Figure 1 below.
  (c) $f(z) = az$, if $a = re^{i\theta}$, then $f(z)$ scales z by r and then rotates by θ degrees. If $a = \alpha + i\beta$, then $u = \alpha x - \beta y$, $v = \alpha y = \beta x$.
  (d) $f(z) = \bar{z}$, this is reflection about the $x-$ axis.
  (e) $f(z) = 1/z$ is inversion, notice that $f\left(re^{i\theta}\right) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$, draw picture.
  (f) $f(z) = e^z = e^{x+iy}$, $u = e^x \cos y$ and $v = e^x \sin y$.
    (i) Show what happens to the line $x = 2$ and the line $y = \pi/4$.
  (g) $f\left(re^{i\theta}\right) = r^{\frac{1}{2}} e^{i\frac{1}{2}\theta}$ for $-\pi < \theta \le \pi$. Somewhat painful to write u, v in this case.

$$f(z) = z^2 = x^2 - y^2 + 2ixy = u + iv$$

$$\mathrm{Re} f(z) = 1 = x^2 - y^2 = u$$



$$\mathrm{Im}\, f(z) = 1 = v = 2xy.$$

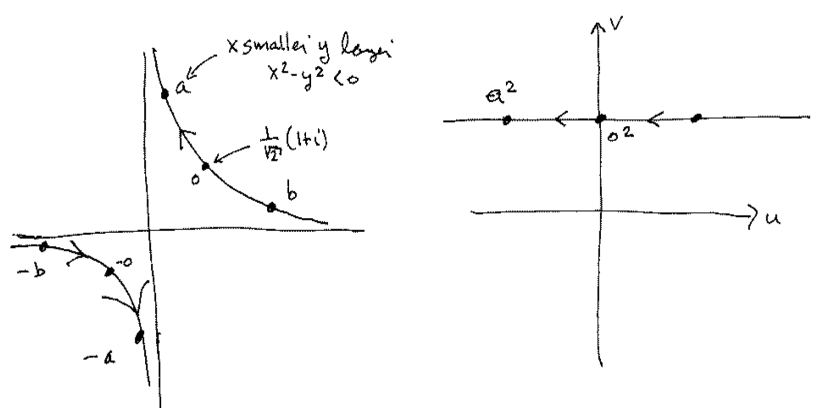
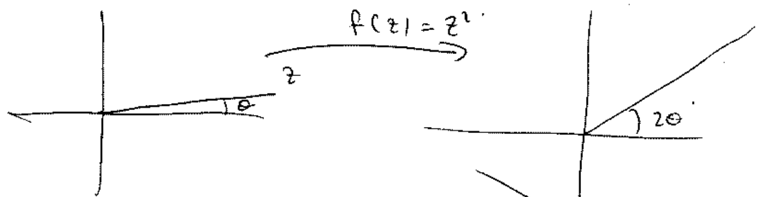
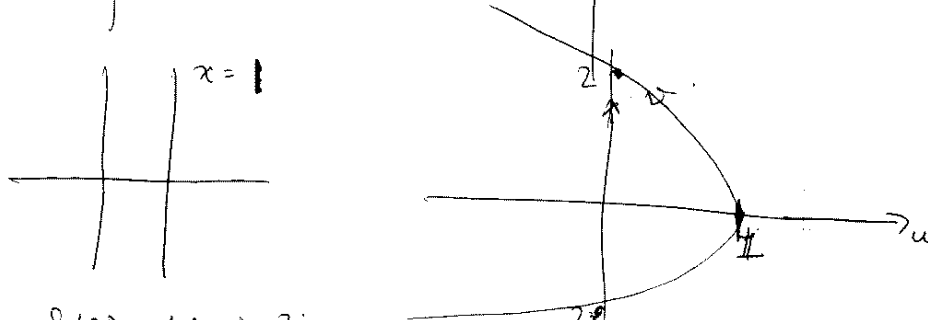


FIGURE 1. Pre-images of lines for $f(z) = z^2$.

Explain. Image of rays.

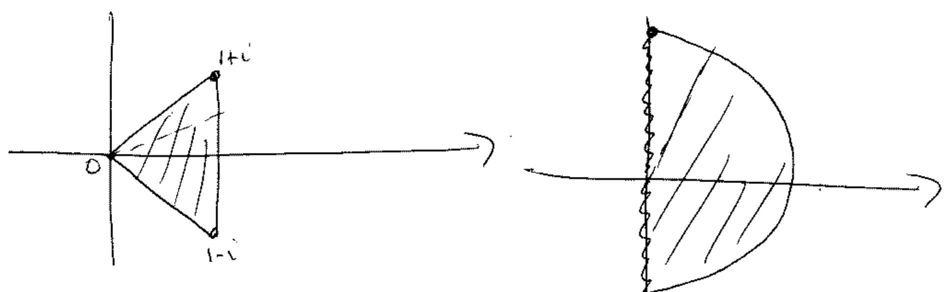




$f(z) = (1+iy)^2 = 1 - y^2 + i2y = u + iv$

$1 - y^2 = u \qquad v = 2y \qquad y = v/2$

$u = 1 - \frac{v^2}{2^2} \qquad u = 0 \qquad 4 \cdot 4^2 = v^2 \qquad v = \sqrt{2}\,4$



Examples

$z \to az$ $a > 0$ dilation

$z \to e^{i\theta} z$ rotation by θ

$z \to z + z_0$ translation

$z \to \frac{1}{z}$ Inversion

5.2. **Continuity and Limits.**

5.3. $\varepsilon -$ **Notation.** In this section, U will be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ a function and $\varepsilon(z)$ will denote a generic function defined for z near zero such that $\lim_{z\to 0} \varepsilon(z) = 0$.

Definition 5.1.
(1) $\lim_{z\to z_0} f(z) = L$ iff $f(z_0 + \Delta z) = L + \varepsilon(\Delta z)$
(2) f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0) = f(\lim_{z\to z_0} z)$.
(3) f is differentiable at z_0 with derivative L iff

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L$$

or equivalently iff

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{z_0 + \Delta z - z_0} = L + \varepsilon(\Delta z) \tag{5.1}$$

or equivalently,

$$f(z_0 + \Delta z) - f(z_0) = (L + \varepsilon(\Delta z))\,\Delta z. \tag{5.2}$$

Proposition 5.2. *The functions $f(z) := \bar{z}$, $f(z) = \operatorname{Re} z$, and $f(z) = \operatorname{Im} z$ are all continuous functions which are **not** complex differentiable at any point $z \in \mathbb{C}$. The following functions are complex differentiable at all points $z \in \mathbb{C}$:*
(1) *$f(z) = z$ with $f'(z) = 1$.*
(2) *$f(z) = \frac{1}{z}$ with $f'(z) = -z^{-2}$.*
(3) *$f(z) = e^z$ with $f'(z) = e^z$.*

Proof. For the first assertion we have

$$\begin{aligned}\left|\overline{z_0 + \Delta z} - \bar{z}_0\right| &= |\Delta z| \to 0 \\ |\operatorname{Re}(z_0 + \Delta z) - \operatorname{Re} z_0| &= |\operatorname{Re} \Delta z| \le |\Delta z| \to 0 \text{ and} \\ |\operatorname{Im}(z_0 + \Delta z) - \operatorname{Im} z_0| &= |\operatorname{Im} \Delta z| \le |\Delta z| \to 0.\end{aligned}$$

For differentiability,

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

which has no limit as $\Delta z \to 0$. Indeed, consider what happens for $\Delta z = x$ and $\Delta z = iy$ with $x, y \in \mathbb{R}$ and $x, y \to 0$. Similarly

$$\frac{\operatorname{Re}(z_0 + \Delta z) - \operatorname{Re} z_0}{\Delta z} = \frac{\operatorname{Re} \Delta z}{\Delta z}$$

as no limit as $\Delta z \to 0$.

(1)
$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = 1 \to 1 \text{ as } \Delta z \to 0.$$

(2) Let us first shows that $1/z$ is continuous, for this we have

$$\begin{aligned}\left|(z + \Delta z)^{-1} - z^{-1}\right| &= \left|\frac{z - (z + \Delta z)}{z(z + \Delta z)}\right| = \left|\frac{1}{z}\right| \left|\frac{1}{z + \Delta z}\right| |\Delta z| \\ &\le \left|\frac{1}{z}\right| \left|\frac{1}{|z| - |\Delta z|}\right| |\Delta z| \le \frac{2}{|z|^2} |\Delta z| \to 0.\end{aligned}$$

We now use this to compute the derivative,

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{(z+\Delta z)^{-1}-z^{-1}}{\Delta z} \\ &= \frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z} = \frac{1}{\Delta z}\frac{z-(z+\Delta z)}{z(z+\Delta z)} = -\frac{1}{z(z+\Delta z)} \to -\frac{1}{z^2}.\end{aligned}$$

where the continuity of $1/z$ was used in taking the limit.

(3) Since

$$\frac{e^{z+\Delta z}-e^z}{\Delta z} = e^z\frac{e^{\Delta z}-1}{\Delta z}$$

it suffices to show

$$\frac{e^{\Delta z}-1}{\Delta z} \to 1 \text{ as } \Delta z \to 0.$$

This follows from,

$$\frac{e^{\Delta z}-1}{\Delta z} = \frac{1}{\Delta z}\int_0^1 \frac{d}{dt}e^{t\Delta z}dt = \frac{1}{\Delta z}\Delta z\int_0^1 e^{t\Delta z}dt = \int_0^1 e^{t\Delta z}dt$$

which implies

$$e^{\Delta z}-1 = \Delta z\int_0^1 e^{t\Delta z}dt = \varepsilon(\Delta z)$$

and therefore

$$\left|\frac{e^{\Delta z}-1}{\Delta z}-1\right| = \left|\int_0^1\left[e^{t\Delta z}-1\right]dt\right| \le \int_0^1\left|e^{t\Delta z}-1\right|dt = \int_0^1|\varepsilon(t\Delta z)|\,dt \to 0 \text{ as } \Delta z\to 0.$$

Alternative 1.,

$$\begin{aligned}\int_0^1 e^{t\Delta z}dt &= \int_0^1 e^{t\Delta z}d[t-1] \\ &= \left(e^{t\Delta z}[t-1]\right)_0^1 - \int_0^1 \frac{d}{dt}e^{t\Delta z}[t-1]\,dt \\ &= 1-\Delta z\int_0^1 e^{t\Delta z}[t-1]\,dt\end{aligned}$$

from which it should be clear that

$$\frac{e^{\Delta z}-1}{\Delta z}-1 = \epsilon(\Delta z).$$

Alternative 2. Write $\Delta z = x+iy$, then and use the definition of the real derivative to learn

$$\begin{aligned}e^{\Delta z} &= e^{x+iy} = e^x(\cos y + i\sin y) = \left(1+x+O\left(x^2\right)\right)\left(1+iy+O\left(y^2\right)\right) \\ &= 1+x+iy+O\left(|\Delta z|^2\right) = 1+\Delta z+O\left(|\Delta z|^2\right).\end{aligned}$$

■

6. (10/08/2003 and 10/10/2003) Lectures 6-7

Go over the function $f(z) = e^z$ in a bit more detail than was done in class using **Alternative 2** above to show

$$\lim_{\Delta z \to 0} \frac{e^{\Delta z} - 1}{\Delta z} = 1 \tag{6.1}$$

To do this write $\Delta z = x + iy$, then and use Taylor's formula with remainder for real functions to learn

$$\begin{aligned} e^{\Delta z} &= e^{x+iy} = e^x (\cos y + i \sin y) = \left(1 + x + O\left(x^2\right)\right)\left(1 + iy + O\left(y^2\right)\right) \\ &= 1 + x + iy + O\left(|\Delta z|^2\right) = 1 + \Delta z + O\left(|\Delta z|^2\right) \end{aligned}$$

which implies Eq. (6.1).

