Application of the BRST Method to Geometric Quantization

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1 Introduction

We discuss the quantization of gauge theories via the Becchi-Rouet-Store-Tyutin (BRST) quantization method and its application to the geometric quantization of constrained Hamiltonian systems.

We begin with the traditional description of the BRST method within the path integral framework, as a method to gauge fix the path integral. As we are familiar with, this involves the introduction of nonphysical ghost fields to enforce the gauge fixing condition. Although the theory extended with the ghost fields no longer has the original gauge symmetry, we will see that it has a remnant of the gauge symmetry, the BRST symmetry. We will subsequently discuss how the physical states of the gauge theory lie in the cohomology of the charge operator associated to the BRST symmetry. This is one of the reasons why the BRST method has been so successful and powerful; it converts functional analytic problems into algebraic problems.

Subsequently, we will discuss how the BRST method can be applied to the geometric quantization of constrained systems. To do this, we will briefly discuss the ideas of geometric quantization and symplectic reduction. In this setting, the zeroth cohomology of the BRST operator is isomorphic to the space of smooth functions on the phase spaced reduced by the gauge group action (i.e. the physical phase space). Subsequently, we will very roughly sketch how the BRST method was used to (partially) prove the Guillemin-Sternberg conjecture, which states the geometric quantization commutes with symplectic reduction. This is an important result in the quantization of constrained Hamiltonian system.

My interest in this topic (other than the fact that the topic is fascinating in its own right) stems from the fact that it should provide useful insights into my research. My research involves geometric discretization of gauge field theories and constrained Hamiltonian systems, and the BRST method (both in its classical and quantum description) is a powerful method for analyzing such systems. In particular, it allows one to describe the (classical and quantum) dynamics of the symplectically reduced systems (which when dealt with directly involves delicate analytical issues) to be described more simply in terms of algebra and topology.

2 The BRST Method in the Path Integral Framework

Our starting point is the path integral description of a quantum theory. Let $S[\phi]$ be the (Euclidean) action describing a gauge theory for ϕ (note: I am using the compact notation ϕ for the field(s) ϕ^a, although it does not necessarily need to be a scalar field or even a single field. Also, our discussion applies to mechanics by taking the number of spacetime dimensions $n = 1$).

The quantum theory is described by the path integral

$$Z = \int (D\phi/\mathcal{G}) e^{-S[\phi]},$$

where $\mathcal{G}$ is the gauge group (consisting of maps from Euclidean spacetime to some compact Lie group G) and the quotient of the path integral measure arises from the fact that we identify gauge group orbits as equivalent physical states. One way to formally achieve this quotient of the path integral measure is to gauge fix the path integral, i.e. choosing a gauge fixing condition which intersects each gauge orbit once. Let us fix the gauge fixing conditions $F[\phi] = 0$ (I've suppressed the color indices $F_a \to F$ of the gauge group for simplicity; I'll restore them later). To gauge fix the path integral, we want to introduce $\delta(F[\phi])$. This is accomplished by inserting into the path integral

$$1 = \int d\omega \delta(F[\phi]) \det \frac{\delta F[\phi]}{\delta \omega}.$$

We then employ the Faddeev-Popov method (discussed in class) of exponentiating the determinant by introducing the ghost c and its canonically conjugate anti-ghost $\bar{c}$. Ignoring normalization constants, this gives the gauge-fixed path integral

$$Z = \int D\phi^a Dc^b D\bar{c}^b \delta(F[\phi]) \exp(-S[\phi] - S_g[\phi, c, \bar{c}]),$$

(with an implicit product over the color indices a in $D\phi^a$) where the ghost action

$$S_g = \int d^n x \mathrm{Tr}(\bar{c} \frac{\delta F[\phi]}{\delta \omega}\Big|_{\omega=0} c).$$

