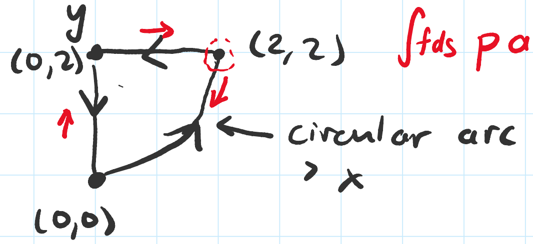
Path Integrals in the plane

$$\int_C f\,ds = \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

$$\vec{c}(t) = (x(t), y(t)), \qquad \vec{c}\,'(t) = (x'(t), y'(t))$$

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

ex/ Parameterize and find the arclength of:



$\int f\,ds$ param. independent

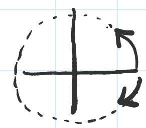
Param $\vec{c}$ as a piecewise map

Circular arc

$t \in [0,1]$ $\qquad (x-a)^2 + (y-b)^2 = r^2$ circle of radius r centered at (a,b)

$$\Rightarrow x^2 + (y-2)^2 = 4$$

$$\vec{c}(t) = (a + r\cos(kt),\ b + r\sin(kt))$$



ccw $(k>0)$
cw $(k<0)$

$$\vec{c}(t) = (2\cos(kt),\ 2+2\sin(kt))$$

$$= \left(2\cos\left(-\tfrac{\pi}{2}t\right),\ 2+2\sin\left(-\tfrac{\pi}{2}t\right)\right), \quad t \in [0,1]$$

Linear arc $(0,0)$ to $(0,2)$

$t \in [1,2]$ $\qquad t_0 = 1,\ \vec{x}_0 = \begin{pmatrix}0\\0\end{pmatrix} = (0,0)$

$$\vec{c}(t) = \vec{x}_0 + \vec{v}(t-t_0) \qquad \vec{c}(t_0) = \vec{x}_0$$

$$\vec{v} = \begin{pmatrix}0\\2\end{pmatrix} = (0,2)$$

$$= (0,0) + (0,2)(t-1)$$

$$= (0,\ 2(t-1)). \qquad \vec{c}(1) = (0,0),\quad \vec{c}(2) = (0,2)$$

Linear arc: $(0,2)$ to $(2,2)$

$t \in [2,3]$

$$\vec{c}(t) = \vec{x}_0 + \vec{v}(t-t_0) \qquad t_0 = 2,\quad \vec{x}_0 = (0,2)$$

$\vec{c}(t) = \vec{x}_0 + \vec{v}(t - t_0)$ $t_0 = 2$, $x_0 = (0, 2)$

$= (0, 2) + (2, 0)(t - 2)$ $\vec{v} = (2, 0)$

$= (2(t-2), 2)$ $\vec{c}(2) = (0, 2)$, $\vec{c}(3) = (2, 2)$

$$\vec{c}(t) = \begin{cases} \left(2\cos(-\tfrac{\pi}{2}t),\ 2 + 2\sin(-\tfrac{\pi}{2}t)\right), & t \in [0, 1] \\ (0,\ 2(t-1)), & t \in [1, 2] \\ (2(t-2),\ 2), & t \in [2, 3]. \end{cases}$$

$$\int_C \underset{\overset{\shortparallel}{1}}{f}\, ds = \int_0^3 \underbrace{f(\vec{c}(t))}_{=1} \|\vec{c}\,'(t)\|\, dt$$ Arclength $f = 1$.

$$= \int_0^1 \|\vec{c}\,'(t)\| dt + \int_1^2 \|\vec{c}\,'(t)\| dt + \int_2^3 \|\vec{c}\,'(t)\| dt.$$

ex) Consider $\vec{c} : t \mapsto (\cos^3 t, \sin^3 t)$ $t \in [0, \pi]$.
Integrate $f(x, y) = 1$ along C.

$$\int_C f ds = \int_0^\pi f(\vec{c}(t)) \|\vec{c}\,'(t)\| dt.$$

$f(\vec{c}(t)) = 1$ Arclength

$\vec{c}\,'(t) = (-3\cos^2 t \sin t,\ 3\sin^2 t \cos t)$

Mistake:

$$\|\vec{c}\,'(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3\sqrt{\cos^2 t \sin^2 t \underbrace{(\sin^2 t + \cos^2 t)}_{=1}}$$

$$= 3\cos t \sin t$$

$$\Rightarrow \int_C ds = \int_0^\pi 3\cos t \sin t\, dt = 3\int_0^\pi \frac{d}{dt}\left(\frac{\sin^2 t}{2}\right) dt$$

$$\Rightarrow \int_C ds = \int_0^{\pi} 3\cos t \sin t \, dt = 3\int_0^{\pi} \frac{d}{dt}\left(\frac{\sin^2 t}{2}\right) dt$$

$$= \frac{3}{2}\sin^2 t \Big|_0^{\pi} = 0.$$

$a \in \mathbb{R}$ $\qquad \sqrt{a^2} = |a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$