Exercise 6.1. Suppose that $f'(0) = 5$ and $g(z) = f(\bar{z})$. Show $g'(0)$ does not exists.

Solution:

$$\frac{g(z) - g(0)}{z} = \frac{f(\bar{z}) - f(0)}{z} = \frac{(5 + \epsilon(\bar{z}))\bar{z}}{z}$$

and the latter does not have a limit by Proposition 5.2.

BRUCE: Do **examples** in this section before giving proofs.

Definition 6.2. Limits involving ∞,

1. $\lim_{z\to\infty} f(z) = w$ iff $\lim_{z\to 0} f(1/z) = w$.
2. $\lim_{z\to w} f(z) = \infty$ iff $\lim_{z\to w} \frac{1}{f(z)} = 0$.
3. $\lim_{z\to\infty} f(z) = \infty$ iff $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$.

BRUCE: Explain the motivation via stereographic projection.

Theorem 6.3. *If* $\lim_{z\to z_0} f(z) = L$ *and* $\lim_{z\to z_0} g(z) = K$ *then*

1. $\lim_{z\to z_0} [f(z) + g(z)] = L + K$.
2. $\lim_{z\to z_0} [f(z)\, g(z)] = LK$
3. *If* $z \to h(z) = f(g(z))$ *is continuous at* z_0 *if* g *is continuous at* z_0 *and* f *is continuous at* $w_0 = g(z_0)$.
4. $\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{L}{K}$ *provided* $K \neq 0$.
5. *We also have* $\lim_{z\to z_0} f(z) = L$ *iff* $\lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} L$ *and* $\lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} L$.

Proof.

(1)

$$f(z_0 + \Delta z) + g(z_0 + \Delta z) = L + \varepsilon(\Delta z) + K + \varepsilon(\Delta z) = (L + K) + \varepsilon(\Delta z).$$

(2)

$$\begin{aligned} f(z_0 + \Delta z) \cdot g(z_0 + \Delta z) &= [L + \varepsilon(\Delta z)] \cdot [K + \varepsilon(\Delta z)] \\ &= LK + K\varepsilon(\Delta z) + L\varepsilon(\Delta z) + \varepsilon(\Delta z)\varepsilon(\Delta z) = LK + \varepsilon(\Delta z). \end{aligned}$$

(3) Well,

$$\begin{aligned} h(z_0 + \Delta z) - h(z_0) &= f(g(z_0 + \Delta z)) - f(g(z_0)) \\ &= f(g(z_0) + \varepsilon(\Delta z)) - f(g(z_0)) = \varepsilon(\varepsilon(\Delta z)) \to 0 \text{ as } \Delta z \to 0. \end{aligned}$$

(4) This follows directly using

$$\frac{f(z_0+\Delta z)}{g(z_0+\Delta z)} - \frac{L}{K} = \frac{L+\varepsilon(\Delta z)}{K+\varepsilon(\Delta z)} - \frac{L}{K} = \frac{(L+\varepsilon(\Delta z))K - L(K+\varepsilon(\Delta z))}{K^2+K\varepsilon(\Delta z)}$$
$$= \frac{\varepsilon(\Delta z)}{K^2+\varepsilon(\Delta z)} = \varepsilon(\Delta z).$$

or more simply using item 3. and the fact $1/z$ is continuous so that $\lim_{z\to z_0}\left[\frac{1}{g(z)}\right] = \frac{1}{K}$.

(5) This follows from item 1. and the continuity of the functions $z \to \operatorname{Re} z$ and $z \to \operatorname{Im} z$.

■

Theorem 6.4. *If $f'(z_0) = L$ and $g'(z_0) = K$ then*

(1) *f is continuous at z_0,*

(2) $\frac{d}{dz}[f(z)+g(z)]|_{z=z_0} = L+K$

(3) $\frac{d}{dz}[f(z)g(z)]|_{z=z_0} = f'(z_0)g(z_0)+f(z_0)g'(z_0)$

(4) *If $w_0 = f(z_0)$ and $g'(w_0)$ exists then $h(z) := g(f(z))$ is differentiable as z_0 and*

$$h'(z_0) = g'(f(z_0))f'(z_0).$$

(5)
$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'g-g'f}{g^2}|_{z=z_0}.$$

(6) *If $z(t)$ is a differentiable curve, then $\frac{d}{dt}f(z(t)) = f'(z(t))\dot{z}(t)$.*

Proof. To simplify notation, let $\Delta f = f(z+\Delta z) - f(z)$ and $\Delta g = g(z+\Delta z) - g(z)$ and recall that recall that $\Delta f \to 0$ and $\Delta g \to 0$ as $\Delta z \to 0$, i.e. $\Delta f = \varepsilon(\Delta z)$.

(1) This follows from Eq. (5.2).

(2)
$$\frac{[f(z+\Delta z)+g(z+\Delta z)] - [f(z)+g(z)]}{\Delta z} = \frac{\Delta f}{\Delta z} + \frac{\Delta g}{\Delta z}$$
$$\to f'(z)+g'(z)$$

(3)
$$\frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} = \frac{(f(z)+\Delta f)(g(z)+\Delta g) - f(z)g(z)}{\Delta z}$$
$$= \frac{f(z)\Delta g + \Delta f g(z) + \Delta f\Delta g}{\Delta z}$$
$$= f(z)\frac{\Delta g}{\Delta z} + g(z)\frac{\Delta f}{\Delta z} + \frac{\Delta g}{\Delta z}\Delta f \to f(z)g'(z) + g(z)f'(z).$$

(4) Recall that $\Delta f = \varepsilon(\Delta z)$ and so

$$\Delta h := h(z+\Delta z) - h(z) = g(f(z+\Delta z)) - g(f(z))$$
$$= g(f(z)+\Delta f) - g(f(z))$$
$$= [g'(f(z)) + \varepsilon(\Delta f)]\Delta f = [g'(f(z)) + \varepsilon(\Delta z)]\Delta f.$$

Therefore

$$\frac{\Delta h}{\Delta z} = [g'(f(z)) + \varepsilon(\Delta z)]\Delta f \to g'(f(z))f'(z).$$

(5) This follows from the product rule, the chain rule and the fact that $\frac{d}{dz}z^{-1} = -z^{-2}$.
(6) In order to verify this item, we first need to observe that $\dot{z}(t)$ exists iff $\lim_{\Delta t\to 0}\frac{z(t+\Delta t)-z(t)}{\Delta t}$. Recall that we defined

$$\dot{z}(t) = \frac{d}{dt}\operatorname{Re} z(t) + i\frac{d}{dt}\operatorname{Im} z(t).$$

Since the limit of a sum is a sum of a limit if $\frac{d}{dt}\operatorname{Re} z(t)$ and $\frac{d}{dt}\operatorname{Im} z(t)$ exist then $\lim_{\Delta t\to 0}\frac{z(t+\Delta t)-z(t)}{\Delta t}$ exists. Conversely if $w = \lim_{\Delta t\to 0}\frac{z(t+\Delta t)-z(t)}{\Delta t}$ exists, then

$$\begin{aligned}\lim_{\Delta t\to 0}\left|\frac{\operatorname{Re} z(t+\Delta t)-\operatorname{Re} z(t)}{\Delta t} - \operatorname{Re} w\right| &= \lim_{\Delta t\to 0}\left|\operatorname{Re}\left(\frac{z(t+\Delta t)-z(t)}{\Delta t} - w\right)\right| \\ &\le \lim_{\Delta t\to 0}\left|\frac{z(t+\Delta t)-z(t)}{\Delta t} - w\right| = 0\end{aligned}$$

which shows $\frac{d}{dt}\operatorname{Re} z(t)$ exists. Similarly one shows $\frac{d}{dt}\operatorname{Im} z(t)$ exists as well.

Now for the proof of the chain rule: let $\Delta z := z(t+\Delta t) - z(t)$

$$\begin{aligned}\frac{f(z(t+\Delta t)) - f(z(t))}{\Delta t} &= \frac{[f'(z(t)) + \varepsilon(\Delta z)]\Delta z}{\Delta t} \\ &= [f'(z(t)) + \varepsilon(\Delta z)]\frac{\Delta z}{\Delta t} \to f'(z(t))\dot{z}(t).\end{aligned}$$

■

Example 6.5.

