Lecture 9 - Line Integrals cont.

- The next two lectures will be asynchronous (Friday 10/15 and Monday 10/18); you can find the videos uploaded in the Media Gallery on Canvas by this afternoon. Please watch the two lectures before next Wednesday's in-person lecture (on 10/20).

- Homework 2 is due tonight at 11:59 pm via Gradescope. Please make sure to assign the pages on your submission to the corresponding problem. For the 13th problem "Completion Grade", assign every page to this problem. Also, double check your submission to make sure that it was properly submitted to Gradescope.

- Homework 3 is now posted, covering path and line integrals.

- Midterm 1 is on Monday 10/25 of Week 5. It will cover everything in the course up to line integrals (sections 5.2, 5.3, 5.4, 5.5, 6.1, 6.2, 4.3, 7.1, 7.2, homework 1, 2, 3, and lectures 1 - 10). Although we will start talking about surfaces for the lectures of Week 4, the first midterm will not cover surfaces.

- I have OH today after class, 9 - 10 am, and also tomorrow from 11 am to 12. If you watch the two asynchronous lectures before tomorrow and have questions, feel free to ask during tomorrow's OH.

Line integral: $\vec{c}:[a,b]\to\mathbb{R}^n$, $\vec{F}$ vector field

$$\underbrace{\int_{\vec{c}} \vec{F}\cdot dr}_{\parallel} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt$$

$$(n=3)\quad \int_{\vec{c}} (F_x, F_y, F_z)\cdot(dx, dy, dz) = \int_{\vec{c}} \underbrace{F_x\,dx + F_y\,dy + F_z\,dz}_{\text{differential 1-form}}$$

Along curves $x=x(t)$, $y=y(t)$, $z=z(t)$

chain rule $\quad dx = \frac{dx}{dt}\,dt = x'(t)\,dt$

ex/ Let $\vec{c}(t) = (\overbrace{\cos t}^{x(t)}, \overbrace{\sin t}^{y(t)}, \overbrace{2t}^{z(t)})$, $t\in[0,\pi]$

$\vec{F}(x,y,z) = (-x, y, x^2+y^2+z^2)$

Evaluate $\displaystyle\int_{\vec{c}} \vec{F}\cdot d\vec{r} = \int_0^{\pi} \underbrace{\vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)}_{f:[0,\pi]\to\mathbb{R}}\,dt$

$\vec{c}\,'(t) = \frac{d}{dt}\vec{c}(t) = (-\sin t, \cos t, 2)$

$\vec{F}(\vec{c}(t)) = \vec{F}(x(t), y(t), z(t))$

$= (-\cos t, \sin t, 1+4t^2)$

$$\vec{F}(\vec{c}(t)) = \vec{F}(x(t), y(t), z(t))$$
$$= (-\cos t, \sin t, 1+4t^2)$$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{r} = \int_0^{\pi} (-\cos t, \sin t, 1+4t^2)\cdot(-\sin t, \cos t, 2)\,dt$$
$$= \int_0^{\pi} (2\sin t\cos t + 2 + 8t^2)\,dt$$
$$= \int_0^{\pi} \frac{d}{dt}\left(\sin^2 t + 2t + \frac{8}{3}t^3\right)dt$$
$$= 2\pi + \frac{8}{3}\pi^3 \qquad \square$$

ex/ $n=2$ (in the plane)

$$\vec{c}(t) = (\cos t, \sin t) \qquad t\in[0, 2\pi]$$
$$\vec{F}(x,y) = (x^2, e^y)$$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$$
$$= \int_0^{2\pi} \vec{F}(x(t), y(t))\cdot(x'(t), y'(t))\,dt$$

$$\vec{c}\,'(t) = (-\sin t, \cos t)$$
$$\vec{F}(\vec{c}(t)) = (\cos^2 t, e^{\sin t})$$

$\Big\{$ dot product $= -\cos^2 t\sin t + \cos t\, e^{\sin t}$

$$= \int_0^{2\pi} \left(-\cos^2 t\sin t + \cos t\, e^{\sin t}\right)dt$$
$$= \int_0^{2\pi} \frac{d}{dt}\left(\frac{\cos^3 t}{3} + e^{\sin t}\right)dt = 0.$$