$$\|\vec{c}\,'(t)\| = 3|\cos t \sin t|$$

$t \in [0, \pi]$

$t \in [0, \pi/2]$, $\cos t \geq 0$

$t \in [\pi/2, \pi]$, $\cos t \leq 0$

$$= 3\begin{cases} \cos t \sin t, & t \in [0, \pi/2] \\ -\cos t \sin t, & t \in [\pi/2, \pi] \end{cases}$$

$$\int_C ds = \int_0^{\pi} \|\vec{c}\,'(t)\| \, dt = \int_0^{\pi/2} 3\cos t \sin t \, dt + \int_{\pi/2}^{\pi} -3\cos t \sin t \, dt$$

$$= 3\left[\int_0^{\pi/2} \frac{d}{dt}\left(\frac{\sin^2 t}{2}\right) dt + \int_{\pi/2}^{\pi} -\frac{d}{dt}\left(\frac{\sin^2 t}{2}\right) dt\right]$$

$$= \frac{3}{2}\left(\sin^2 t \Big|_0^{\pi/2} - \sin^2 t \Big|_{\pi/2}^{\pi}\right)$$

$$= \frac{3}{2}\left(1 - 0 - 0 - (-1)\right) = 3 > 0.$$

Line Integrals:

Idea: integrate a vector field over a path.

Review: dot products

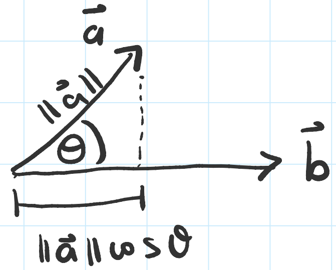
In $\mathbb{R}^3$,

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \sum_{j=1}^{3} a_j b_j$$

(In $\mathbb{R}^n$, $\vec{a} \cdot \vec{b} = \sum_{j=1}^{n} a_j b_j$. $n = 2, 3$)

$\vec{a}$



$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

When $\vec{a} \perp \vec{b}$ (θ is $\pi/2$ or $3\pi/2$), $\vec{a}\cdot\vec{b} = 0$

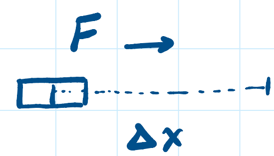
When they are parallel (θ is 0) $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|$

When $\vec{a} = \vec{b}$, $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

Motivation for line integrals

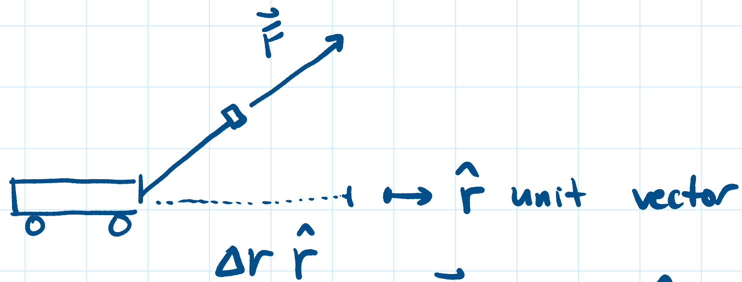
"Work" in physics

- Move an object w/ constant force F a distance Δx, work is defined $W = F\Delta x$



(constant)

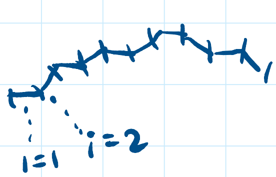
- What if $\vec{F}$ does not necessarily point in the direction of motion



Work is defined to be $W = \vec{F}\cdot(\Delta r\,\hat{r})$

What if $\vec{F}$ is not constant?

$$W \approx \sum_i \vec{F}_i\cdot(\Delta r_i\,\hat{r}_i)$$



$\lim \Delta r_i \to 0$

$W = \int \vec{F}\cdot d\vec{r}$ work done by a force $\vec{F}$

$$W = \int_{\vec{C}} \vec{F} \cdot d\vec{r}$$ work done by a force F on a particle moving in a path $\vec{c}$.

text

$d\mathbf{s}$ ds

in $n=3$

$$\int_{\vec{C}} \vec{F}(x,y,z) \cdot (dx, dy, dz)$$

$\vec{C} : [a,b] \to \mathbb{R}^3$ param, $\vec{C}(t) = (x(t), y(t), z(t))$

$$\to \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))\, dt$$

$$\int_{\vec{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}\,'(t)\, dt.$$

ex/ Find the work done moving a particle from $(0,0,0)$ to $(1,1,1)$ along a line by the force

$$\vec{F}(x,y,z) = (x,y,z)$$

$$\vec{C}(t) = \vec{x}_0 + \vec{v}_0 t = (1,1,1)t = (t,t,t), \quad t \in [0,1]$$

$$\vec{C}\,'(t) = \vec{v}_0 = (1,1,1)$$

$$W = \int_0^1 \vec{F}(t,t,t) \cdot (1,1,1)\, dt = \int_0^1 (t,t,t) \cdot (1,1,1)\, dt$$

$$= \int_0^1 3t\, dt = \frac{3}{2}.$$