The starting point for the BRST method is introducing auxiliary scalar fields B^a whose equations of motion enforces the gauge fixing condition $F^a[\phi] = 0$, i.e. it acts as a Lagrange multiplier:

$$\delta(F^a[\phi]) \to \int DB^a \exp(-\underbrace{i \int d^n x B^a F^a}_{\equiv S_{gf}}),$$

where S_{gf} denotes the gauge-fixing action. The path integral then becomes

$$Z = \int D\phi^a DB^a Dc^b D\bar{c}^b \exp(-S - S_g - S_{gf}).$$

The upshot is that our gauge fixed theory for ϕ, described by the action $S[\phi]$ and the gauge-fixing condition $F = 0$, can be replaced by a field theory on an extended field space, involving $\phi, B, c, \bar{c}$ and described by the action $S + S_g + S_{gf}$. Of course, this extended theory no longer obeys the original gauge symmetry of the original theory, since it was constructed to explicitly gauge fix the theory. However, we know that the ghosts and auxiliary fields do not contribute to the physical content of the theory, so there should be a symmetry which ensures that the unphysical degrees of freedom do not contribute to physical quantities, e.g. amplitudes. This symmetry is the BRST symmetry, given by

$$\begin{aligned}
\delta_\epsilon \phi^a &= -i\epsilon c^b \delta_b \phi^a, \\
\delta_\epsilon B^a &= 0, \\
\delta_\epsilon \bar{c}^a &= \epsilon B^a, \\
\delta_\epsilon c^a &= \frac{i}{2} \epsilon c^b c^c f^{bca},
\end{aligned}$$

where ϵ is a (global) grassmann parameter and f^{bca} are the structure constants corresponding to the Lie algebra of G. Several remarks about this symmetry: first, note that the transformation for the fields ϕ^a are just the (infinitesimal) gauge transformation, with the Lie algebra parameter ω^b replaced by ϵc^b. In this sense, the BRST symmetry is a remnant of the original gauge symmetry. Furthermore, a direct computation shows that the BRST transformation is nilpotent, $\delta^2 = 0$ (i.e. $\delta_\epsilon \delta_{\epsilon'} = 0$).

By Noether's theorem, the BRST transformation gives rise to a conserved charge Q (given by $Q = c^a F^a - \frac{1}{2} f^{bca} c^b c^c \bar{c}^a$), which generates the transformation in the sense that

$$\begin{aligned} \delta_\epsilon \phi^a &= \epsilon[Q, \phi^a], \\ \delta_\epsilon B^a &= \epsilon[Q, B^a], \\ \delta_\epsilon \bar{c}^a &= \epsilon\{Q, \bar{c}^a\}, \\ \delta_\epsilon c^a &= \epsilon\{Q, c^a\}, \end{aligned}$$

and nilpotency of δ implies that as an operator, $Q^2 = 0$. We now show that physical states must lie in the kernel of Q. To show this, recall that we chose an arbitrary gauge-fixing condition; physical amplitudes must be independent of this choice of gauge-fixing. Hence, under an infinitesimal shift in the gauge-fixing $F[\phi] \to F[\phi] + \tilde{\delta} F[\phi]$, any physical amplitude must be invariant, $\tilde{\delta}\langle f|i\rangle = 0$. This then gives

$$0 = \tilde{\delta}\langle f|i\rangle \overset{(*)}{=} -\langle f|\{Q, \int d^n x \bar{c}^a \tilde{\delta} F^a\}|i\rangle,$$

(for a proof of $(*)$, see [9] page 19). Since the variation in F was arbitrary, this implies

$$Q|\psi\rangle = 0$$

for any physical state $|\psi\rangle$ (and using the fact that $Q = Q^\dagger$). This shows that $\{\text{physical states}\} \subset \ker Q$, as claimed, i.e. physical states are BRST closed. Now, consider a BRST exact state, i.e. a state in the image of Q: $|\psi\rangle = Q|\chi\rangle$. Let $|\psi'\rangle$ be any physical state (i.e. BRST closed but not necessarily exact). Then,

$$\langle \psi'|\psi\rangle = \langle \psi'|Q|\chi\rangle = (Q|\psi'\rangle)^\dagger|\chi\rangle = 0.$$

Thus, the BRST exact state $Q|\chi\rangle$ has no overlap with any of the physical states, so is unphysical. Thus, we see that the physical states of the theory are precisely the cohomology of Q:

$$\{\text{physical states}\} = \frac{\ker Q}{\text{im} Q} \equiv H(Q).$$

As discussed in the introduction, this is one of the strengths of the BRST method. It converts functional analytic computations (e.g. working with the gauge-fixed path integral) to algebraic computations involving the cohomology of Q. Although I won't go into much into examples in this paper, two examples include: (1) the relativistic particle (which has reparametrization as a gauge symmetry), where it can be shown that the cohomology of Q precisely selects the states which satisfy the energy-momentum relation. (2) non-abelian Yang-Mills, where the cohomology of Q selects out states satisfying the non-abelian analog of Gauss' law.

