Multisymplectic Geometry and Classical Field Theories

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Contents

1 Motivation

In this part, we will motivate multisymplectic geometry as a generalization of the variational principle in passing from classical mechanics to classical field theory. Although we will later give the general definition of a multisymplectic manifold, we will not explore the details of this geometry in generality (see references mentioned in the paper) but rather focus on a particular multisymplectic manifold; namely, the multiphase space of classical field theory.

1.1 The Variational Principle in Time-Dependent Classical Mechanics

To begin, we will briefly review time-dependent classical mechanics. Let $I \subset \mathbb{R}$ be an open interval and Q the configuration space of a particle. Let θ be the canonical form on $T^*(I \times Q)$ which in local coordinates $((t, \xi), (p_i, q^i))$ has the form $\theta = p_i dq^i + \xi dt$. Define the symplectic form $\omega = -d\theta$. Let $H : I \times T^*Q \to \mathbb{R}$ be a (time-dependent) Hamiltonian function. Viewing $T^*(I \times Q) \cong \mathbb{R} \times (I \times T^*Q)$, we define the contact form $\theta_H = (H, \mathbf{1}_{I\times T^*Q})^*\theta$ where $(H, \mathbf{1}_{I\times T^*Q})$ is interpreted as a map $I \times T^*Q \to \mathbb{R} \times (I \times T^*Q)$. In local coordinates, $\theta_H = p_i dq^i + H dt$ and similarly define $\omega_H = -d\theta_H$. Consider the action functional

$$S[\psi] = \int_{\psi(I)} \theta_H,$$

on the space of curves on $I \times T^*Q$ covering the identity on I, i.e. in coordinates the curve has the form $\psi(t) = (t, q^i(t), p_i(t))$ (for the rest of this section when we refer to curves, we mean those of this form). To see that this is the usual action one sees in Hamiltonian mechanics, using the above coordinate expressions,

$$S[\psi] = \int_{\psi(I)} \theta_H = \int_I \psi^* \theta_H = \int_I \psi^*(p_i dq^i + H dt) = \int_I \Big(p_i(t)\dot{q}^i(t) + H(t, q(t), p(t)) \Big) dt.$$

Note one can similarly obtain the Lagrangian picture of classical mechanics: using a Lagrangian $L : TQ \to \mathbb{R}$ and its Legendre transform $\mathbb{F}L : TQ \to T^*Q$, defined via the fiber derivative

$$\langle \mathbb{F}L(u), v \rangle = \frac{d}{d\epsilon}\Big|_0 L(u + \epsilon v).$$

Returning to the Hamiltonian picture, the variational principle asserts that the trajectory which a particle takes (i.e. the equations of motion) is a critical point of the action S, with respect to vertical variations compact over I; denote this space $\mathfrak{X}_c^V$.

Proposition (Variational Principle). The following are equivalent:

(i) A curve ψ is a critical point of S (with respect to $\mathfrak{X}_c^V$ variations).

(ii) $\psi^*(i_X \omega_H) = 0$ for all $X \in \mathfrak{X}_c^V$.

(iii) $i_{\dot{\psi}} \omega_H = 0$.

(iv) Hamilton's equations: $i_{X_H} \omega_{T^*Q} = -d_V H$ (where X_H is the vertical component of $\dot{\psi}$ with respect to $T(I \times T^*Q) = \mathrm{span}(\partial_t) \oplus V(I \times T^*Q)$ and ω_{T^*Q} is the canonical symplectic form on T^*Q pulled back to $I \times T^*Q$).

Proof. (i) and (iii) are equivalent via the routine calculation in Hamiltonian mechanics (we'll omit this proof). For the equivalence between (i) and (ii), let $X \in \mathfrak{X}_c^V$ and let φ_ϵ denote its associated flow. Then,

$$\begin{aligned} dS[\psi] \cdot X &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} S[\varphi_\epsilon \circ \psi] = \int_I \frac{d}{d\epsilon}\Big|_0 \psi^* \varphi_\epsilon^* \theta_H = \int_I \psi^* \frac{d}{d\epsilon}\Big|_0 \varphi_\epsilon^* \theta_H = \int_I \psi^* \mathcal{L}_X \theta_H \\ &= -\int_I \psi^* \iota_X \omega_H + \int_I \psi^* d(\iota_X \theta_H) = -\int_I \psi^* \iota_X \omega_H + \cancel{\int_I d(\psi^* \iota_X \theta_H)}; \end{aligned}$$

since X was arbitrary, (i) implies (ii) (and clearly (ii) implies (i)). Note we implicitly used that X was vertical here, since the action is defined only on curves which cover the identity on I. For the equivalence between (iii) and (iv), observe that in coordinates $\dot{\psi} = \frac{d}{dt}(t, q^i(t), p_i(t)) = \frac{\partial}{\partial t} + X_H$, where $X_H = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_j \frac{\partial}{\partial p_j}$. Hence,

$$i_{\dot{\psi}} \omega_H = 0 \Longleftrightarrow i_{\frac{\partial}{\partial t} + X_H}(-dp_i \wedge dq^i - dH \wedge dt) = 0 \Longleftrightarrow i_{X_H} \omega_{T^*Q} = -d_V H + \cancel{X_H(H)} dt.$$

□

Remark. The statement (iv) looks like the usual condition for X_H to be a Hamiltonian vector field on (T^*Q, ω_{T^*Q}) with respect to H; note though that this is interpreted in a time-dependent sense (the equality holds at each time).

Remark. Physicists usually use functional derivatives for the variational principle instead of vertical vector fields, i.e. one demands criticality $0 = \delta S[\psi] \cdot \chi \equiv \frac{d}{d\epsilon}\big|_0 S[\psi + \epsilon\chi]$, over the space of curves χ on $I \times T^*Q$ covering the identity on I such that its projection to T^*Q has support $\Subset I$. They are equivalent in this setting. However, the case of vertical vector fields is more general, since we can think of $\psi \mapsto \psi + \epsilon\chi$ as defining a flow and hence a vector field. Furthermore, in the case that the configuration bundle does not have a linear structure, the addition $\psi + \epsilon\chi$ is not well-defined although a general flow $\varphi_\epsilon \psi$ still is. In particular, for classical field theories, we will allow our configuration space to be a fiber bundle, so we will need this more general formulation.

Of these equivalent conditions, pay particular attention to (ii); this will generalize readily to the case of classical fields. We now explain several important features of the variational principle that will generalize to the case of classical field theories; namely, we consider symmetries, boundary conditions, and symplecticity.