(1) z is continuous, $\bar{z}$, $\operatorname{Re} z$, $\operatorname{Im} z$ are continuous and polynomials in these variables.
(2) $\lim_{z\to z_0} z^n = z_0^n$, Proof by induction.
(3) $\lim_{z\to 1}\frac{z^2-1}{z-1} = \lim_{z\to 1}(z+1) = 2$.
(4) $\lim_{z\to 1}\frac{1}{z^3-1} = \infty$, where by definition $\lim_{z\to z_0} f(z) = \infty$ iff $\lim_{z\to z_0}\frac{1}{f(z)} = 0$.
(5) $\lim_{z\to\infty}\frac{z^2+1}{z^2-1} = 1$ where by definition $\lim_{z\to\infty} f(z) = L$ iff $\lim_{z\to 0} f\left(\frac{1}{z}\right) = L$.
(6) $\lim_{z\to\infty}\frac{z^{25}+1}{z^{24}-z^6+7z^2-5} = \infty$ where by definition $\lim_{z\to\infty} f(z) = \infty$ iff $\lim_{z\to 0}\frac{1}{f\left(\frac{1}{z}\right)} = 0$.
(7) e^z is continuous, proof

$$e^{z_0+\Delta z} - e^{z_0} = e^{z_0}\left(e^{\Delta z} - 1\right) = \Delta z e^{z_0}\int_0^1 e^{t\Delta z}dt = \varepsilon(\Delta z)$$

(8) $\lim_{z\to -1}\frac{z+1}{z^2+(1-i)z-i}$, since $z^2 + (1-i)z - i = 1 - (1-i) - i = 0$ at $z = -1$ we have to factor the denominator. By the quadratic formula we have

$$\begin{aligned}z &= \frac{-(1-i)\pm\sqrt{(1-i)^2+4i}}{2} = \frac{-(1-i)\pm\sqrt{(1+i)^2}}{2} \\ &= \frac{-(1-i)\pm(1+i)}{2} = \{i, -1\}\end{aligned}$$

and thus

$$z^2 + (1-i)z - i = (z-i)(z+1)$$

and we thus have

$$\lim_{z\to -1} \frac{z+1}{z^2+(1-i)\,z-i} = \lim_{z\to -1} \frac{z+1}{(z-i)\,(z+1)} = \frac{1}{-1-i} = -\frac{1-i}{2}.$$

Example 6.6. Describe lots of analytic functions and compute their derivatives: for example z^2, $p(z)$, e^{z^2}, $e^{1/z}$, $\sin(z)\,\cos(z)$, etc.

Example 6.7 (Important Example)**.**

$$\int_0^1 (1+it)^3\,dt = \frac{1}{4i}\,(1+it)^4\,|_0^1 = \frac{1}{4i}\left[(1+i)^4-1\right] = \frac{5}{4}i.$$

If we did this the old fashion way it would be done as follows

$$\int_0^1 (1+it)^3\,dt = \int_0^1 \left[1+3it-3t^2-it^3\right] dt = 1-1+i\left(\frac{3}{2}-\frac{1}{4}\right) = \frac{5}{4}i.$$

Example 6.8.

$$\begin{aligned}\int_0^{\pi/2} e^{(1+i)\pi\sin t}\cos t dt &= \frac{1}{\pi}\frac{e^{(1+i)\pi\sin t}}{1+i} = \frac{1}{\pi}\frac{1}{1+i}\left[e^{\pi(1+i)}-1\right]\\ &= \frac{1}{\pi}\frac{1}{1+i}\left[-e^{\pi}-1\right].\end{aligned}$$

7. Study Guide for Math 120A Midterm 1 (Friday October 17, 2003)

(1) $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$ with $i^2 = -1$ and $\bar{z} = x - iy$. The complex numbers behave much like the real numbers. In particular the quadratic formula holds.
(2) $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$, $|zw| = |z|\,|w|$, $|z + w| \le |z| + |w|$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$, $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$. We also have $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z+w} = \bar{z} + \bar{w}$ and $z^{-1} = \frac{\bar{z}}{|z|^2}$.
(3) $\{z : |z - z_0| = \rho\}$ is a circle of radius ρ centered at z_0.
$\{z : |z - z_0| < \rho\}$ is the open disk of radius ρ centered at z_0.
$\{z : |z - z_0| \ge \rho\}$ is every thing outside of the open disk of radius ρ centered at z_0.
(4) $e^z = e^x (\cos y + i \sin y)$, every $z = |z|\, e^{i\theta}$.
(5) $\arg(z) = \left\{\theta \in \mathbb{R} : z = |z|\, e^{i\theta}\right\}$ and $\operatorname{Arg}(z) = \theta$ if $-\pi < \theta \le \pi$ and $z = |z|\, e^{i\theta}$. Notice that $z = |z|\, e^{i \arg(z)}$
(6) $z^{1/n} = \sqrt[n]{|z|} e^{i \frac{\arg(z)}{n}}$.
(7) $\lim_{z \to z_0} f(z) = L$. Usual limit rules hold from real variables.
(8) Mapping properties of simple complex functions
(9) The definition of complex differentiable $f(z)$. Examples, $p(z)$, e^z, $e^{p(z)}$, $1/z$, $1/p(z)$ etc.
(10) Key points of e^z are is $\frac{d}{dz} e^z = e^z$ and $e^z e^w = e^{z+w}$.
(11) All of the usual derivative formulas hold, in particular product, sum, and chain rules:
$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$
and
$$\frac{d}{dt} f(z(t)) = f'(z(t))\, \dot{z}(t).$$
(12) $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, are nice functions from the real - variables point of view but are **not** complex differentiable.
(13) Integration:
$$\int_a^b z(t)\, dt := \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt.$$
All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.

8. (10/13/2003) Lecture 8

Definition 8.1 (Analytic and entire functions)**.** A function $f : D \to \mathbb{C}$ is said to be **analytic (or holomorphic)** on an open subset $D \subset \mathbb{C}$ if $f'(z)$ exists for all $z \in D$. An analytic function f on $\mathbb{C}$ is said to be **entire.**

8.1. Cauchy Riemann Equations in Cartesian Coordinates.

If $f(z)$ is complex differentiable, then by the chain rule

$$\begin{aligned} \partial_x f(x+iy) &= f'(x+iy) \text{ while} \\ \partial_y f(x+iy) &= if'(x+iy). \end{aligned}$$

So in order for $f(z)$ to be complex differentiable at $z = x + iy$ we must have

$$f_y(x+iy) := \partial_y f(x+iy) = i\partial_x f(x+iy) = if_x(x+iy). \tag{8.1}$$

Writing $f = u + iv$, Eq. (8.1) is equivalent to $u_y + iv_y = i(u_x + iv_x)$ and thus equivalent to

$$u_y = -v_x \text{ and } u_x = v_y. \tag{8.2}$$

Theorem 8.2 (Cauchy Riemann Equations)**.** *Suppose $f(z)$ is a complex function. If $f'(z)$ exists then $f_x(z)$ and $f_y(z)$ exists and satisfy Eq. (8.1), i.e.*

$$\partial_y f(z) = i\partial_x f(z)$$

Conversely if f_x and f_y exists and are continuous in a neighborhood of z, then $f'(z)$ exists iff Eq. (8.1) holds.

Proof. (I never got around to giving this proof.) We have already proved the first part of the theorem. So now suppose that f_x and f_y exists and are continuous in a neighborhood of z and Eq. (8.1) holds. To simplify notation let us suppose that $z = 0$ and $\Delta z = x + iy$, then

$$\begin{aligned} f(x+iy) - f(0) &= f(x+iy) - f(x) + f(x) - f(0) \\ &= \int_0^1 \frac{d}{dt} f(x+ity)\,dt + \int_0^1 \frac{d}{dt} f(tx)\,dt \\ &= \int_0^1 [yf_y(x+ity) + xf_x(tx)]\,dt \\ &= \int_0^1 [iyf_x(x+ity) + xf_x(tx)]\,dt \\ &= \int_0^1 [iy(f_x(x+ity) - f_x(0)) + x(f_x(tx) - f_x(0)) + f_x(0)(x+iy)]\,dt \\ &= zf_x(0) + \int_0^1 [iy(f_x(x+ity) - f_x(0)) + x(f_x(tx) - f_x(0))]\,dt \\ &= zf_x(0) + \int_0^1 [iy\varepsilon(z) + x\varepsilon(z)]\,dt = zf_x(0) + |z|\,\varepsilon(|z|). \end{aligned}$$

■

Fact 8.3 (Amazing Fact). We we will eventually show, that if f is analytic on an open subset $D \subset \mathbb{C}$, then f is infinitely complex differentiable on D, i.e. f analytic implies f' is analytic!!! **Note well:** it is important that D is open here. See Remark 8.6 below.

Example 8.4. Consider the following functions:

(1) $f(z) = x + iby$. In this case $f_x = 1$ while $f_y = ib$ so $f_y = if_x$ iff $b = 1$. In this case $f(z) = z$.
(2) $f(z) = z^2$, then $u = x^2 - y^2$ and $v = 2xy$, $u_y = -2y = -v_x$ and $v_y = 2x = u_x$, which shows that $f(z) = z^2$ is complex differentiable.
(3) $f(z) = e^z = e^x (\cos y + i \sin y)$, so $f_x = f$ while
$$f_y = e^{ix} (-\sin y + i \cos y) = if = if_x$$
which again shows that f is complex differentiable.
(4) Also work out the example $f(z) = 1/z = \frac{x-iy}{x^2+y^2}$,
$$f_x = \frac{x^2 + y^2 - 2x(x - iy)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 + 2ixy}{|z|^4}$$
Note
$$-\left(\frac{1}{z}\right)^2 = -\frac{(x - iy)^2}{(x^2 + y^2)^2} = -\frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}$$
$$= \frac{y^2 - x^2 + 2ixy}{|z|^4} = f_x.$$
Similarly
$$f_y = \frac{-i\left(x^2 + y^2\right) - 2y(x - iy)}{(x^2 + y^2)^2} = \frac{-ix^2 + iy^2 - 2yx}{(x^2 + y^2)^2} = if_x$$
and all of this together shows that $f'(z) = -\frac{1}{z^2}$ for $z \neq 0$.

Corollary 8.5. *Suppose that $f = u + iv$ is complex differentiable in an open set D, then u and v are harmonic functions, i.e. that real and imaginary parts of analytic functions are harmonic.*

Proof. The C.R. equations state that $v_y = u_x$ and $v_x = -u_y$, therefore
$$v_{yy} = u_{xy} = u_{yx} = -v_{xx}.$$
A similar computation works for u. ■

Remark 8.6. The only harmonic functions $f : \mathbb{R} \to \mathbb{R}$ are straight lines, i.e. $f(x) = ax + b$. In particular, any harmonic function $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable. This should shed a little light on the Amazing Fact in Example 8.3.

Example 8.7 (The need for continuity in Theorem 8.2)**.** Exercise 6, on p. 69. Consider the function
$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$$
Then
$$f_x(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$
while
$$f_y(0) = \lim_{y \to 0} \frac{f(iy) - f(0)}{y} = \lim_{y \to 0} \frac{-y^2/iy}{y} = -\frac{1}{i} = i = if_x(0).$$
Thus the Cauchy Riemann equations hold at 0. However,
$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\bar{z}^2}{z^2}$$

does not exist. For example taking $z = x$ real and $z = xe^{i\pi/4}$ we get

$$\lim_{z=x\to 0} \frac{\bar{z}^2}{z^2} = 1 \text{ while } \lim_{z=xe^{i\pi/4}\to 0} \frac{\bar{z}^2}{z^2} = \lim_{z=xe^{i\pi/4}\to 0} \frac{-ix^2}{ix^2} = -1.$$

9. (10/15/2003) Lecture 9

Example 9.1. Show $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic when $f(z) = z^2$ and $f(z) = e^z$.