At this point, we could say more about using this method, such as a discussion of renormalization, its use in proving the Ward identities, or using it to compute amplitudes for specific examples of gauge theories. This is fairly standard in the literature (see references below and the references therein), so I will instead focus on how this method is powerful tool in the context of geometric quantization of constrained systems.

3 BRST and Geometric Quantization

In this section, we discuss the application of the BRST method to geometric quantization, and in particular to the Guillemin-Sternberg conjecture. Since this topic involves technical material (including non-trivial applications of homology theory, vector bundles, bigraded complexes, symplectic geometry, etc.) which is outside the scope of this course, I will have to be heuristic in certain parts of the subsequent discussion. For more rigorous details, see [4] and [6].

3.1 A Quick Intro to Symplectic Reduction and Geometric Quantization

Symplectic Reduction: Unlike the previous covariant formulation of our theory, we will now work in the Hamiltonian formulation where the fields or particles evolve in time with values in some symplectic manifold M. We first recall that gauge theories are in correspondence with constrained Hamiltonian systems. This is one formulation of Noether's second theorem: the conserved charge Q corresponding to a gauge symmetry must vanish. Heuristically, the proof is that the charge at time t_1 and time $t_2 \neq t_1$ must be the same, but since the gauge group action is local, it is possible to construct a infinitesimal generator of a gauge transformation which can take any value at time t_1 but be zero at time t_2. Thus, the gauge theory corresponds to a constrained theory. Conversely, given a constrained Hamiltonian system, the (first-class) constraints generate gauge transformations.

Consequently, given a theory evolving on a symplectic manifold M subject to constraints $\Phi = 0$ (in components, $\Phi^a = 0$, where the indices run over the color indices of the gauge group), we are not interested in the full theory on M but rather on the constraint manifold $\Phi^{-1}(0)$ modulo gauge transformations.

The previous paragraph is geometrically described by the process of symplectic reduction. Let (M, ω) be a symplectic manifold (i.e. ω is a closed non-degenerate 2-form on M) and suppose we have a (free and proper) $G-$action on M by symplectomorphisms, which admits a (Ad^* equivariant) momentum map, i.e. a map $\Phi : M \to \mathfrak{g}^* \equiv \text{Lie}(G)$ such that $d\langle \Phi, \xi \rangle = i_{\xi_M}\omega$ for any $\xi \in \mathfrak{g}$, where ξ_M is the infinitesimal generator corresponding to the G action (in more familiar language, the momentum map is just the conserved charge Q arising from Noether's theorem). Then, the quotient $\widetilde{M} = \Phi^{-1}(0)/G$ is a symplectic manifold. Furthermore, the symplectic form $\tilde{\omega}$ on $\widetilde{M}$ is given as follows: let $\pi : \Phi^{-1}(0) \to \widetilde{M}$ be the natural quotient projection and $i : \Phi^{-1}(0) \to M$ be the inclusion; then $i^*\omega = \pi^*\tilde{\omega}$. Intuitively, this means that the symplectic form on the reduced space $\widetilde{M}$ pulled back to $\Phi^{-1}(0)$ is just the restriction of the original symplectic form on M to $\Phi^{-1}(0)$, so $\tilde{\omega}$ is the 'natural' symplectic form. As mentioned before, we are not concerned with the Hamiltonian dynamics on (M, ω) but rather $(\widetilde{M}, \tilde{\omega})$, and the physical observables of the theory lie in $C^\infty(\widetilde{M})$, which can be viewed as the space of gauge-invariant functions on the constraint manifold $\Phi^{-1}(0)$. More precise details on symplectic reduction can be found in [1] and references therein.

We will return to this later when discussing its connections to BRST theory.