1.1.1 Symmetries, Boundary Conditions, and Symplecticity from the Variational Principle

Consider a Lie group action of G on T^*Q. Lift this action to $I \times T^*Q$ via $g \hookrightarrow (I, g)$. Then, through this action, an element $X \in \mathfrak{g}$ induces a vertical vector field X' on $I \times T^*Q$. Now, we say G is a symmetry for our theory if $dS_U[\psi] \cdot X' = \int_U d(\psi^* i_{X'} K)$ for some one-form K on $T^*(I \times Q)$ (in physics, this is usual stated as 'the action transforms up to a total derivative'), and we require this to hold for arbitrary subintervals $U \subset I$. Combining this with our previous computation in the proposition, we see that for a critical point ψ of S, $\int_U d(\psi^* i'_X(K - \theta_H)) = 0$. Defining the charge $\langle J, X\rangle = i_{X'}(K - \theta_H)$, we have Noether's theorem:

$$d(\psi^*\langle J, X\rangle) = 0.$$

Of course, the exterior derivative here is with respect to the time variable, so this is just the statement that $\langle J, X\rangle$ is conserved along ψ (but we formulate it in this language because the field theory case is similar). A special case of this theorem is when G acts by canonical symplectomorphisms, i.e. $g^*\theta_H = \theta_H$ (i.e. $\mathcal{L}_{X'}\theta_H = 0$). Then, $K \equiv 0$ as can be seen from the above computation in the proposition, so $\langle J, X\rangle = -i_{X'}\theta_H$ is conserved along ψ critical, and in fact this is the momentum map corresponding to the G action, since

$$d\langle J, X\rangle = -di_{X'}\theta_H = i_{X'}d\theta_H - \mathcal{L}_{X'}\theta_H = -i_{X'}\omega_H.$$

Now, we turn to boundary conditions in our theory. Recall that we restricted our variation vector fields to be compactly supported. This ensured that the boundary term vanishes. Suppose $X \in \mathfrak{X}^V$ is not necessarily compactly supported, then

$$dS[\psi] \cdot X = -\int_I \psi^* i_X\omega_H + \int_{\partial I} \psi^* i_X\theta_H.$$

We want the equations of motion associated to this action to be the same, so we need to place a restriction on $\psi^* i_X\theta_H$ so that it vanishes on ∂I. Expressing $X(t, q, p) = V_i(t, q, p)\frac{\partial}{\partial p_i} + W^j(t, q, p)\frac{\partial}{\partial q^j}$ and parameterizing $\psi(t) = (t, q^j(t), p_i(t))$, we have

$$\psi^*(i_X\theta_H) = \psi^*(p_j W^j(t, q, p)) = p_j(t)W^j(t, q(t), p(t)).$$

Thus, if we don't wish to make X compactly supported, we can instead place the restriction that $p(t_\pm) = 0$ at the endpoints $t_\pm$ of the interval. These are 'Neumann' boundary conditions (in this case, the discussion of a normal direction is extraneous since we are on I, but this discussion will generalize to field theory). The case of considering compactly supported vector fields corresponds to Dirichlet boundary conditions, i.e. fixing ψ on ∂I (hence any variation should also preserve this boundary condition, so should vanish at the boundary). This shows how boundary conditions are related to the variational principle, and the same will be true in field theory.

Finally, we discuss symplecticity from the viewpoint of the variational principle. We already know that the flow of a Hamiltonian vector field is symplectic, so I will try to formally rephrase this symplecticity in a way more related to the variational principle. We would like to state this in a way that does not explicitly involve time: in the case of mechanics, there is a specified time direction so it is precise what it means for a quantity to be conserved; in field theory, we don't want to single out a time direction and break the (Lorentz) covariance of our theory, so we need to reformulate the above statement. The idea is to consider first variation vector fields, i.e. vertical vector fields whose flow on a critical point ψ of S is still a critical point.

Proposition (Symplectic Form Formula). Let $U \Subset I$. Let ψ be a critical point of S. Then, for any first variations V, W,

$$\int_{\partial U} \psi^*(i_V i_W \omega_H) = 0.$$

Proof. Recall $dS[\psi]$ can be decomposed into two one-forms, one which involves integration on the interior (which vanishes when ψ is a solution) and one which involves integration on the boundary. Since V and W are first variations, the former vanishes. So, it suffices to consider the second one-form. Call it α,

$dS[\psi] \cdot X = \int_{\partial U} \psi^* i_X \theta_H \equiv \alpha[\psi] \cdot X$. Then, using the formula $(d\alpha)[\psi] \cdot (V, W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W])$, compute

$$
\begin{aligned}
0 = d^2 S[\psi] \cdot (V, W) = (d\alpha)[\psi] \cdot (V, W) &= \int_{\partial U} \psi^* \mathcal{L}_V i_W \theta_H - W(\alpha(V)) - \alpha([V, W]) \\
&= \int_{\partial U} \psi^* (i_V d + \cancel{d i_V}) i_W \theta_H - W(\alpha(V)) - \alpha([V, W]) \\
&= -\int_{\partial U} \psi^* i_V i_W d\theta_H + \int_{\partial U} \psi^* i_V \mathcal{L}_W \theta_H - W(\alpha(V)) - \alpha([V, W]) \\
&= \int_{\partial U} \psi^* i_V i_W \omega_H + \int_{\partial U} \psi^* \mathcal{L}_W i_V \theta_H + \int_{\partial U} \psi^* i_{[V,W]} \theta_H - W(\alpha(V)) - \alpha([V, W]) \\
&= \int_{\partial U} \psi^* i_V i_W \omega_H,
\end{aligned}
$$

where in the second to last line, the second term cancels the fourth term, and the third term cancels the last term. □

This will generalize to the multisymplectic case and is the covariant way of stating that multisymplecticity is conserved.

As a last note for this section, it is also possible to incoporate/enforce constraints through the variational principle.

1.2 Analogy with Classical Field Theory

We will be more rigorous about the definition of a classical field theory after having discussed its multisymplectic geometry. For now, we will informally discuss the parallels between mechanics and field theory. Let X be an oriented manifold over which the fields are defined (the 'spacetime' of our theory; let $\dim(X) = n + 1$ with n interpreted as the spatial dimension) and let Y be the configuration space of the fields (Y is a fiber bundle over X). Our action is of the form

$$S[\phi] = \int_X \mathcal{L}(x, \phi, \partial\phi),$$

where $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$ is a bundle map (here J^1Y is the first jet bundle of Y; we will define this rigorously later, but for now note that it is coordinatized by $(x, \phi, \partial\phi)$). Comparing this setup to classical mechanics, I is analogous to X, Q to Y, and $I \times TQ$ to J^1Y. Then, informally, we want a form Θ which allows us to express our action

$$S[\phi] = \int_{(\phi,\partial\phi)(X)} \Theta_{\mathcal{L}} = \int_X (\phi, \partial\phi)^* \Theta_{\mathcal{L}},$$

in analogy with classical mechanics. From the above, we see an immediate difference between mechanics and field theory: in a general field theory, we will need Θ to be an $n+1$ form in order for its (pull-backed) integration over X to make sense; so the associated (multisymplectic) form $\Omega = -d\Theta$ will need to be an $n+2$ form. This motivates the study of (closed, non-degenerate) forms of 2 or higher, which is the study of multisymplectic geometry.