Definition 9.2. A function $f : D \to \mathbb{C}$ is **analytic on an open set** D iff $f'(z)$ is complex differentiable at all points $z \in D$.

Definition 9.3. For $z \neq 0$, let $\log z = \{w \in \mathbb{C} : e^w = z\}$.

Writing $z = |z| e^{i\theta}$ we and $w = x + iy$, we must have $|z| e^{i\theta} = e^x e^{iy}$ and this implies that $x = \ln|z|$ and $y = \theta + 2\pi n$ for some n. Therefore

$$\log z = \ln|z| + i \arg z.$$

Definition 9.4. $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$, so $\operatorname{Log}(re^{i\theta}) = \ln r + i\theta$ if $r > 0$ and $-\pi < \theta \leq \pi$. Note this function is **discontinuous** at points z where $\operatorname{Arg}(z) = \pi$.

Definition 9.5. Given a multi-valued function $f : D \to \mathbb{C}$, we say a $F : D_0 \subset D \to \mathbb{C}$ is a branch of f if $F(z) \in f(z)$ for all $z \in D_0$ and F is continuous on D_0. Here D_0 is taken to be an open subset of D.

Example 9.6 (A branch of $\log(z)$: a new analytic function)**.** A branch of $\log(z)$. Here we take $D = \{z = x + iy : x > 0\}$.

$$f(z) = \operatorname{Log}(z) = \frac{1}{2}\ln\left(x^2 + y^2\right) + i\tan^{-1}(y/x).$$

Recall that $\frac{d}{dt}\tan^{-1}(t) = \frac{1}{t^2+1}$ so we learn

$$f_x = \frac{1}{2}\frac{2x}{x^2+y^2} + i\frac{-\frac{y}{x^2}}{1+(y/x)^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{|z|^2}\bar{z} = \frac{1}{z}$$
$$f_y = \frac{1}{2}\frac{2y}{x^2+y^2} + i\frac{\frac{1}{x}}{1+(y/x)^2} = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} = i\frac{1}{z} = if_x$$

from which it follows that f is complex differentiable and $f'(z) = \frac{1}{z}$.

Note that for $\operatorname{Im} z > 0$, we have $\operatorname{Log}(z) = f\left(\frac{1}{i}z\right) + i\pi/2$ which shows $\operatorname{Log}(z)$ is complex differentiable for $\operatorname{Im} z > 0$.

Similarly, if $\operatorname{Im} z < 0$, we have $\operatorname{Log}(z) = f(iz) - i\pi/2$ which shows $\operatorname{Log}(z)$ is complex differentiable for $\operatorname{Im} z < 0$.

Combining these remarks shows that $\operatorname{Log}(z)$ is complex differentiable on $\mathbb{C} \setminus (-\infty, 0]$.

Example 9.7 (Homework Problem: Problem 7a on p.74)**.** Suppose that f is a complex differentiable function such that $\operatorname{Im} f = 0$. Then f_x and f_y are real and $f_y = if_x$ can happen iff $f_x = f_y = 0$. But this implies that f is constant.

Example 9.8 (Problem 7b on p.74 in class!)**.** Now suppose that $|f(z)| = c \neq 0$ for all z is a domain D. Then

$$\overline{f(z)} = \frac{|f(z)|^2}{f(z)} = \frac{c^2}{f(z)}$$

which shows $\bar{f}$ is complex differentiable and from this it follows that $\operatorname{Re} f = \frac{f+\bar{f}}{2}$ and $\operatorname{Im} f = \frac{f-\bar{f}}{2i}$ are real valued complex differentiable functions. So by the previous example, both $\operatorname{Re} f$ and $\operatorname{Im} f$ are constant and hence f is constant.

Test #1 was on 10/17/03. This would have been lecture 10.

10. (10/20/2003) Lecture 10

Definition 10.1 (Analytic Functions)**.** A function $f : D \to \mathbb{C}$ is said to be **analytic (or holomorphic)** on an open subset $D \subset \mathbb{C}$ if $f'(z)$ exists for all $z \in D$.

Proposition 10.2. *Let $f = u + iv$ be complex differentiable, and suppose the level curves $u = a$ and $v = b$ cross at a point z_0 where $f'(z_0) \neq 0$ then they cross at a right angle.*

Proof. The normals to the level curves are given by ∇u and ∇v, so it suffices to observe from the Cauchy Riemann equations that

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = v_y v_x + (-v_x) v_y = 0.$$

Draw Picture.

Alternatively: Parametrize $u = a$ and $v = b$ by $z(t)$ and $w(t)$ so that $z(0) = z_0 = w(0)$. Then $f(z(t)) = a + iv(z(t))$ and $f(w(t)) = u(w(t)) + ib$ and

$$i\beta = \frac{d}{dt}|_0 f(z(t)) = f'(z_0)\,\dot{z}(0) \text{ while}$$
$$\alpha = \frac{d}{dt}|_0 f(w(t)) = f'(z_0)\,\dot{w}(0)$$

where $\alpha = \frac{d}{dt}|_0 u(w(t))$ and $\beta = \frac{d}{dt}|_0 v(z(t))$. Therefore

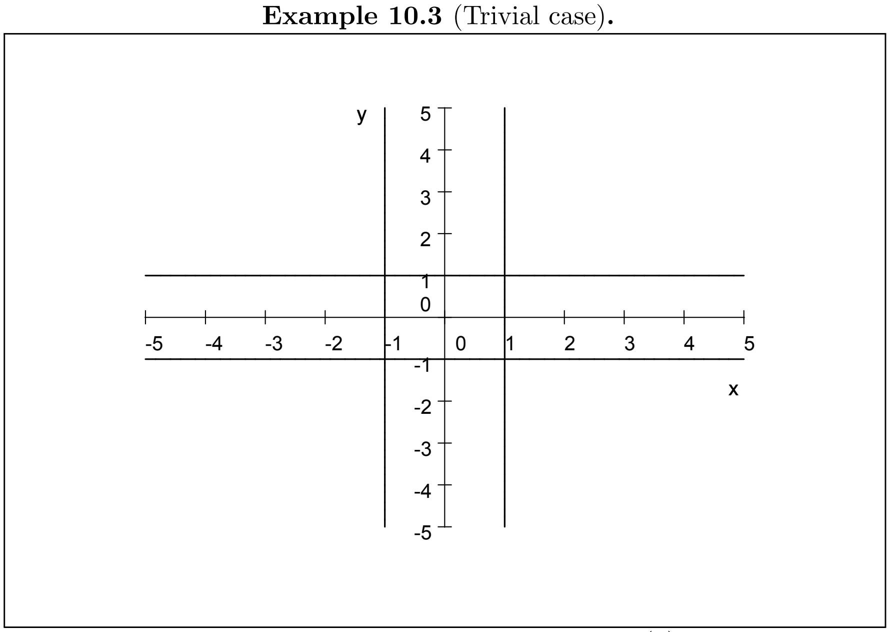
$$\operatorname{Re}\left[\dot{z}(0)\,\overline{\dot{w}(0)}\right] = \operatorname{Re}\left[\frac{i\beta}{f'(z_0)}\overline{\frac{\alpha}{f'(z_0)}}\right] = 0.$$

Alternatively,

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = v_y v_x + (-v_x) v_y = 0.$$

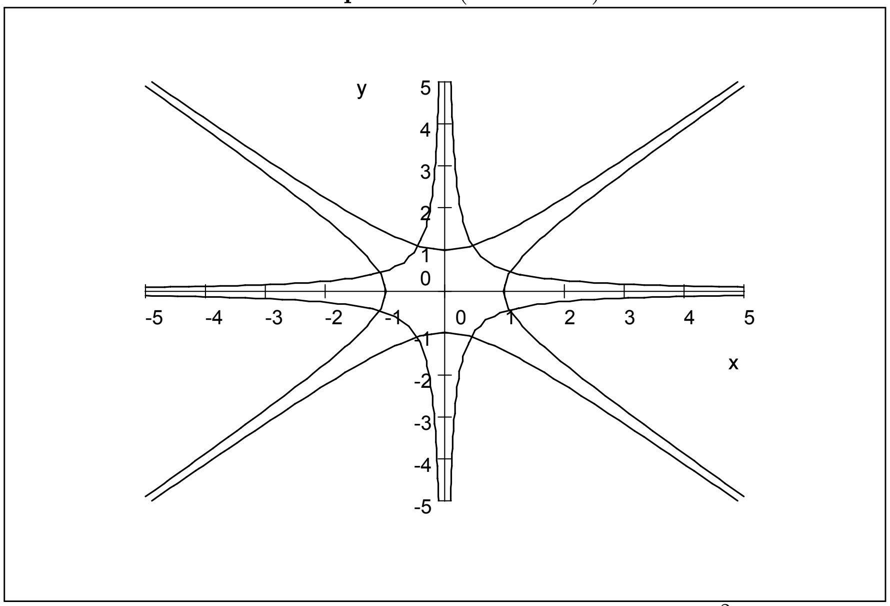
■

Example 10.3 (Trivial case)**.**



Some Level curves of $\operatorname{Re} f$ and $\operatorname{Im} f$ for $f(z) = z$.

Example 10.4 (Homework)**.**



Some Level curves of $\operatorname{Re} f$ and $\operatorname{Im} f$ for $f(z) = z^2$.

10.1. **Harmonic Conjugates.**

Definition 10.5. Given a harmonic function u on a domain $D \subset \mathbb{C}$, we say v is a harmonic conjugate to u if v is harmonic and u and v satisfy the C.R. equations.

Notice that v is uniquely determined up to a constant since if w is another harmonic conjugate we must have

$$w_y = u_x = v_y \text{ and } w_x = -u_y = v_x.$$

Therefore $\frac{d}{dt} w(z(t)) = \frac{d}{dt} v(z(t))$ for all paths z in D and hence $w = v + C$ on D.

Proposition 10.6. *$f = u + iv$ is complex analytic on D iff u and v are harmonic conjugates.*

Example 10.7. Suppose $u(x, y) = x^2 - y^2$ we wish to find a harmonic conjugate. For this we use

$$\begin{aligned} v_y &= u_x = 2x \text{ and} \\ v_x &= -u_y = 2y \end{aligned}$$

to conclude that $v = 2xy + C(x)$ and then $2y = v_x = 2y + C'(x)$ which implies $C'(x) = 0$ and so $C =$const. Thus we find

$$f = u + iv = x^2 - y^2 + i2xy + iC = z^2 + iC$$

is analytic.