Geometric Quantization: The idea for geometric quantization originated from the canonical quantization of a theory. Recall that a symplectic manifold (M, ω) admits a Poisson structure,

$$\{f, g\} = \omega(X_f, X_g),$$

where X_f is the Hamiltonian vector field determined by f: $i_{X_f}\omega = -df$ (given f, X_f exists and is uniquely determined since ω is non-degenerate). The usual Hamilton's equations are $\frac{d}{dt}f = \{f, H\}$. A familiar example is where $M = T^*Q$ is the cotangent bundle of some manifold with local coordinates (q^i, p_i) such that $\omega = dq^i \wedge dp_i$. Then, $\{q^i, p_j\} = \delta^i_j$ with the other brackets vanishing. The idea of canonical quantization is to promote functions f on M to operators on $O(f)$ a Hilbert space $\mathcal{H}$ such that the quantization map O is (up to a constant factor) a Lie algebra homomorphism,

$$[O_f, O_g] = i\hbar O_{\{f,g\}},$$

i.e. the Poisson bracket gets replaced by (a constant times) the operator commutator. When $M = T^*Q$, canonical quantization achieves this by taking $\mathcal{H}$ to be (a subspace of) the space $L^2(Q, \mathbb{C})$, and defining the quantization map $O : q^i \mapsto q^i$ (the multiplication operator) and $O : p_i \mapsto -i\hbar\partial/\partial q^i$. There are however, several issues with this: first, the above method is coordinate dependent and relies on global coordinates on Q; furthermore, it is not clear from the above procedure how to generalize to an arbitrary symplectic manifold M which is not the cotangent bundle of some other manifold; finally, given a generally non-linear function $f(q^i, p_j)$, it is not clear how one should order the operators in $f(q^i, -i\hbar\partial/\partial q^j)$. It is from these (and other) considerations that geometric quantization arose, providing a geometric way to formulate the quantization of a symplectic manifold.

I will now informally discuss the steps in geometric quantization (and glance over many details; for more rigorous discussion, see [2]). First, we state the axioms of the quantization of a symplectic manifold (M, ω):

Definition. A quantization of (M, ω) is a Hilbert space $\mathcal{H}$ and a quantization map O, such that

- $\mathcal{H}$ is a (separable) complex Hilbert space,
- O maps functions on M to operators on $\mathcal{H}$ ($f \mapsto O_f$) such that
  1. $O_{f+g} = O_f + O_g$
  2. $O_{\alpha f} = \alpha O_f, \alpha \in \mathbb{C}$
  3. $O_1 = \mathbf{1}_{\mathcal{H}}$
  4. $[O_f, O_g] = i\hbar O_{\{f,g\}}$
  5. If $\{f_i\}$ are a complete set of functions with respect to the Poisson bracket, then $\{O_{f_i}\}$ are a complete set of operators with respect to the commutator. Here, complete means any function (resp. operator) which Poisson commutes (resp. operator commutes) with the complete set must be a constant function (resp. proportional to the identity operator).

To achieve this, geometric quantization begins with a complex line bundle $\pi : L \to M$. Informally, the line bundle L is a space which locally (on local open sets $U \subset M$) looks like $U \times \mathbb{C}$. The sections of this bundle, for our purposes, can then be locally thought of as complex valued functions, i.e. as maps from $U \to \mathbb{C}$ (more rigorously, a section $s : M \to L$ satisfies $\pi \circ s = \mathbf{1}_M$).

Let ∇ be a connection on the line bundle (satisfying $\nabla_{fX} = f\nabla_X$, $\nabla_X(fs) = (Xf)s + f\nabla_X s$ for functions $f \in C^\infty(M)$, vector fields $X \in \mathfrak{X}(M)$, and sections s of the line bundle). Locally,

$$\nabla_X(fs) = (X(f) + (2\pi i)i_X\alpha)s.$$

Here, α is the locally defined connection $1-$form, and we define curvature $2-$form associated to the connection, $\Omega \equiv d\alpha$ (which can be shown to be, in fact, globally defined). We will also assume that our line bundle is equipped with a metric and that the connection is metric compatible (analogous to the covariant derivative one studies in general relativity); this allows us to define the space of square integrable sections of the line bundle. At this point, we will take this as our Hilbert space; however, as we will discuss, in order to achieve property 5, we must reduce the Hilbert space further using a process known as polarization.