1.3 Why Study the Geometry of Classical Field Theories?

In the previous section, we saw that if want to define an action (and associated variational structure) analogous to classical mechanics, we need to introduce a higher dimensional analog of symplectic geometry. Of course this point is moot if we gain nothing by introducing this additional structure. However, as in the case of classical mechanics, understanding the underlying geometry provides deeper insights into the workings of the physics of interest.

Several examples. The language of multisymplectic geometry is sufficient to encapsulate arbitrary field theories since the configuration bundle Y is not restricted to just e.g. functions, vector fields, tensors, etc on X. This allows one to discuss e.g. gauge theories (where Y is the space of gauge connections over X). Furthermore, the multisymplectic formulation allows for a covariant Hamiltonian description of classical field theories, in the sense that it respects the Lorentzian symmetries of the underlying spacetime, as compared to the canonical formulation of Hamiltonian field theories which assumes an underlying space-like foliation which breaks Lorentz invariance. This geometry provides a rigorous setting for the covariant Hamiltonian framework introduced by De Donder-Weyl. In fact, given a space-like foliation, the multisymplectic geometry can be actually reduced to a symplectic geometry on the space of time-evolving fields (if there's enough time, I'll explain this in the later section). This can be related to the Covariant Poisson brackets introduced by Crnković -Witten and Zuckerman (see [1]). Furthermore, the geometric language allows one to simply relate symmetries of the theory with conservation laws (Noether's theorem) in a similar manner to classical mechanics. Another interesting question is how to (rigorously?) quantize a field theory; this question is plagued with functional-analytic and algebraic issues. Multisymplectic geometry provides one possible path forward, with the observation that the geometric structures introduced provide a Lie ∞-algebra structure on a suitable space of observables (see [4]).

Finally (and most directly related to my own research), many PDEs are so-called multisymplectic (e.g. wave equation, KdV), meaning they can be written in a generalized Hamiltonian form and admit a conservation law of the form $\partial_0\omega^0 + \partial_i\omega^i = 0$ where the ω^μ are each symplectic forms. In fact, this conservation law can be derived from the multisymplectic conservation laws we will discuss below. In the computational mathematics community, there has been recent interest in developing discretization methods which discretely respect these multisymplectic conservation laws (analogous to symplectic integrators for mechanics), and understanding the geometry of classical field theories is essential in developing such methods. (Explain some of the ideas of my research).

Of course, there is also the study of multisymplectic geometry in its own right, such as questions concerning multisymplectomorphisms and differences/similarities between symplectic geometry (e.g. Darboux type theorems); see [4] for a survey of several results in this vein. We will now move on to defining multisymplectic geometry and study its use in classical field theories.

2 Multisymplectic Geometry, the Geometry of Jet Bundles, Classical Fields

In this part, we will first briefly discuss general multisymplectic manifolds. Subsequently, we will develop the machinery of jet bundles and use this to give the dual jet bundle a multisymplectic structure. Using this structure, we will define Lagrangian field theory in an analogous fashion to classical mechanics and prove analogous results (and in particular, in the $n = 0$ case, this formulation reduces to classical mechanics).

2.1 Multisymplectic Manifolds

Definition. A multisymplectic manifold (M, ω) of degree $k \geq 1$ (also called a k-plectic manifold) is a manifold M with a closed differential $(k+1)-$form ω which is non-degenerate: i.e.

$$X \mapsto i_X\omega$$

is an injective map $\mathfrak{X}(M) \to \Lambda^k(M)$.

Some examples:

- A symplectic manifold (M, ω) is $1-$plectic.
- A manifold M with a volume form is $(\dim(M) - 1)-$plectic.
- The affine dual of the jet bundle, J^1Y^*, over X, $\dim(X) = n + 1$, is $(n + 1)-$plectic, whose multisymplectic form is the derivative of a canonical form. This is the "(multi-)phase space" of a classical field theory, analogous to the extended phase space $I \times T^*Q$. We will focus on this example in our discussion of field theory.

- Semi-simple Lie groups are $2-$plectic. Let G be semi-simple. Define the Maurer-Cartan 1-form with values in $\mathfrak{g}$ by $\theta_g = (L_{g^{-1}})_*$. Recall the non-degenerate symmetric and bilinear Killing form $K(X,Y) = \text{tr}(ad_X \circ ad_Y)$ from $\mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$. Then, $\omega(X,Y,Z) = K(\theta_L(X), [\theta_L(Y), \theta_L(Z)])$ is a non-degenerate $3-$form (the fact that ω is fully antisymmetric follows from symmetry of K and associativity $K(u,[v,w]) = K([u,v],w)$). Closedness of ω follows from the fact that it is bi-invariant by construction. This multisymplectic form is actually an example of the previous case, for the particular classical field theory known as the Wess-Zumino-Witten model.

The first two examples are the extreme ends of multisymplectic forms; minimum and maximum dimensional respectively. These two extreme cases have the feature that the map $X \mapsto i_X\omega$ is an isomorphism (by rank-nullity). These are also special in that they admit a Darboux type theorem, i.e. these multisymplectic forms are locally multisymplectomorphic to the canonical symplectic and volume forms respectively. These Darboux type theorems are not generally true for multisymplectic forms of intermediate degree, i.e. there may exist local invariants in these cases. Multisymplectic manifolds which admit such a Darboux theorem are called flat. See [4] and references therein for several results on when specific multisymplectic manifolds are flat and examples of non-flat multisymplectic manifolds.