Example 10.8. Now suppose that $u = 2xy$. In this case we have

$$\begin{aligned} v_y &= u_x = 2y \text{ and} \\ v_x &= -u_y = -2x \end{aligned}$$

and so $v = \frac{y^2}{2} + C(x)$ and so $-2x = v_x = C'(x)$ from which we learn that $C(x) = -x^2 + k$. Thus we find

$$f = 2xy + i\left(y^2 - x^2\right) + ik = -iz^2 + ik$$

is complex analytic.

Recall the following definitions:

Definition 10.9. For $z \neq 0$, let $\log z = \{w \in \mathbb{C} : e^w = z\}$.

Writing $z = |z|\, e^{i\theta}$ we and $w = x + iy$, we must have $|z|\, e^{i\theta} = e^x e^{iy}$ and this implies that $x = \ln |z|$ and $y = \theta + 2\pi n$ for some n. Therefore

$$\log z = \ln |z| + i \arg z.$$

Definition 10.10. $\mathrm{Log}(z) = \ln |z| + i\mathrm{Arg}(z)$, so $\mathrm{Log}(re^{i\theta}) = \ln r + i\theta$ if $r > 0$ and $-\pi < \theta \leq \pi$. Note this function is **discontinuous** at points z where $\mathrm{Arg}(z) = \pi$.

Example 10.11. Find $\log 1$, $\log i$, $\log\left(-1 - \sqrt{3}i\right)$.

Theorem 10.12 (Converse Chain Rule: Optional)**.** *Suppose $f : D \subset_o \mathbb{C} \to U \subset_o \mathbb{C}$ and $g : U \subset_o \mathbb{C} \to \mathbb{C}$ are functions such that f is continuous, g is analytic and $h := g \circ f$ is analytic, then f is analytic on the set $D \setminus \{z : g'(f(z)) = 0\}$. Moreover $f'(z) = h'(z)/g'(f(z))$ when $z \in D$ and $g'(f(z)) \neq 0$.*

Proof. Suppose that $z \in D$ and $g'(f(z)) \neq 0$. Let $\Delta f = f(z + \Delta z) - f(z)$ and notice that $\Delta f = \varepsilon(\Delta z)$ because f is continuous at z. On one hand

$$h(z + \Delta z) = h(z) + (h'(z) + \varepsilon(\Delta z))\Delta z$$

while on the other

$$\begin{aligned} h(z + \Delta z) &= g(f(z + \Delta z)) = g(f(z) + \Delta f) \\ &= g(f(z)) + [g'(f(z) + \varepsilon(\Delta f)]\,\Delta f \\ &= h(z) + [g'(f(z) + \varepsilon(\Delta z)]\,\Delta f. \end{aligned}$$

Comparing these two equations implies that

$$(h'(z) + \varepsilon(\Delta z))\Delta z = [g'(f(z)) + \varepsilon(\Delta z)]\,\Delta f \tag{10.1}$$

and since $g'(f(z)) \neq 0$ we may conclude that

$$\frac{\Delta f}{\Delta z} = \frac{h'(z) + \varepsilon(\Delta z)}{g'(f(z)) + \varepsilon(\Delta z)} \to \frac{h'(z)}{g'(f(z))} \text{ as } \Delta z \to 0,$$

i.e. $f'(z)$ exists and $f'(z) = \frac{h'(z)}{g'(f(z))}$. ■

Definition 10.13 (Inverse Functions)**.** Given a function $f : \mathbb{C} \to \mathbb{C}$ we let $f^{-1}(w) := \{z \in \mathbb{C} : f(z) = w\}$. In general this is a multivalued function and we will have to choose a branch when we need an honest function.

Example 10.14. Since $e^{\mathrm{Log}(z)} = z$ and $\mathrm{Log}(z)$ is continuous on $D := \mathbb{C} \setminus (-\infty, 0]$, $\mathrm{Log}(z)$ is complex analytic on D and

$$1 = \frac{d}{dz}z = \frac{d}{dz}e^{\mathrm{Log}(z)} = e^{\mathrm{Log}(z)}\frac{d}{dz}\mathrm{Log}(z) = z\frac{d}{dz}\mathrm{Log}(z),$$

i.e. we have

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}.$$

11. (10/22/2003) Lecture 11

Example 11.1. In fact the above example generalizes, suppose $\ell(z)$ is any branch of $\log(z)$, that is ℓ is a continuous function on an open set $D \subset \mathbb{C}$ such that $e^{\ell(z)} = z$, then $\ell'(z) = 1/z$. Indeed, this follows just as above using the converse to the chain rule.

- Give the proof of Theorem 10.12.

Lemma 11.2. *The following properties of* log *hold.*

(1) $e^{\log z} = z$
(2) $\log e^z = z + i2\pi\mathbb{Z}$
(3) $z^n = e^{n\log(z)} = e^{\log z + \log z + \cdots + \log z}$ *(n – times.)*
(4) $z^{1/n} = e^{\frac{1}{n}\log z}$
(5) $\log z^{\pm 1/n} = \pm\frac{1}{n}\log z$ *but be careful:*
(6) $\log z^n \neq n\log z$
(7) $\log(wz) = \log w + \log z$ *and in particular*

$$\log z^n = \overbrace{\log z + \log z + \cdots + \log z}^{n\text{ – }times}.$$

Proof.

(1) This is by definition.
(2) $\log e^z = \log e^{z+i2\pi\mathbb{Z}} = x + i(y + 2\pi\mathbb{Z}) = z + i2\pi\mathbb{Z}$.
(3) If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$ for any $\theta \in \arg z$, therefore

$$z^n = r^n e^{in\arg(z)} = e^{n\ln r}e^{in\arg(z)} = e^{n\log z}.$$

Better proof, if $w \in \log z$, then $z = e^w$ so that $z^n = e^{nw}$ for any $w \in \log z$, so $z^n = e^{n\log(z)}$.

(4) We know

$$z^{1/n} = |z|^{1/n} e^{i\frac{1}{n}\arg z} = e^{\frac{1}{n}\ln|z|}e^{i\frac{1}{n}\arg z} = e^{\frac{1}{n}\log z}.$$

(5) Now

$$\log z^{\pm 1/n} = \ln\left(|z|^{1/n}\right) \pm i\frac{1}{n}\arg z + i2\pi\mathbb{Z} = \ln\left(|z|^{1/n}\right) \pm i\frac{1}{n}\arg z = \pm\frac{1}{n}\log z.$$

(6) On the other hand if $z = |z|e^{i\theta}$, then

$$\log z^n = \ln|z|^n + i\arg(z^n) = n\ln|z| + i(n\theta + 2\pi\mathbb{Z}) = n\ln|z| + in\theta + i2\pi\mathbb{Z}$$

while

$$n\log z = n(\ln|z| + i\theta + i2\pi\mathbb{Z}) = n\ln|z| + in\theta + i2\pi n\mathbb{Z}.$$

(7) This follows from the corresponding property $\arg(wz)$ and for ln,

$$\log(wz) = \ln|wz| + i\arg(wz) = \ln|w| + \ln|z| + i[\arg(w) + \arg(z)] = \log w + \log z.$$

■

Definition 11.3. For $c \in \mathbb{C}$, let $z^c := e^{c\log z}$.

As an example let us work out i^i :

$$i^i = e^{i\log i} = e^{ii(\pi/2 + n2\pi)} = e^{-(\pi/2+n2\pi)}.$$

Example 11.4. Let ℓ be a **branch of** $\log(z)$, i.e. a continuous choice $\ell : D \to \mathbb{C}$ such that $\ell(z) \in \log(z)$ for all $z \in D$ then we define

$$\begin{aligned}\frac{d}{dz} z_\ell^c &= \frac{d}{dz} e^{c\ell(z)} = e^{c\ell(z)} c\ell'(z) \\ &= ce^{c\ell(z)}\frac{1}{z} = ce^{c\ell(z)}e^{-\ell(z)} = ce^{(c-1)\ell(z)} = cz_\ell^{c-1}.\end{aligned}$$

The book writes $P.V.z^c = z_+^c := z_{\mathrm{Log}}^c := e^{c\mathrm{Log}(z)}$ for the **principal value** choice. Note with these definitions we have

$$z_\ell^{-c} = e^{-c\ell(z)} = \frac{1}{e^{c\ell(z)}} = \frac{1}{z_\ell^c}.$$

and when $n \in \mathbb{N}$, then

$$(z_\ell^c)^n = e^{nc\mathrm{Log}(z)} = z_\ell^{nc}$$

however

$$\left(z_{\mathrm{Log}}^c\right)_{\mathrm{Log}}^d = \left(e^{c\mathrm{Log}(z)}\right)_{\mathrm{Log}}^d = e^{d\mathrm{Log}\left(e^{c\mathrm{Log}(z)}\right)} = e^{d(c\mathrm{Log}(z)+2\pi in))} = z_{\mathrm{Log}}^{dc}e^{i2\pi nd}$$

for some integer n.

Definition 11.5 (Trig. and Hyperbolic Trig. functions:)**.**

$$\begin{aligned}\sin(z) &:= \frac{e^{iz}-e^{-iz}}{2i} \\ \cos(z) &:= \frac{e^{iz}+e^{-iz}}{2} \\ \tan(z) &= \frac{\sin(z)}{\cos(z)} = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} \\ \sinh(z) &:= \frac{e^{z}-e^{-z}}{2} \\ \cosh(z) &:= \frac{e^{z}+e^{-z}}{2} \\ \tanh(z) &= \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}.\end{aligned}$$

Example 11.6. Basic properties of Trig. functions.

(1) $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\sinh z = \cosh z$
(2) $\frac{d}{dz}\cos z = -\sin z$ and $\frac{d}{dz}\cosh z = \sinh z$
(3) $\sin z = -i\sinh(iz)$ or $\sin iz = -i\sinh(iiz) = -i\sinh(-z)$, i.e.

$$\sin iz = i\sinh z.$$

Alternatively

$$\sin iz = \frac{e^{iiz}-e^{-iiz}}{2i} = -\frac{e^{z}-e^{-z}}{2i} = i\sinh z.$$

(4) $\cos z = \cosh(iz)$ or $\cosh(z) = \cos(iz)$.
(5) All the usual identities hold. For example

$$\cos(w+z) = \cos w\cos z - \sin w\sin z \tag{11.1}$$

$$\sin(w+z) = \sin w\cos z + \cos w\sin z. \tag{11.2}$$

Indeed,

$$\begin{aligned}\cos w \cos z - \sin w \sin z &= \frac{e^{iw}+e^{-iw}}{2}\frac{e^{iz}+e^{-iz}}{2} - \frac{e^{iw}-e^{-iw}}{2i}\frac{e^{iz}-e^{-iz}}{2i} \\ &= \frac{1}{4}\left[2e^{i(w+z)}+2e^{-i(w+z)}\right] = \cos\left(w+z\right)\end{aligned}$$

and (this one is homework)

$$\begin{aligned}\sin w \cos z + \cos w \sin z &= \frac{e^{iw}-e^{-iw}}{2i}\frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iw}+e^{-iw}}{2}\frac{e^{iz}-e^{-iz}}{2i} \\ &= \frac{1}{4i}\left[2e^{i(w+z)}-2e^{-i(w+z)}\right] = \sin\left(w+z\right).\end{aligned}$$

12. (10/24/2003) Lecture 12

Remark 12.1 (Roots Remarks).