In order to have a connection satisfying the required quantization properties (1-4 in the definition above), one can show that the correct choice for the quantization map O is

$$O_f = -i\hbar\nabla_{X_f} + f.$$

where $\nabla_{X_f} = (X_f + \frac{i}{\hbar} i_{X_f}\theta)$ and θ is the local symplectic potential, $\omega = d\theta$. Comparing this expression to our previous local expression for the connection, we see that this choice is possible only when the symplectic form and curvature are related by $\omega/2\pi\hbar = \Omega$. Clearly O_f satisfies conditions 1-3 (3 follows from the fact that $X_1 = 0$ since $d1 = 0$). For condition 4, a direct computation yields

$$[O_f, O_g]\psi = [-i\hbar\nabla_{X_f} + f, -i\hbar\nabla_{X_g} + g]\psi = (\hbar^2\nabla_{X_{\{f,g\}}} + i\hbar\{f, g\})\psi = i\hbar O_{\{f,g\}}\psi.$$

We say that a symplectic manifold is prequantizable if the above construction satisfies conditions 1-4 of the quantization axioms. A theorem due to Weil asserts that a symplectic manifold is prequantizable if and only if $[\omega/2\pi\hbar]$ is an integral cohomology class, which is the familiar Bohr-Sommerfeld quantization condition, stated in the framework of geometric quantization.

At this point, we have prequantized our system, i.e. constructed a Hilbert space and quantization map satisfying properties 1-4. To see that 5 fails, it suffices to consider a counterexample. The simplest case is $M = T^*\mathbb{R}^n$ with the usual symplectic form and $L = M \times \mathbb{C}$ being the trivial line bundle. Note that in this case, the quantization map O coincides with the canonical quantization procedure, as one would expect. Then, the Hilbert space corresponding to our construction is $L^2(T^*\mathbb{R}^n, \mathbb{C})$, when it should be $L^2(\mathbb{R}^n, \mathbb{C})$ as our previous discussion of canonical quantization indicates. That is, we should only be considering square integrable functions of only the position, or only the momenta, but not both (and one can readily check that the taking $L^2(T^*\mathbb{R}^n, \mathbb{C})$ does not satisfy the irreducibility condition 5). Choosing only the subspace of the Hilbert space which depends on either the momenta or position (but not both) is known as polarization, in the simple case of $M = T^*\mathbb{R}^n$.

The general case of polarization is inspired from simple case: for a $2n-$dimensional symplectic manifold, we should choose an $n-$dimensional distribution $D \subset TM$ such that the quantum states are the square integrable sections of the line bundle which are covariantly constant in the direction of the distribution, $\nabla_X \psi = 0$ for any $X \in D$. It follows that the distribution should be involutive $[D, D] \subset D$ so that the covariant constancy condition is preserved for commutators of vector fields in D. Finally, a computation yields

$$0 \overset{!}{=} [\nabla_D, \nabla_D]\psi = \nabla_{[D,D]}\psi + \frac{i}{\hbar}\omega(D, D)\psi.$$

Of course, since $[D, D] \subset D$, $\nabla_{[D,D]}\psi = 0$ which implies that $\omega(D, D) = 0$. In other words, D must be a Lagrangian distribution (i.e. the restriction of the symplectic form to D, viewed as a bilinear form on D, must vanish). With this final polarization step, the quantization of the symplectic manifold is complete and satisfies all of the above axioms.

3.2 The BRST Method applied to Geometric Quantization

We are now in position to (very roughly) discuss the application of the BRST theory to the geometric quantizataion of constrained systems. Our main concern here is the Guillemin-Sternberg conjecture: that reduction (i.e. incorporating constraints) commutes with (geometric) quantization. In other words, that the following diagram commutes:

$$\begin{tikzcd}
(M, \omega) \arrow[r, "\text{quantization}"] \arrow[d, "\text{reduction}"'] & \mathcal{H} \arrow[d, "\text{reduction}"] \\
(\widetilde{M}, \tilde{\omega}) \arrow[r, "\text{quantization}"'] & \widetilde{\mathcal{H}}
\end{tikzcd}$$

Intuitively, this means that we could either (i) start with the full phase space, quantize, and then restrict the states of H to the physical states $\widetilde{H}$ (i.e. by imposing that they satisfy $O_{\Phi^a}|\psi\rangle = 0$ for the constraints Φ^a); or we could (ii) start by reducing the dynamics to the symplectic manifold $\widetilde{M}$ and subsequently quantize, and the result should be the same. This commutativity is essential for the quantization of constrained Hamiltonian systems since, if the processes did not commute, it would not be clear which order we should take, i.e. which resulting theory is the correct physical theory. Furthermore, it is often the case that one route is easier in practice than the other. For example, often the reduced manifold $\widetilde{M}$ is topologically complicated (and one may not even have an explicit characterization of the manifold and symplectic form), so it might be preferable to first quantize M and then constrain the Hilbert space. On the other hand, if one does have an explicit characterization of $(\widetilde{M}, \tilde{\omega})$, then one can reduce first and then quantize, without having the need to go through the process of reducing the Hilbert space or introducing ghosts, etc. This has the advantage that we could in practice perform the easier route, while knowing that the theoretical properties and tools of the other route still apply to the route we chose.