One last remark regarding general multisymplectic manifolds. Let (M,ω) be $k-$plectic. We define the algebra of observable $(k-1)-$forms as follows: The algebra $\mathcal{A}$ consists of all $(k-1)-$forms α such that there exists a solution $X_\alpha \in \mathfrak{X}(M)$ to $i_{X_\alpha}\omega = -d\alpha$ (remember, in general, this map is not necessarily an isomorphism so there does not always exist such an X_α), equipped with the bracket

$$\{\alpha, \beta\} = i_{X_\alpha} i_{X_\beta}\omega.$$

To show that this bracket is closed on $\mathcal{A}$, one performs an identical calculation to the exercise we did in class; namely, one can use the homotopy formula for the Lie derivative $i_{[u,v]} = \mathcal{L}_u i_v - i_v \mathcal{L}_u$, Cartan's magic formula $\mathcal{L}_u = i_u d + d i_u$, and closedness of ω to show $X_{\{\alpha,\beta\}} = [X_\alpha, X_\beta]$. In general, $\mathcal{A}$ is not a Lie algebra, but it does satisfy the Jacobi identity up to an exact term; this provides an example of a Lie $\infty-$algebra. As mentioned before, this provides a way forward towards quantization of classical field theories in the context of multisymplectic geometry. See [7] for an introduction to quantization in this setting.

We now move away from talking about general multisymplectic structures to the specific case of multiphase space in classical field theory. To do this, we will need the machinery of jet bundles.

2.2 The Jet Bundle and its Dual

Throughout the rest of this paper, let X be an oriented manifold of dimension $n+1$ and $\pi : Y \to X$ be a fiber bundle over X; we often refer to sections of Y as fields. When we refer to a point $y \in Y$, we will call the basepoint $x = \pi(y)$ unless stated otherwise.

Definition. The first jet bundle J^1Y of Y is constructed as follows: We say two local sections ϕ_1, ϕ_2 of Y agree to first-order at $x \in X$ if $\phi_1(x) = \phi_2(x)$ and their linearizations at x agree, $(\phi_1)_{*x} = (\phi_2)_{*x}$ (which are maps from $T_xX \to T_{\phi_1(x)}Y$). The set of equivalence classes defines the fiber $(J^1Y)_x$. In local coordinates (x^μ, y^A) on Y, J^1Y has coordinates (x^μ, y^A, y^A_μ).

From the above definition, we see that the first jet bundle consists of the fields and their first derivatives, which is precisely the setting for (Lagrangian) field theory. Note that J^1Y can be viewed as a bundle over Y or X. As a bundle over Y, J^1Y is an affine bundle; to see this, using coordinates or the definition of a section $\pi \circ \phi = \mathbf{1}_X$, one sees that $\phi \in J^1_yY$ induces a splitting $T_yY = \text{im}(\phi) \oplus V_yY$ and so sections of the jet bundle can be viewed as elements of the space of Ehresmann connections on Y (this is a 1:1 correspondence), which is an affine space. The affine structure of this bundle is modeled over the space of linear bundle maps from TX to VY.

Example. $Y = I \times Q$ as a bundle over I (using the notation from our discussion of classical mechanics). Then, the sections of J^1Y are precisely the tangents of curves covering I, $\psi : I \to Y$, so $J^1Y = \mathbb{R} \times TQ$.

Let ϕ be a (local) section of $Y \to X$; then $x \mapsto (\phi_*)_x$ is a (local) section of $J^1Y \to X$. In coordinates, this section is given by $x^\mu \mapsto (x^\mu, \phi^A(x), \partial_\mu\phi^A(x))$. This lifted section is called the first jet prolongation of ϕ, denoted $j^1\phi$.

We now introduce the affine dual of J^1Y:

Definition. The dual jet bundle J^1Y^* is the affine dual of J^1Y twisted with $\Lambda^{n+1}X$; that is, its fiber over $y \in Y$ consists of affine maps from $J^1_yY \to \Lambda^{n+1}_xX$.

In fact, J^1Y^* is a vector bundle over Y; since the space of affine maps into a vector space is a vector space (in this case one has a zero map). In coordinates, if $v^A_\mu \in J^1_yY$, then an element of $J^1Y^*_y$ has the form $v^A_\mu \mapsto (p + p^\mu_A v^A_\mu)d^{n+1}x$ where $d^{n+1}x = dx^0 \wedge \cdots \wedge dx^n$. Thus, we coordinatize $J^1Y^*_y$ with (x^μ, y^A, p, p^μ_A). To define the canonical form on J^1Y^*, it is useful to have an alternate description. Let Z be the vector bundle of n-horizontal $(n+1)-$forms over Y, $Z = \Lambda^{n+1,0}(Y) \oplus \Lambda^{n,1}(Y)$, i.e. having fiber,

$$Z_y = \{z \in \Lambda^{n+1}_y(Y) : i_U i_V z = 0 \text{ for all } U, V \in V_yY\}.$$

Clearly, $z \in Z_y$ can be uniquely expressed $z = pd^{n+1}x + p^\mu_A dy^A \wedge d^n x_\mu$ where $d^n x_\mu \equiv i_{\partial_\mu} d^{n+1}x$, which gives coordinates (x^μ, y^A, p, p^μ_A) on Z. It's clear that Z is isomorphic as a vector bundle to J^1Y^*, by equating the coordinates (p, p^μ_A) on both.

We thus view $J^1Y^* \cong Z$ and can now define the canonical form on Z (hence the dual jet bundle) as follows. Let $\pi_{\Lambda Y} : \Lambda^{n+1}(Y) \to Y$ be the bundle projection and $i : Z \to \Lambda^{n+1}(Y)$ be the inclusion. Then, we define the canonical $n+1$ form on Z: for $z \in Z$,

$$\Theta(z) = i^* \pi^*_{\Lambda Y} z.$$

This form is canonical in the sense that for a section $\beta : X \to Z$, $\beta^*\Theta = (\pi_{YZ} \circ \beta)^*\beta$. We define the $(n+2)-$form $\Omega = -d\Theta$. The pair $(Z \cong J^1Y^*, \Omega)$ is the multisymplectic analog of phase space (and in fact reduces to our discussion of classical mechanics when $Y = I \times Q$ as a bundle over I).

Proposition. (Z, Ω) is $(n+1)-$plectic.