Def: (Reparametrization)

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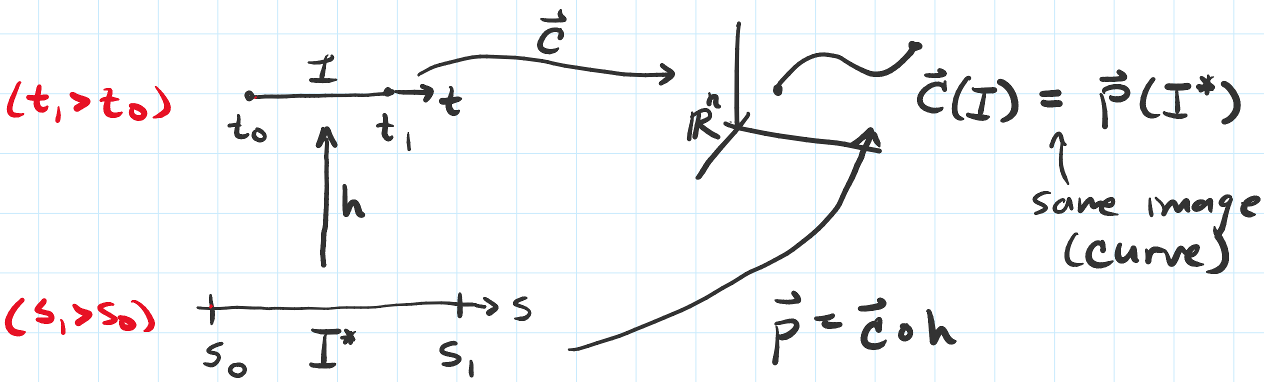
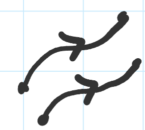
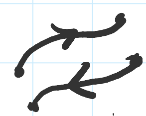
- Let $h: I^* \to I$ be a C^1 bijection of an interval $I^* = [s_0, s_1]$ to $I = [t_0, t_1]$. Let $\vec{c}: I \to \mathbb{R}^n$ be a (piecewise) C^1 path. Then the function
$$\vec{p} = \vec{c} \circ h : I^* \to \mathbb{R}^n$$
is called a reparametrization of $\vec{c}$.



- A reparametrization is orientation-preserving if $\vec{c}(t_0) = \vec{p}(s_0)$, $\vec{c}(t_1) = \vec{p}(s_1)$
  $h(s_0) = t_0$, $h(s_1) = t_1$



- A reparametrization is orientation-reversing if $\vec{c}(t_0) = \vec{p}(s_1)$, $\vec{c}(t_1) = \vec{p}(s_0)$
  $h(s_0) = t_1$ $h(s_1) = t_0$



Theorem:

Let $\vec{F}$ be a (cont.) vector field on the C^1 path $\vec{c}: [t_0, t_1] \to \mathbb{R}^n$ and let $\vec{p} = \vec{c} \circ h : [s_0, s_1] \to \mathbb{R}^n$ be a reparametrization of $\vec{c}$. Then:

If $\vec{p}$ is orientation-preserving,
$$\int_{\vec{p}} \vec{F} \cdot d\vec{r} = \int_{\vec{c}} \vec{F} \cdot d\vec{r}$$
If $\vec{p}$ is orientation-reversing,
$$\int_{\vec{p}} \vec{F} \cdot d\vec{r} = -\int_{\vec{c}} \vec{F} \cdot d\vec{r}$$

$$\int_{\vec{p}} \vec{F}\cdot d\vec{r} = -\int_{\vec{c}} \vec{F}\cdot d\vec{r}$$

proof: $\vec{p} = \vec{c}\circ h \qquad \vec{p}(s) = \vec{c}(h(s))$

chain rule $\quad \vec{p}\,'(s) = \vec{c}\,'(h(s))\, h'(s)$

$$\int_{\vec{p}} \vec{F}\cdot d\vec{r} \underset{\text{def.}}{=} \int_{s_0}^{s_1} \vec{F}(\vec{p}(s))\cdot \vec{p}\,'(s)\,ds = \int_{s_0}^{s_1} \vec{F}(\vec{c}(h(s)))\cdot \vec{c}\,'(h)\, \underbrace{h'(s)\,ds}$$