(1) **Warning:** $1^i = e^{i\log 1} = e^{i(i2\pi\mathbb{Z})} = \left\{1, e^{\pm 2\pi}, e^{\pm 4\pi}, \dots\right\} \neq 1$

(2) $i^i = e^{i\log i} = e^{ii(\pi/2+n2\pi)} = e^{-(\pi/2+n2\pi)} = e^{-\pi/2}\left\{1, e^{\pm 2\pi}, e^{\pm 4\pi}, \dots\right\}$.

(3) On the positive side we do have $\left(w^2 z\right)^{1/2} = wz^{1/2}$ or more generally that

$$\left(w^n z\right)^{1/n} = wz^{1/n}$$

for any integer n. To prove this, $\xi \in \left(w^n z\right)^{1/n}$ iff $\xi^n = w^n z$ iff $\left(\frac{\xi}{w}\right)^n = z$ iff $\frac{\xi}{w} \in z^{1/n}$ iff $\xi \in wz^{1/n}$.

Alternatively,

$$\left(w^2 z\right)^{1/2} = e^{\frac{1}{2}\log\left(w^2 z\right)} = e^{\frac{1}{2}\left[\log(z)+\log\left(w^2\right)\right]} = e^{\frac{1}{2}\log(z)}e^{\frac{1}{2}\log\left(w^2\right)} = z^{1/2}e^{\frac{1}{2}\log\left(w^2\right)}.$$

Now if $w = re^{i\theta}$, then

$$\log w^2 = 2\ln r + i\left(2\theta + 2\pi\mathbb{Z}\right)$$

and therefore,

$$e^{\frac{1}{2}\log w^2} = e^{[\ln r + i(\theta+\pi\mathbb{Z})]} = \pm re^{i\theta} = \pm w.$$

But $\pm wz^{1/2} = z^{1/2}$.

(4) $\log z^{1/2} = \frac{1}{2}\log z$. Indeed,

$$\log z^{1/2} = \log\left(e^{\frac{1}{2}[\ln|z|+i\arg z]}\right) = \frac{1}{2}\left[\ln|z| + i\arg z\right] + i2\pi\mathbb{Z} = \frac{1}{2}\left[\ln|z| + i\arg z\right] = \frac{1}{2}\log z.$$

Example 12.2. Continuing Example 11.6 above.

(1)

$$\sin^2 z + \cos^2 z = \left[\frac{e^{iz} - e^{-iz}}{2i}\right]^2 + \left[\frac{e^{iz} + e^{-iz}}{2}\right]^2 = \frac{1}{4}4 = 1.$$

(2) Taking $w = x$ and $z = iy$ in the above equations shows

$$\begin{aligned}\cos z &= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \cosh y - i \sin x \sinh y\end{aligned}$$

and

$$\begin{aligned}\sin z &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

(3) From this it follows that $\sin z = 0$ iff $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$. Since, $\cosh y$ is never zero we must have $\sin x = 0$ in which case $\cos x \neq 0$ so that $\sinh y = 0$ i.e. $y = 0$. So the only solutions to $\sin z = 0$ happen when z is real and hence $z = \pi\mathbb{Z}$. A similar argument works for $\cos z$.

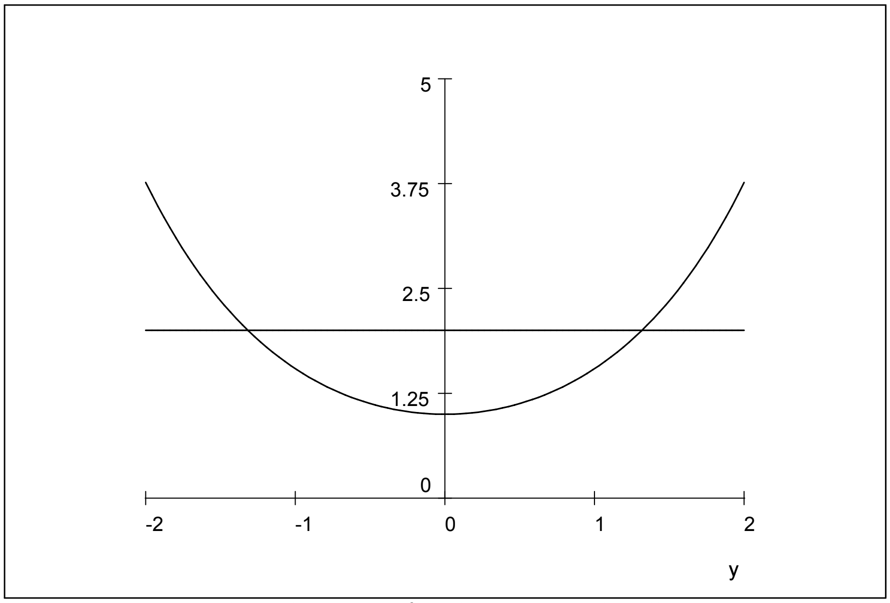
(4) Lets find all the roots of $\sin z = 2$,

$$2 = \sin z = \sin x \cosh y + i \cos x \sinh y$$

and so

$$\cos x \sinh y = 0 \text{ and } \sin x \cosh y = 2.$$

Hence either $y = 0$ and $\sin x = 2$ which is impossible of $\cos x = 0$, i.e. $x = \frac{\pi}{2} + n\pi$ for some integer n, and in this case $\sin x = (-1)^n$ and we must have $(-1)^n \cosh y = 2$ which can happen only for **even** n. Now $\cosh y = 2$



Finding the roots of $\cosh y = 2$ graphically.

iff (with $\xi = e^y$) $2 = \frac{\xi+\xi^{-1}}{2}$, i.e.

$$\xi^2 + 1 - 4\xi = 0$$

or

$$\xi = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}.$$

Therefore $y = \ln\left(2 \pm \sqrt{3}\right)$ and we have

$$\sin z = 2 \text{ iff } z = \frac{\pi}{2} + 2n\pi + i\ln\left(2 \pm \sqrt{3}\right) \text{ for some } n \in \mathbb{Z}. \tag{12.1}$$

It should be noted that

$$\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 4 - 3 = 1$$

so that the previous equation may be written as

$$z = \frac{\pi}{2} + 2n\pi \pm i\ln\left(2+\sqrt{3}\right).$$

Theorem 12.3. *The inverse trig. functions*

$$\begin{aligned}
\sin^{-1}(z) &= -i\log\left(iz + \left(1-z^2\right)^{1/2}\right) \\
\cos^{-1}(z) &= -i\log\left(z + i\left(1-z^2\right)^{1/2}\right) \\
\tan^{-1}(z) &= \frac{i}{2}\log\left(\frac{i+z}{i-z}\right).
\end{aligned}$$

.

Moreover we have

$$\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$$
$$\frac{d}{dz}\cos^{-1}(z) = \frac{-1}{\sqrt{1-z^2}}$$
$$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$$

with appropriate choices of branches being specified.

Example 12.4.

$$\cos^{-1}(0) = \frac{1}{i}\log(\pm i) = \frac{1}{i}i\left(\pm\frac{\pi}{2} + 2\pi\mathbb{Z}\right) = \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\right\}$$

so the zeros of the complex cosine function are precisely the zeros of the real cosine function. Similarly

$$\begin{aligned}
\sin^{-1}(2) &= -i\log\left(i2 + (1-4)^{1/2}\right) = -i\log\left(i\left(2\pm\sqrt{3}\right)\right) \\
&= -i\left[\log i + \log\left(2\pm\sqrt{3}\right)\right] = -i\left[\frac{\pi}{2} + 2\pi n + \ln\left(2\pm\sqrt{3}\right)\right] \\
&= -i\left[i\left(\frac{\pi}{2} + 2\pi n\right) + \ln\left(2\pm\sqrt{3}\right)\right] \\
&= \frac{\pi}{2} + 2\pi n - i\ln\left(2\pm\sqrt{3}\right) = \frac{\pi}{2} + 2\pi n \pm i\ln\left(2+\sqrt{3}\right)
\end{aligned}$$

as before.

Proof.

- $\cos^{-1}(w)$: We have $z \in \cos^{-1}(w)$ iff
$$w = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \frac{\xi + \xi^{-1}}{2}$$
where $\xi = e^{iz}$. Thus
$$\xi^2 - 2w\xi + 1 = 0$$
or
$$\xi = \frac{2w + \left(4w^2 - 4\right)^{1/2}}{2} = w + \left(w^2 - 1\right)^{1/2}$$
and therefore
$$iz = \log\xi = \log\left(w + \left(w^2-1\right)^{1/2}\right)$$
and we have shown
$$\begin{aligned}
\cos^{-1}(w) &= -i\log\left(w + \left(w^2-1\right)^{1/2}\right) \\
&= -i\log\left(w + i\left(1-w^2\right)^{1/2}\right).
\end{aligned}$$

■

13. (10/31/2003) Lecture 13 (Contour Integrals)

Lost two Lectures because of the big fire!!

(Here I only computed $\frac{d}{dz}\tan^{-1}(z)$ in the proof below.)

Proof. Continuation of the proof.