Proof. Clearly since Ω is exact, it is closed. So, we just need to show that the map $V \mapsto i_V\Omega$ is injective. Let $z \in Z$ (then, by construction, we can view z as a local section) and let V be a local vector field about $z \in Z$. We can express $z = pd^{n+1}x + p^\mu_A dy^A \wedge d^n x_\mu$ and $V = V^\alpha \frac{\partial}{\partial x^\alpha} + W^A \frac{\partial}{\partial y^A} + U \frac{\partial}{\partial p} + U^\alpha_A \frac{\partial}{\partial p^\mu_A}$. Then,

$$\begin{aligned} i_V\Omega(z) = -i_V i^*\pi^* dz &= -(V^\alpha \frac{\partial}{\partial x^\alpha} + W^B \frac{\partial}{\partial y^B} + U \frac{\partial}{\partial p} + U^\alpha_B \frac{\partial}{\partial p^\mu_B}) \lrcorner (dp \wedge d^{n+1}x + dp^\mu_A \wedge dy^A \wedge d^n x_\mu) \\ &= -\Big(- V^\alpha dp \wedge d^n x_\alpha + V^\alpha dp^\mu_A \wedge dy^A \wedge d^{n-1}x_{\mu\alpha} - W^B dp^\mu_B \wedge d^n x_\mu + U d^{n+1}x + U^\alpha_B dy^B \wedge d^n x_\alpha \Big). \end{aligned}$$

Finally, note $dp \wedge d^n x_\alpha, dp^\mu_B \wedge d^n x_\mu, d^{n+1}x, dy^B \wedge d^n x_\alpha$ are all linearly independent. Thus, $i_V\Omega = 0 \implies V = 0$. $\square$

Remark. From the above proof, we see that in coordinates on Z, the forms Θ and Ω have the expressions

$$\begin{aligned} \Theta &= pd^{n+1}x + p^\mu_A dy^A \wedge d^n x_\mu, \\ \Omega &= -dp \wedge d^{n+1}x - dp^\mu_A \wedge dy^A \wedge d^n x_\mu. \end{aligned}$$

2.3 Lagrangian Field Theory

We will now study classical field theories within the Lagrangian perspective. We could just as well study it from the Hamiltonian perspective, and in fact the two are equivalent when the Legendre transform is a fiber bundle diffeomorphism onto its image (see [6]). We choose the Lagrangian perspective since this theory uses the jet bundle, where the operations such as jet extensions are straight-forward to define. The Hamiltonian case is not much more complicated; for a treatment, see [6]. Our goal for this section is to phrase the variational principle using the multisymplectic machinery previously constructed, derive the multisymplectic form formula, and look at symmetries and currents. I will also briefly discuss the reduction to canonical field theory (introducing temporal evolution).

Let $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$ be a bundle map over X, called the Lagrangian density (we write $\mathcal{L} = Ld^{n+1}x$ with respect to the volume form). We define the Legendre transform $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$: for $v \in J^1_yY$, we take $\mathbb{F}\mathcal{L}$ to be the first-order (affine) approximation to $\mathcal{L}$:

$$\langle \mathbb{F}\mathcal{L}(v), w\rangle = \mathcal{L}(v) + \frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{L}(v + \epsilon(w - v)).$$

In the usual coordinates, we have

$$\langle \mathbb{F}\mathcal{L}(v), w\rangle = \Big(L(v) + \frac{\partial L(v)}{\partial v^A_\mu}(w^A_\mu - v^A_\mu)\Big)d^{n+1}x,$$

so that in our coordinates on J^1Y^*, the constant term $p = L - \frac{\partial L}{\partial v^A_\mu}v^A_\mu$ and the linear term $p^\mu_A = \frac{\partial L}{\partial v^A_\mu}$, which are called the covariant Hamiltonian and multimomenta respectively. In canonical Hamiltonian field theory, one usually considers the momenta for only the time derivatives: $p^0_A = \frac{\partial L}{\partial v^A_0}$ and similarly a Hamiltonian which sums over only these momenta: $H = L - \frac{\partial L}{\partial v^A_0}v^A_0$. This canonical formalism breaks the Lorentz covariance of a theory, unlike our covariant formulation.

Now, recalling the multisymplectic form Θ on $Z \cong J^1Y^*$, we use the Legendre transform to pullback this form to the jet bundle, $\Theta_\mathcal{L} = \mathbb{F}\mathcal{L}^*\Theta$, and similarly define $\Omega_\mathcal{L}$. Note that the coordinate expressions for Θ, Ω combined with the coordinate expression for $\mathbb{F}\mathcal{L}$ give coordinate expressions for $\Theta_\mathcal{L}, \Omega_\mathcal{L}$.

$$\Theta_\mathcal{L} = (L - \frac{\partial L}{\partial y^A_\mu}y^A_\mu)d^{n+1}x + \frac{\partial L}{\partial y^A_\mu}dy^A \wedge d^nx_\mu,$$

and similarly for $\Omega_\mathcal{L}$.

Let ϕ be a section of Y. Then $j^1\phi$ is a section of J^1Y over X, so $\mathcal{L}(j^1\phi)$ is a section of $\Lambda^{n+1}(X)$, so its integral over X is well-defined. Define the action $S[\phi] = \int_X \mathcal{L}(j^1\phi)$. By using the coordinate expressions, it is easy to see that $\mathcal{L}(j^1\phi) = (j^1\phi)^*\Theta_\mathcal{L}$, so

$$S[\phi] = \int_X (j^1\phi)^*\Theta_\mathcal{L}.$$

We now state a proposition analogous to the proposition we saw in the case of classical mechanics.

Proposition (Variational Principle). The following are equivalent

(i) ϕ is a stationary point of S, with respect to vertical vector fields on Y with compact support.

(ii) $(j^1\phi)^*(i_U\Omega_\mathcal{L}) = 0$ for all compactly supported vertical vector fields U on J^1Y over X.

(iii) The Euler-Lagrange equations

$$\frac{\partial L}{\partial y^A}(j^1\phi) - \partial_\mu\frac{\partial L}{\partial y^A_\mu}(j^1\phi) = 0$$

hold in coordinates $x^\mu, y^A = \phi^A, y^A_\mu = \partial_\mu\phi^A$.

Proof. We will show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Let V be a vertical vector field on Y with compact support. Denote its flow φ_ϵ. We can extend this to J^1Y as follows. Let $v \in J^1_yY$. Then recall we can view $v : T_xX \to T_yY$ linear. Since $(\varphi_\epsilon)_{*y} : T_yY \to T_{\phi_\epsilon(y)}Y$, we have $(\varphi_\epsilon)_{*y}v \in J^1_{\varphi_\epsilon(y)}Y$. We denote this jet extension $j^1\varphi_\epsilon$. Observe for a section ϕ of Y, by the chain rule, we have $j^1(\varphi_\epsilon \circ \phi) = j^1\varphi_\epsilon \circ j^1\phi$. The vector field which generates this flow is defined by $j^1V(v) = \frac{d}{d\epsilon}\big|_{\epsilon=0}j^1\varphi_\epsilon(v)$. We can now proceed with a calculation formally identical to our calculation in classical mechanics:

$$\begin{aligned} dS[\phi]\cdot U &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}S[\varphi_\epsilon \circ \phi] = \int_X \frac{d}{d\epsilon}\Big|_{\epsilon=0}(j^1(\varphi_\epsilon\circ\phi))^*\Theta_\mathcal{L} = \int_X (j^1\phi)^*\frac{d}{d\epsilon}\Big|_{\epsilon=0}(j^1\varphi_\epsilon)^*\Theta_\mathcal{L} \\ &= \int_X (j^1\phi)^*\mathcal{L}_{j^1V}\Theta_\mathcal{L} = -\int_X (j^1\phi)^* i_{j^1V}\Omega_\mathcal{L} + \cancel{\int_{\partial X}(j^1\phi)^* i_{j^1V}\Theta_\mathcal{L}}. \end{aligned}$$