(change variables $t = h(s)$, $dt = h'(s)\,ds$)

$$= \int_{h(s_0)}^{h(s_1)} \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$$

preserving $\quad h(s_0) = t_0, \quad h(s_1) = t_1$

$$= \int_{t_0}^{t_1} \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt = \int_{\vec{c}} \vec{F}\cdot d\vec{r}$$

reversing $\quad h(s_0) = t_1, \quad h(s_1) = t_0$

$$= \int_{t_1}^{t_0} \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt = -\int_{t_0}^{t_1} \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$$

$$= -\int_{\vec{c}} \vec{F}\cdot d\vec{r}$$

$\square$

Thm:

Let f be a (cont.) function on the C^1 path $\vec{c}: [t_0, t_1] \to \mathbb{R}^n$ and let $\vec{p} = \vec{c}\circ h : [s_0, s_1] \to \mathbb{R}^n$ be any reparametrization. Then,

$$\int_{\vec{c}} f\,ds = \int_{\vec{p}} f\,ds.$$

$$\int_{\vec{c}} f\,ds = \int_{t_0}^{t_1} f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\,dt$$

proof: $\vec{p} = \vec{c}\circ h, \quad \vec{p}\,'(s) = \vec{c}\,'(h(s))\,h'(s)$

$$\int_{\vec{p}} f\,ds = \int^{s_1} f(\vec{p}(s))\, \|\vec{p}\,'(s)\|\,ds$$

$\|a\vec{v}\| \qquad a \in \mathbb{R}$

$$\int_{\vec{p}} f\, d\underset{\sim}{s} = \int_{s_0}^{s_1} f(\vec{p}(s))\, \|\vec{p}\,'(s)\|\, ds$$

$\|a\vec{v}\| = |a|\,\|\vec{v}\|$ $a \in \mathbb{R}$, $v \in \mathbb{R}^n$

$$= \int_{s_0}^{s_1} f(\vec{c}(h(s)))\, \|\vec{c}\,'(h(s))\, h'(s)\|\, ds$$

$$= \int_{s_0}^{s_1} f(\vec{c}(h(s)))\, \|\vec{c}\,'(h(s))\|\, |h'(s)|\, ds$$

$$= \int_{I^*} f(\vec{c}(h(s)))\, \|\vec{c}\,'(h(s))\|\, |h'(s)|\, ds$$

$$= \int_{I} f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt = \int_{\vec{c}} f\, d\underset{\sim}{s}. \qquad \square$$

ex/ Let $\vec{c}(t) = (\cos t, \sin t, 2t)$, $t \in [\overset{t_0}{0}, \overset{t_1}{\pi}]$

$\vec{F}(x,y,z) = (-x, y, x^2+y^2+z^2)$

Evaluate $\displaystyle\int_{\vec{c}} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$

Reparam $h(s) = \pi - s$, $s \in [\overset{s_0}{0}, \overset{s_1}{\pi}]$

$h(s_0) = h(0) = \pi = t_1$

$h(s_1) = h(\pi) = 0 = t_0$

orientation-reversing

$\vec{p}(s) = \vec{c}(h(s))$

$= (\cos(\pi - s), \sin(\pi - s), 2(\pi - s))$

$\vec{p}\,'(s) = (\sin(\pi - s), -\cos(\pi - s), -2)$

$\vec{F}(\vec{p}(s)) = (-\cos(\pi - s), \sin(\pi - s), 1 + 2(\pi - s)^2)$

$$\int_{\vec{p}} \vec{F} \cdot d\vec{r} = \int_0^{\pi} \left[-2\sin(\pi - s)\cos(\pi - s) - 2 - 4(\pi - s)^2\right] ds$$

let $u = \pi - s$ ...

$\square$