We now describe the steps taken in both routes of the above diagram. Given the symplectic manifold (M, ω), we perform the above geometric quantization procedure, with the caveat that we choose a polarization D such that the gauge group $\mathcal{G}$ leaves D invariant:

$$[D, \mathfrak{g}_M] \subset D$$

(where $\mathfrak{g}_M$ is the space of infinitesimal generator vector fields of the gauge group action on M). This ensures that the quantized constraint operators form a representation of the Lie algebra. Then, if Φ_ξ is the Hamiltonian function corresponding to ξ_M (i.e. the constraint function), we reduce the Hilbert space

$\mathcal{H}$ by the condition that $O_{\Phi_\xi}|\psi\rangle = 0$ for any $\xi \in \mathfrak{g}$. In the other direction, we perform the geometric quantization procedure on the reduced phase space $\widetilde{M} = \Phi^{-1}(0)/\mathcal{G}$. This of course requires choosing an admissible line bundle and polarization, and one would not expect that any arbitrary choice would give the same result as the other direction. Naturally, one should choose the induced line bundle. The induced bundle is given by restricting the line bundle on M and projecting to the quotient, i.e. if $\pi_0 : L_0 \to \Phi^{-1}(0)$ is the restricted bundle and $P : \Phi^{-1}(0) \to \widetilde{M}$ is the quotient projection, then the induced bundle is given by the projection $P \circ \pi_0$. Furthermore, the above condition $[D, \mathfrak{g}_M] \subset D$ ensures that a polarization is also induced on the reduced space. One can then ask if the result quantized theories are equivalent.

To answer this, one uses the BRST theory in the following way (see [4] and [7] for precise details): first, one constructs the classical BRST operator Q, which acts on $C^\infty(M) \otimes \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ (this is the full space of observables including graded ghosts and anti-ghosts) via the Poisson (super)bracket and this action has the property that its zeroth cohomology is isomorphic to $C^\infty(\widetilde{M})$, the space of observable functions (we already saw this in the path integral formulation; the cohomology of Q selects physical observables). This is known as the classical BRST cohomology. Subsequently, one naturally extends the prequantization procedure to the BRST extended phase space and uses this to define the quantum BRST operator O_Q (which acts on the quantum line bundle extended by the bi-graded ghost space, as above in the classical case). Using these elements, one can then prove that the zeroth cohomology of O_Q (known as the "quantum BRST cohomology") is isomorphic to the space of square integrable sections on the line bundle over $\widetilde{M}$. This shows that prequantization commutes with reduction. For the full Guillemin-Sternberg conjecture, one then must also incorporate the polarization step in the geometric quantization procedure; this additional step has been proven with a few additional assumptions (and in specific examples/theories).

More abstractly, this commutativity of reduction and quantization can be phrased as the following commutative diagram

$$\begin{tikzcd}
(M, \omega, C^\infty(M), \{\cdot,\cdot\}) \arrow[r, "\text{quantization}"] \arrow[d, "\text{classical BRST cohomology}"'] & (\mathcal{H}, \mathrm{Op}(L), [\cdot,\cdot]) \arrow[d, "\text{quantum BRST cohomology}"] \\
(\widetilde{M}, \tilde{\omega}, C^\infty(\widetilde{M}), \{\cdot,\cdot\}) \arrow[r, "\text{quantization}"'] & (\widetilde{\mathcal{H}}, \mathrm{Op}(\widetilde{L}), [\cdot,\cdot])
\end{tikzcd},$$

where $Op(L)$ denotes the space of quantum operators on the line bundle. The above proof then reduces to showing that (geometric) quantization is functorial with respect to the reduction given by (classical and quantum) BRST cohomology in the category of Poisson (super)algebras. This is an abstract manifestation of the fact that the BRST theory allows one to convert (functional) analytic issues to purely algebraic issues. This is extremely powerful, since the proof can be extended to abstrasct Poisson (super)algebras without requiring that they arise as the Poisson structures of symplectic manifolds, and allows one to construct a quantization theory for constrained Poisson algebras purely algebraically.

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