So, a critical section ϕ satisfies $(j^1\phi)^*(i_{j^1V}\Omega_{\mathcal{L}}) = 0$. Now, we want to show that this implies $(j^1\phi)^*(i_U\Omega_{\mathcal{L}}) = 0$ for any compactly supported vertical vector field. This follows from two observations. First, observe $V(J^1Y \to X) = V(J^1Y \to Y) \oplus j^1(V(Y \to X))$. To see this, let $W \in V(J^1Y \to X)$. Define $W_2 = j^1\Pi_*W \in j^1(V(Y \to X))$ where Π is the projection $J^1Y \to Y$. Then, define $W_1 = W - W_2$. Observe $W_1 \in V(J^1Y \to Y)$ since $\Pi_*W = \Pi_*W - \Pi_*W_2 = \Pi_*W - \Pi_*W = 0$. Furthermore, this decomposition preserves the compact support, in the sense that both W_1 and W_2 have compact support if W does. The second observation is that $(j^1\phi)^*(i_W\Omega_{\mathcal{L}}) = 0$ for any $J^1Y \to Y$ vertical vector field W. To see this, observe that if W_1 has coordinates W^A_μ,

$$i_W\Omega_{\mathcal{L}} = -W^A_\mu \frac{\partial^2 L}{\partial y^A_\mu \partial y^B_\sigma}(dy^A \wedge d^n x_\mu - y^A_\mu d^{n+1}x),$$

which pulls back by any jet-extended section to zero. Thus, (i) implies (ii).

To see that (ii) implies (iii), let V be a vertical vector field on Y with compact support. If in coordinates $V = V^A\frac{\partial}{\partial y^A}$, then by construction of j^1V, its coordinates are given by the chain rule:

$$j^1V = V^A\frac{\partial}{\partial y^A} + \Big(\partial_\mu V^A + \frac{\partial V^A}{\partial y^B}y^B_\mu\Big)\frac{\partial}{\partial y^A_\mu}.$$

A direct calculation in coordinates gives

$$(j^1\phi)^*(i_{j^1V}\Omega_{\mathcal{L}}) = -V^A \mathrm{EL}_A(j^1\phi)d^{n+1}x,$$

where $\mathrm{EL}_A(j^1\phi)$ is the left hand side of the Euler-Lagrange equation above. The expressions we've derived also clearly show that (iii) $\implies$ (i). $\square$

Remark. Once we know (ii), it then follows that $(j^1\phi)^*(i_U\Omega_{\mathcal{L}}) = 0$ for all vertical vector fields on J^1Y over X (not necessarily compactly supported), since the above is a pointwise equality. It can actually be extended to all vector fields (i.e. to non $J^1Y \to X$ vertical vector fields). This follows from the decomposition $\mathfrak{X}(J^1Y) = V(X, J^1Y) \oplus \mathrm{Tan}(j^1\phi)$ where $\mathrm{Tan}(j^1\phi)$ are vector fields tangent to the image of $j^1\phi$, and the observation $(j^1\phi)^*(i_{(j^1\phi)_*W}\Omega_{\mathcal{L}}) = i_W(j^1\phi)^*\Omega_{\mathcal{L}} = 0$, since $(j^1\phi)^*\Omega_{\mathcal{L}}$ is an $n+2$-form on X, $\dim(X) = n+1$.

Now, we can do a similar generalization as we did in the case of mechanics. Namely, we can ask the question of what happens when we remove the compact support condition. Then, the boundary term $\int_{\partial X}(j^1\phi)^* i_{j^1V}\Theta_{\mathcal{L}}$ remains. We want to choose boundary conditions such that the integrand vanishes at the boundary for arbitrary V. To see what this amounts to, compute in coordinates for $j^1V = (0, V^A, V^A_\mu)$ (the expression for V^A_μ was given before, but we won't need it since $\Theta_{\mathcal{L}}$ is horizontal viewing J^1Y^* as a bundle over Y):

$$i_{j^1V}\Theta_{\mathcal{L}} = V^A\frac{\partial L}{\partial y^A_\mu}d^n x_\mu,$$

so we require $\frac{\partial L(j^1\phi)}{\partial y^A_\mu}d^n x_\mu$ to vanish (for each A) at the boundary, which is precisely the requirement that the normal component of the multimomentum vanishes $n_\mu p^\mu_A = 0$, i.e. Neumann boundary conditions as we alluded to before.

2.3.1 Multisymplectic Form Formula and the Instantaneous Trace

We now state the analog of symplecticity of the flow in the field theoretic setting.

Theorem (Multisymplectic Form Formula). Let $U \subset X$. Let V, W be first variations, i.e. vector fields on Y such that given a critical point ϕ to the action S, their flows on ϕ are still critical points. Then,

$$\int_{\partial U}(j^1\phi)^* i_{j^1(V)} i_{j^1(W)}\Omega_{\mathcal{L}} = 0$$

The proof is formally identical to the proof of the symplectic form formula that we saw in classical mechanics, so we will not repeat the derivation. Instead, we will look at an application of this theorem. Namely, we will use it to show that the induced (pre)symplectic form on the space of instantaneous evolving fields is independent of the choice of Cauchy surface.