Let us now compute the derivative of this $\cos^{-1}(z)$. For this we will need to take a branch of $f(z)$ of $\cos^{-1} z$, say

$$f(z) = -i\ell\left(z + iQ\left(1 - z^2\right)\right) \tag{13.1}$$

where ℓ is a branch of log and Q is a branch of the square-root. Then $\cos f(z) = z$ and differentiating this equations gives, $-\sin f(z) \cdot f'(z) = 1$ or equivalently that

$$f'(z) = \frac{1}{-\sin f(z)} \in -\frac{1}{\left(1 - z^2\right)^{1/2}}.$$

since $\sin f(z) \in \left(1 - z^2\right)^{1/2}$. The question now becomes which branch do we take. To determine this let us differentiate Eq. (13.1);

$$\begin{aligned} f'(z) &= \frac{-i}{z + iQ\left(1 - z^2\right)}\left\{1 - \frac{i}{2Q\left(1 - z^2\right)}(2z)\right\} \\ &= \frac{-i}{z + iQ\left(1 - z^2\right)}\left\{\frac{Q\left(1 - z^2\right) - iz}{Q\left(1 - z^2\right)}\right\} \\ &= \frac{-1}{z + iQ\left(1 - z^2\right)}\left\{\frac{z + iQ\left(1 - z^2\right)}{Q\left(1 - z^2\right)}\right\} = \frac{-1}{Q\left(1 - z^2\right)} \end{aligned}$$

so we must use the same branch of the square-root used in Eq. (13.1). Hence we have shown

$$\text{“}\frac{d}{dz}\cos^{-1}(z) = \frac{-1}{\sqrt{1 - z^2}}\text{,”}$$

with the branch conditions determined as above.

- $\tan^{-1}(w)$: We have $z \in \tan^{-1}(w)$ iff

  $$w = \tan(z) = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i\frac{\xi - \xi^{-1}}{\xi + \xi^{-1}} = -i\frac{\xi^2 - 1}{\xi^2 + 1}$$

  where $\xi = e^{iz}$. Thus

  $$\left(\xi^2 + 1\right)w + i\left(\xi^2 - 1\right) = 0$$

  or

  $$\xi^2(w + i) = i - w$$

  that is

  $$\xi = \left(\frac{i - w}{i + w}\right)^{\frac{1}{2}}$$

  and hence

  $$iz = \log \xi = \log\left(\frac{i - w}{i + w}\right)^{\frac{1}{2}} = -\frac{1}{2}\log\left(\frac{i + w}{i - w}\right)$$

  so that

  $$\tan^{-1}(w) = \frac{i}{2}\log\left(\frac{i + w}{i - w}\right).$$

We have used here that $\log\left(\eta^{-\frac{1}{n}}\right) = -\frac{1}{n}\log\eta$ which happens because n is an integer, see Lemma 11.2. Let us now compute the derivative of $\tan^{-1}(w)$. In order to do this, let ℓ be a branch of log, and the $f(w) = \frac{i}{2}\ell\left(\frac{i+w}{i-w}\right)$ be a Branch of $\tan^{-1}(w)$, then

$$\begin{aligned}\frac{d}{dw}f(w) &= \frac{i}{2}\frac{1}{\frac{i+w}{i-w}}\frac{d}{dw}\frac{i+w}{i-w} = \frac{i}{2}\frac{i-w}{i+w}\frac{(i-w)+(i+w)}{(i-w)^2} \\ &= -\frac{1}{(i+w)(i-w)} = \frac{1}{(w+i)(w-i)} = \frac{1}{1+w^2}.\end{aligned}$$

Thus we have

$$\frac{d}{dw}\tan^{-1}(w) = \frac{1}{1+w^2}$$

where the formula is valid for any branch of $\tan^{-1}(w)$ that we have chosen.

- $\sin^{-1}(w)$: (This is done in the book so do not do in class.) We have $z \in \sin^{-1}(w)$ iff

$$w = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{\xi - \xi^{-1}}{2i}$$

where $\xi = e^{iz}$. Thus

$$\xi^2 - 1 - 2iw\xi = 0$$

or

$$\xi = \frac{2iw + \left(-4w^2 + 4\right)^{1/2}}{2} = iw + \left(1 - w^2\right)^{1/2}$$

and therefore

$$iz = \log\xi = \log\left(iw + \left(1 - w^2\right)^{1/2}\right)$$

and we have shown

$$\sin^{-1}(w) = -i\log\left(iw + \left(1 - w^2\right)^{1/2}\right).$$

Example if $w = 0$, we have

$$\sin^{-1}(0) = \frac{1}{i}\log(\pm 1) = \frac{1}{i}i\pi\mathbb{Z} = \pi\mathbb{Z}.$$

Suppose that

$$f(w) = -i\ell\left(iw + Q\left(1 - w^2\right)\right) \tag{13.2}$$

where ℓ is a branch of log and Q is a branch of the square-root, then $\sin f(w) = w$ and so differentiating this equation in w gives $\cos f(w) f'(w) = 1$ or equivalently that

$$f'(w) = \frac{1}{\cos f(w)}.$$

Now $\cos f(w) \in \left(1-w^2\right)^{1/2}$, the question is which branch do we take. To determine this let us differentiate Eq. (13.2). Here we have

$$\begin{aligned} f'(w) &= \frac{-i}{iw + Q\left(1-w^2\right)}\left\{i + \frac{1}{2Q\left(1-w^2\right)}(-2w)\right\} \\ &= \frac{1}{iw + Q\left(1-w^2\right)}\left\{1 + i\frac{w}{Q\left(1-w^2\right)}\right\} \\ &= \frac{1}{iw + Q\left(1-w^2\right)}\left\{\frac{iw + Q\left(1-w^2\right)}{Q\left(1-w^2\right)}\right\} = \frac{1}{Q\left(1-w^2\right)} \end{aligned}$$

so we must use the same branch of the square-root used in Eq. (13.2). Hence we have shown

$$\text{“}\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}},\text{”}$$

where one has to be careful about the branches which are used.

■

13.1. Complex and Contour integrals:

Definition 13.1. A path or contour C in $D \subset \mathbb{C}$ is a piecewise C^1 – function $z : [a,b] \to \mathbb{C}$. For a function $f : D \to \mathbb{C}$, we let

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$

Example 13.2 (Some Contours).

(1) $z(t) = z_0 + re^{it}$ for $0 \le t \le \pi$ is a semi-circle centered at z_0.
(2) If $z_0, z_1 \in \mathbb{C}$ then $z(t) = z_0(1-t) + z_1 t$ for $0 \le t \le 1$ parametrizes the straight line segment going from z_0 to z_1.
(3) If $z(t) = t + it^2$ for $-1 \le t \le 1$, then $z(t)$ parametrizes part of the parabola $y = x^2$. More generally $z(t) = t + if(t)$ parametrizes the graph, $y = f(x)$.
(4) $z(t) = t + i\sqrt{1-t^2}$ for $-1 \le t \le 1$ parametrizes the semicircle of radius 1 centered at 0 as does $z(t) = e^{-i\pi t}$ for $-1 \le t \le 0$.

Example 13.3. Integrate $f(z) = z - 1$ along the two contours

(1) $C_1 : z = x$ for $x = 0$ to $x = 2$ and
(2) $C_2 : z = 1 + e^{i\theta}$ for $\pi \le \theta \le 2\pi$.

For the first case we have

$$\int_{C_1}(z-1)\,dz = \int_0^2 (x-1)\,dx = \frac{1}{2}(x-1)^2|_0^2 = 0$$

and for the second

$$\begin{aligned} \int_{C_2}(z-1)\,dz &= \int_\pi^{2\pi}\left(1 + e^{i\theta} - 1\right)ie^{i\theta}d\theta \\ &= \int_\pi^{2\pi} ie^{i2\theta}d\theta = \frac{i}{2}e^{i2\theta}|_\pi^{2\pi} = 0 \end{aligned}$$

Example 13.4. Repeat the above example for $f(z) = \bar{z} - 1$.

For the first case we have

$$\int_{C_1} (\bar{z} - 1)\, dz = \int_0^2 (x-1)\, dx = \frac{1}{2}(x-1)^2 \,|_0^2 = 0$$

and for the second

$$\int_{C_2} (\bar{z} - 1)\, dz = \int_\pi^{2\pi} \left(1 + e^{-i\theta} - 1\right) ie^{i\theta} d\theta$$
$$= \int_\pi^{2\pi} i d\theta = i\pi \neq 0$$

Example 13.5 (I skipped this example.)**.** Here we consider $f(z) = y - x - i3x^2$ along the contours

(1) C_1 : consists of the straight line paths from $0 \to i$ and $i \to 1 + i$ and
(2) C_2 : consists of the straight line path from $0 \to 1 + i$.

1. For the first case $z = iy$, $dz = idy$ and $z = x + i$ and $dz = dx$, so

$$\int_{C_1} f(z)\, dz = \int_0^1 \left(y - x - i3x^2\right) |_{x=0} idy + \int_0^1 \left(y - x - i3x^2\right) |_{y=1} dx$$
$$= i \int_0^1 ydy + \int_0^1 \left(1 - x - i3x^2\right) dx$$
$$= \frac{i}{2} + \left(1 - \frac{1}{2} - i\right) = \frac{1}{2}(1 - i).$$

2. For the second contour, $z = t(1+i) = t + it$, then $dz = (1+i)\, dt$,

$$\int_{C_2} f(z)\, dz = \int_0^1 \left(y - x - i3x^2\right) |_{x=y=t}(1+i)dt$$
$$= (1+i) \int_0^1 \left(t - t - i3t^2\right) dt$$
$$= (1+i)(-i) = 1 - i.$$

Notice the answers are different.

Example 13.6. Now lets use the same contours but with the function, $f(z) = z^2$ instead. In this case

$$\int_{C_1} z^2 dz = \int_0^1 (iy)^2\, idy + \int_0^1 (x+i)^2\, dx$$
$$= -i\frac{1}{3} + \frac{1}{3}(x+i)^3 \,|_0^1 = -i\frac{1}{3} + \frac{1}{3}(1+i)^3 - \frac{1}{3}i^3$$
$$= \frac{1}{3}(1+i)^3$$

while for the second contour,

$$\int_{C_2} z^2 dz = \int_0^1 t^2 (1+i)^2 (1+i)\, dt = \frac{1}{3}(1+i)^3.$$

14. (11/3/2003) Lecture 14 (Contour Integrals Continued)

Proposition 14.1. *Let us recall some properties of complex integrals*

(1)
$$\int_a^b w(\phi(t))\,\dot{\phi}(t)\,dt = \int_{\phi(a)}^{\phi(b)} w(\tau)\,d\tau$$

(2) *If $f(z)$ is continuous in a neighborhood of a contour C, then $\int_C f(z)\,dz$ is independent of how C is parametrized as long as the orientation is kept the same.*

(3) *If $-C$ denotes C traversed in the opposite direction, then*
$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

Proof.

(1) The first fact follows from the change of variable theorem for real variables.