To do this, we introduce some terminology. Suppose we have a foliation of X by Cauchy surfaces $\{\Sigma_\tau\}$ with inclusions $i_\tau : \Sigma_\tau \to X$. For a given bundle, we will use a subscript τ to denote the restriction of the bundle to Σ_τ and curly script to denote the space of sections. So, for example, Y_τ denotes the bundle Y restricted to Σ_τ and $\mathcal{Y}_\tau$ denotes the space of sections on Y_τ. We will assume that our foliation is compatible with this bundle, in the sense that there exists a vector field ξ on Y which is transverse to each Y_τ. This gives a way to decompose the first jet: $\beta_\tau : j^1(\mathcal{Y})_\tau \to j^1(\mathcal{Y}_\tau) \times \mathcal{V}Y_\tau$ via $\beta_\tau(j^1(\phi)_\tau) = (j^1\varphi, \dot{\varphi})$, where $j^1(\phi)_\tau = j^1(\phi) \circ i_\tau$, $\varphi = \phi \circ i_\tau$, and $\dot{\varphi} = \mathcal{L}_\xi \varphi$. In coordinates, this map decomposes the spacetime derivatives of ϕ into its spatial derivatives and temporal derivative. It is clear that this map induces an isomorphism $j^1(\mathcal{Y})_\tau \to T\mathcal{Y}_\tau$ (which is given by equating the time-derivative with the tangent vector; note that the spatial derivatives of an element of $j^1(\mathcal{Y})_\tau$ are completely determined, whereas the temporal derivative is not).

Proposition (Symplectic Trace of Multisymplectic Structure). Consider the (pre)symplectic form on $j^1(\mathcal{Y})_\tau$ defined by

$$\Omega^{\mathcal{L}}_\tau(j^1(\phi)_\tau)(V, W) = \int_{\Sigma_\tau} (j^1(\phi)_\tau)^*(i_W i_V \Omega_{\mathcal{L}}).$$

Then,

- for all first variation vector fields $V, W \in T_\phi\mathcal{Y}$, at two time-slices τ, σ, we have

$$\Omega^{\mathcal{L}}_\tau(j^1(\phi)_\tau)(j^1(V)_\tau, j^1(W)_\tau) = \Omega^{\mathcal{L}}_\sigma(j^1(\phi)_\sigma)(j^1(V)_\sigma, j^1(W)_\sigma).$$

- $\Omega^{\mathcal{L}}_\tau$ viewed as a form on $T\mathcal{Y}_\tau$ (isomorphic to $j^1(\mathcal{Y})_\tau$ via β_τ) is equal to the form

$$\omega^L_\tau = \mathbb{F}L^*_\tau \omega_\tau,$$

  where ω_τ is the canonical form on $T^*\mathcal{Y}_\tau$ and L_τ is the instantaneous Lagrangian, given by integrating the Lagrangian density over the spatial degrees of freedom:

$$L_\tau(\varphi, \dot{\varphi}) = \int_{\Sigma_\tau} i^*_\tau(i_\xi \mathcal{L}(\beta^{-1}_\tau(j^1(\phi), \dot{\varphi}))).$$

Proof. The first identity follows by using the multisymplectic form formula applied to the domain $U = \cup_{s\in[\tau,\sigma]}\Sigma_s$. One can prove the second identity in coordinates; this is a straight-forward application of all of the definitions we made above. □

As a corollary, we see that the Euler-Lagrange flow of the instantaneous Lagrangian preserves $\Omega^{\mathcal{L}}_\tau$. Thus, by this proposition, we explicitly see that the fully covariant multisymplectic structure carries more information than the canonical (time-evolving) symplectic structure. In this sense, the symplectic structure is a trace of the multisymplectic structure (given by integrating out spatial degrees of freedom).

2.3.2 Covariant Momentum Maps and Noether's Theorem

Given a group action on Y, we would like to extend this to group actions on J^1Y^* and J^1Y. Let η_Y be a bundle automorphism of Y to itself which covers a diffeomorphism η_X of X (often, but not always, η_X is the identity). We lift this action to $J^1Y^* \cong Z$ via $\eta_Z(z) = (\eta_Y^{-1})^*Z$. Similarly, lift the action to J^1Y via $\eta_{J^1Y}(v) = T\eta_Y \circ v \circ T\eta_X^{-1}$.

Now, let G be a Lie group acting on Y and lifted to $J^1Y, J^1Y^* \cong Z$ as above. We define covariant momentum maps analogously to the symplectic case. Namely, $J : Z \to \mathfrak{g}^* \otimes \Lambda^n(Z)$ is a covariant momentum map for the G action if $d\langle J, \xi\rangle = i_{\xi_Z}\Omega$; similarly on the Lagrangian side, $J : J^1Y \to \mathfrak{g}^* \otimes \Lambda^n(J^1Y)$ is a covariant momentum map for the G action if $d\langle J, \xi\rangle = i_{\xi_{J^1Y}}\Omega_{\mathcal{L}}$.

Theorem (**Noether's Theorem**)**.** Suppose a Lagrangian density $\mathcal{L}$ is equivariant with respect to the G-action, in the sense that for any $v \in J^1Y$,

$$\mathcal{L}(\eta_{J^1Y}v) = \eta_X^{-1}\mathcal{L}(v).$$

Then, the Cartan form $\Theta_{\mathcal{L}}$ is G-invariant and hence J defined by

$$\langle J, \xi \rangle = i_{\xi_{J^1Y}}\Theta_{\mathcal{L}}$$

is a covariant momentum map. Furthermore, for a solution ϕ of the Euler-Lagrange equations, we have the conservation law

$$d((j^1\phi)^*\langle J, \xi\rangle) = 0.$$

Proof. First, we show that $\eta_Z \circ \mathbb{F}\mathcal{L} = \mathbb{F}\mathcal{L} \circ \eta_{J^1Y}$. First, observe by our definitions

$$\langle \eta_Z \mathbb{F}\mathcal{L}(v), w\rangle = (\eta_X^{-1})^*\langle \mathbb{F}\mathcal{L}(v), \eta_{J^1Y}^{-1}(w)\rangle = (\eta_X^{-1})^*\Big(\mathcal{L}(v) + \frac{d}{d\epsilon}\Big|_0 \mathcal{L}(v + \epsilon(\eta_{J^1Y}^{-1}w - v))\Big).$$

Then,

$$\langle \mathbb{F}\mathcal{L}(\eta_{J^1Y}v), w\rangle = \mathcal{L}(\eta_{J^1Y}v) + \frac{d}{d\epsilon}\mathcal{L}(\eta_{J^1Y}v + \epsilon(w - \eta_{J^1Y}v)),$$

and these two are the same by equivariance. Now, compute

$$\eta_{J^1Y}^*\Theta_{\mathcal{L}} = \eta_{J^1Y}^*\mathbb{F}\mathcal{L}^*\Theta = (\mathbb{F}\mathcal{L}\circ\eta_{J^1Y})^*\Theta = (\eta_Z \circ \mathbb{F}\mathcal{L})^*\Theta = \mathbb{F}\mathcal{L}^* \circ \eta_Z^*\Theta;$$

so if we show $\eta_Z^*\Theta = \Theta$, the first part of the theorem is proved. To do this, observe that by construction $\eta_Y^{-1} \circ (\pi_{\Lambda Y} \circ i) \circ \eta_Z = \pi_{\Lambda Y} \circ i$. Then, compute for $z \in Z, v_i \in \Lambda_z Z$,

$$\begin{aligned}(\eta_Z^*\Theta)_z(v_1, \ldots, v_{n+1}) &= \Theta_{\eta_Z(z)}(\eta_{Z*}v_1, \ldots, \eta_{Z*}v_{n+1}) = \eta_Z(z)((\pi \circ i \circ \eta_Z)_* v_1, \ldots, (\pi\circ i \circ \eta_Z)_* v_{n+1}) \\ &= z((\eta_Y^{-1}\circ\pi\circ i\circ\eta_Z)_*v_1, \ldots, (\eta_Y^{-1}\circ\pi\circ i\circ\eta_Z)_*v_{n+1}) = (\pi\circ i)^*z(v_1,\ldots,v_{n+1}) = \Theta_z(v_1,\ldots,v_{n+1}),\end{aligned}$$

where we denoted $\pi = \pi_{\Lambda Y}$. So $\eta_Z^*\Theta = \Theta$. It immediately follows that J defined above is a covariant momentum map.

Finally, the conservation law follows by the variational principle proposition and the definition of the covariant momentum map. □

We now connect this to the instantaneous formulation we developed earlier.

Proposition (**Instantaneous Trace of Covariant Momentum Map**)**.** Assume as in the previous theorem. Let G_τ denote the subgroup of G which stabilizes Σ_τ, i.e. consisting of all $\eta \in G$ such that $\eta_X(\Sigma_\tau) = \Sigma_\tau$; denote its Lie algebra $\mathfrak{g}_\tau$. Define the instantaneous trace of J as $J_\tau : T\mathcal{Y}_\tau \to \mathfrak{g}_\tau^*$ via

$$\langle J_\tau(\varphi, \dot\varphi), \xi\rangle = \int_{\Sigma_\tau}(\varphi,\dot\varphi)^*\langle J, \xi\rangle,$$

where we view $(\varphi, \dot\varphi) \in j^1(\mathcal{Y})_\tau$ via the isomorphism $j^1(\mathcal{Y})_\tau \cong T\mathcal{Y}_\tau$ discussed earlier. Then, for all compactly supported vector fields $V \in T_{(\varphi,\dot\varphi)}T\mathcal{Y}_\tau$ and all $\xi \in \mathfrak{g}_\tau$,

$$i_V d\langle J_\tau(\varphi,\dot\varphi), \xi\rangle = i_V i_{\xi_{j^1(\mathcal{Y})_\tau}}\Omega_\tau^{\mathcal{L}}|_{\varphi,\dot\varphi}.$$

In particular, if Σ is compact, then J_τ is a momentum map for $(T\mathcal{Y}_\tau, \Omega_\tau^{\mathcal{L}})$ with respect to the G_τ action.

Proof. Compute

$$\begin{aligned} i_V d\langle J_\tau(\varphi,\dot\varphi),\xi\rangle &= \int_{\Sigma_\tau}(\varphi,\dot\varphi)^*\mathcal{L}_V\langle J,\xi\rangle = \int_{\Sigma_\tau}(\varphi,\dot\varphi)^*(i_V d + \cancel{d i_V})\langle J, \xi\rangle \\ &= \int_{\Sigma_\tau}(\varphi,\dot\varphi)^* i_V i_{\xi_{J^1Y}}\Omega_{\mathcal{L}} = \int_{\Sigma_\tau}(\varphi,\dot\varphi)^* i_V i_{\xi_{j^1(\mathcal{Y})_\tau}}\Omega_{\mathcal{L}} = i_V i_{\xi_{j^1(\mathcal{Y})_\tau}}\Omega_\tau^{\mathcal{L}}|_{\varphi,\dot\varphi},\end{aligned}$$

where we used that $\xi \in \mathfrak{g}_\tau$ so ξ_{J^1Y} restricts to a well-defined vector in the tangent space $T_{(\varphi,\dot\varphi)}j^1(\mathcal{Y})_\tau$, which we denoted $\xi_{j^1(\mathcal{Y})_\tau}$. If Σ is compact, the above equality is equivalent to $d\langle J_\tau, \xi\rangle = i_{\xi_{j^1(\mathcal{Y})_\tau}}\Omega_\tau^{\mathcal{L}}$ which is of course the momentum map equation. □

Remark. In physics, J is usually called a conserved current and J_τ is usually called the associated conserved charge. $dJ = 0$ when evaluated on (covariant) solutions, and similarly $\frac{d}{d\tau} J_\tau = 0$ along (canonical) solutions.

2.4 Equivalence between Canonical and Covariant Formulations

To conclude and tie together the canonical and covariant viewpoints, we state (but won't prove) the equivalence between the infinite-dimensional Hamiltonian dynamics (canonical formalism) and the covariant Euler-Lagrange stationarity. Let $T^*\mathcal{Y}_\tau$ be the $\Lambda^n(\Sigma_\tau)$-twisted dual to $T\mathcal{Y}_\tau$, with duality pairing giving by integration

$$\langle \pi, V \rangle = \int_{\Sigma_\tau} \pi(V).$$

In coordinates, we can express $\pi = \pi_A dy^A \otimes d^n x_0$.

Fix a Lagrangian density and define the instantaneous Lagrangian L_τ. The Legendre transform $\mathbb{F}L_\tau : T\mathcal{Y}_\tau \to T^*\mathcal{Y}_\tau$ and for simplicity we assume it is a diffeomorphism (this assumption isn't really crucial; we could just restrict the theory to its image). Define the Hamiltonian

$$H_\tau(\varphi, \pi) = \langle \pi, \dot{\varphi} \rangle - L_\tau(\varphi, \dot{\varphi}),$$

where $\dot{\varphi} = \mathbb{F}L_\tau^{-1}\pi$. As in our discussion of classical mechanics, we say that a curve $\psi : I \to T^*\mathcal{Y}_\tau$ satisfies Hamilton's equations if $i_{\dot{\psi}}(\omega + dH_\tau \wedge d\tau) = 0$. Then,

Theorem. Let ϕ be a section of Y which solves the Euler-Lagrange equations associated to $\mathcal{L}$. Then, $\psi : \tau \mapsto \mathbb{F}L_\tau(\beta_\tau(j^1(\phi)_\tau))$ solves Hamilton's equations.

Conversely, every solution of Hamilton's equations arises as the above decomposition of some section ϕ of Y which solves the Euler-Lagrange equations.

For a proof, see [3].

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