(2) Suppose that $z:[a,b]\to\mathbb{C}$ is a parametrization of C, then other parametrizations of C are of the form
$$w(s) = z(\phi(s))$$
where $\phi:[\alpha,\beta]\to[a,b]$ such that $\phi(\alpha)=a$ and $\phi(\beta)=b$. Hence
$$\int_\alpha^\beta f(w(s))\,w'(s)\,ds = \int_\alpha^\beta f(z(\phi(s)))\,\dot{z}(\phi(s))\,\phi'(s)\,ds$$
and letting $t=\phi(s)$, we find
$$\int_\alpha^\beta f(w(s))\,w'(s)\,ds = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$
as desired.

(3) Suppose that $z:[0,1]\to\mathbb{C}$ is a parametrization of C, then $w(s) := z(1-s)$ parametrizes $-C$, so that
$$\begin{aligned}\int_{-C} f(z)\,dz &= -\int_0^1 f(z(1-s))\,\dot{z}(1-s)\,ds = \int_1^0 f(z(t))\,\dot{z}(t)\,dt \\ &= -\int_0^1 f(z(t))\,\dot{z}(t)\,dt = -\int_C f(z)\,dz,\end{aligned}$$
wherein we made the change of variables, $t=1-s$.

■

Theorem 14.2 (Fundamental Theorem of Calculus)**.** *Suppose C is a contour in D and $f:D\to\mathbb{C}$ is an analytic function, then*
$$\int_C f'(z)\,dz = f(C_{end}) - f(C_{\boldsymbol{begin}}).$$

Proof. Let $z:[a,b]\to\mathbb{C}$ parametrize the contour, then
$$\int_C f'(z)\,dz = \int_a^b f'(z(t))\,\dot{z}(t)\,dt = \int_a^b \frac{d}{dt} f(z(t))\,dt = f(z(t))\,|_{t=a}^{t=b}.$$

■

Example 14.3. Using either of contours in Example 13.5, we again learn (more easily)

$$\int_{C_1} z^2 dz = \frac{1}{3} z^3|_{\partial C_1} = \frac{1}{3}\left[(1+i)^3 - (0)^3\right].$$

Example 14.4. Suppose that C is a closed contour in $\mathbb{C}$ such which does not pass through 0, then

$$\int_C z^n dz = 0 \text{ if } n \neq -1.$$

The case $n = 1$ is different and leads to the winding number. This can be computed explicitly, using a branch of a logarithm. For example if $C : [0, 2\pi] \to \mathbb{C}\setminus\{0\}$ crosses $(-\infty, 0)$ only at $z(0) = z(2\pi)$, then

$$\begin{aligned}
\int_C \frac{1}{z} dz &= \lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} \frac{1}{z} dz = \lim_{\varepsilon \downarrow 0} \left[\mathrm{Log}(z(2\pi - \varepsilon)) - \mathrm{Log}(z(\varepsilon))\right] \\
&= \lim_{\varepsilon \downarrow 0} \left[\ln\left|\frac{z(2\pi - \varepsilon)}{z(\varepsilon)}\right| + i\left(2\pi - O(\varepsilon) - O(\varepsilon)\right)\right] \\
&= i2\pi.
\end{aligned}$$

Also work out explicitly the special case where $C(\theta) = re^{i\theta}$ with $\theta : 0 \to 2\pi$.

Proposition 14.5. *Let us recall some estimates of complex integrals*

(1)
$$\left|\int_\alpha^\beta w(t)\,dt\right| \le \int_\alpha^\beta |w(t)|\,dt.$$

(2) *We also have*
$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz| \le ML$$
where $|dz| = |\dot{z}(t)|\,dt$ *and* $M = \sup_{z\in C} |f(z)|$.

Proof.

(1) To prove this let $\rho \ge 0$ and $\theta \in \mathbb{R}$ be chosen so that
$$\int_\alpha^\beta w(t)\,dt = \rho e^{i\theta},$$
then
$$\begin{aligned}
\left|\int_\alpha^\beta w(t)\,dt\right| = \rho = e^{-i\theta}\int_\alpha^\beta w(t)\,dt &= \int_\alpha^\beta e^{-i\theta} w(t)\,dt \\
&= \int_\alpha^\beta \mathrm{Re}\left[e^{-i\theta} w(t)\right] dt \le \int_\alpha^\beta \left|\mathrm{Re}\left[e^{-i\theta} w(t)\right]\right| dt \\
&\le \int_\alpha^\beta \left|e^{-i\theta} w(t)\right| dt = \int_\alpha^\beta |w(t)|\,dt.
\end{aligned}$$

Alternatively:

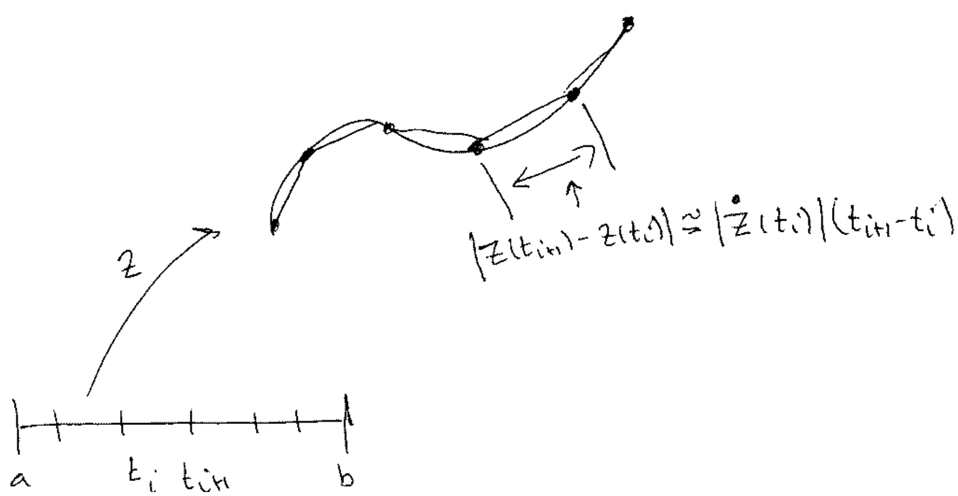
$$\left|\int_\alpha^\beta w(t)\,dt\right| = \left|\lim_{\text{mesh}\to 0}\sum w(c_i)(t_i - t_{i-1})\right| = \lim_{\text{mesh}\to 0}\left|\sum w(c_i)(t_i - t_{i-1})\right|$$
$$\leq \lim_{\text{mesh}\to 0}\sum |w(c_i)|(t_i - t_{i-1}) \text{ (by the triangle inequality)}$$
$$= \int_\alpha^\beta |w(t)|\,dt.$$

(2) For the last item

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,\dot{z}(t)\,dt\right| \leq \int_a^b |f(z(t))|\,|\dot{z}(t)|\,dt$$
$$\leq M\int_a^b |\dot{z}(t)|\,dt \leq ML$$

wherein we have used

$$|\dot{z}(t)|\,dt = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}dt = d\ell.$$



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15. (11/05/2003) Lecture 15

Example 15.1. The goal here is to estimate the integral

$$\left|\int_C \frac{1}{z^4} dz\right|$$

where C is the contour joining i to 1 by a straight line path. In this case $M = \frac{1}{\left|\frac{1}{2}(1+i)\right|^4}$ and $L = |1-i| = \sqrt{2}$ and this gives the estimate

$$\left|\int_C \frac{1}{z^4} dz\right| \le \sqrt{2}\frac{1}{\left|\frac{1}{\sqrt{2}}\right|^4} = 4\sqrt{2}.$$

Example 15.2. Let C be the contour consisting of straight line paths $-4 \to 0$, $0 \to 3i$ and then $3i \to -4$ and we wish to estimate the integral

$$\int_C (e^z - \bar{z})\, dz.$$

To do this notice that on C we have

$$|e^z - \bar{z}| \le |e^z| + |\bar{z}| \le e^{\operatorname{Re} z} + |z| \le e^0 + 4 = 5$$

while

$$\ell(C) = 4 + 3 + |3i - (-4)| = 4 + 3 + \sqrt{3^2 + 4^2} = 3 + 4 + 5 = 12$$

and hence

$$\left|\int_C (e^z - \bar{z})\, dz\right| \le 12 \cdot 5 = 60.$$

Note: The material after this point will not be on the second midterm.

Notation 15.3. Let $D \subset_o \mathbb{C}$ and $\alpha : [a,b] \to D$ and $\beta : [a,b] \to D$ be two piecewise C^1 – contours in D. Further assume that either $\alpha(a) = \beta(a)$ and $\alpha(b) = \beta(b)$ or α and β are loops. We say α is **homotopic to** β if there is a continous map $\sigma : [a,b] \times [0,1] \to D$, such that $\sigma(t,0) = \alpha(t)$, $\sigma(t,1) = \beta(t)$ and either $\sigma(a,s) = \alpha(a) = \beta(a)$ and $\sigma(b,s) = \alpha(b) = \beta(b)$ for all s or $t \to \sigma(t,s)$ is a loop for all s. Draw lots of pictures here.

Definition 15.4 (Simply Connected)**.** A connected region $D \subset_o \mathbb{C}$ is **simply connected** if all closed countours, $C \subset D$ are homotopic to a constant path.

Theorem 15.5 (Cauchy Goursat Theorem)**.** *Suppose that $f : D \to \mathbb{C}$ is an analytic function and α and β are two contours in D which are homotopic relative end-points or homotopic loops in D, then*

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz.$$

In particular if D is simply connected, then

$$\int_C f(z)\, dz = 0$$

for all closed contours in D and complex analytic functions, f, on D.

Example 15.6. Suppose C is a closed contour in $\mathbb{C}$, then

(1) $\int_C e^{\sin z} dz = 0$ and $\int_\alpha e^{\sin z} dz$ depends only on the endpoints of α.
(2) $\int_C z^n dz = 0$ for all $n \in \mathbb{N} \cup \{0\}$.

(3) However if $C(\theta) = re^{i\theta}$ for $\theta : 0 \to 2\pi$, then
$$\int_C \bar{z}dz = \int_0^{2\pi} re^{-i\theta} ire^{i\theta} d\theta = 2\pi i r^2 \neq 0.$$

Example 15.7. Suppose C is a closed contour in $\mathbb{C} \setminus \{0\}$, then

(1) $\int_C e^{\sin z} dz = 0$ and $\int_\alpha e^{\sin z} dz$ depends only on the endpoints of α.
(2) However $\int_C z^{-1} dz = 2\pi i \neq 0$.
(3) On the other hand if C is a loop in $\mathbb{C} \setminus (-\infty, 0]$, then we know
$$\int_C z^{-1} dz = 0$$
this can be checked by direct computation. However it is harder to check directly that
$$\int_C \frac{e^{\sin z}}{z} dz = 0$$
for all closed contours in $\mathbb{C} \setminus (-\infty, 0]$.