# Lecture 7 - Vector Fields cont.; Path Integrals

- Read section 7.1
- Updated course schedule: Week 3 Friday 10/15 and Week 4 Monday 10/18 will be asynchronous lectures. I will not be in town so I will prerecord the lectures and post them to the media gallery on Canvas. I will add an extra office hour sometime week 4 incase anyone has questions that they couldn't ask during these asynchronous lectures (or any other questions).

## Gradient vector field

For $n=2$, $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

For $n=3$, $f: A \subseteq \mathbb{R}^3 \to \mathbb{R}$, $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

**Q1**: Given $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ $(n=2,3)$, how do we know if $\vec{F} = \nabla f$?
(answer: if $\nabla \times \vec{F} = 0$)

**Q2**: If we do know $\vec{F} = \nabla f$, and we're given $\vec{F}$, how do we find $f$?

ex/ $$\vec{F}(x,y,z) = (\overbrace{\cos(x)y}^{F_1(x,y,z)}, \overbrace{\sin(x) + 2yz}^{F_2(x,y,z)}, \overbrace{y^2}^{F_3(x,y,z)})$$

- $\vec{F}$ is a gradient vector field. ✓
- Find $f$ s.t. $\vec{F} = \nabla f$.

$$(F_1, F_2, F_3) = \vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\frac{\partial f}{\partial x}(x,y,z) = F_1(x,y,z) = \cos(x)y$$

$$\frac{\partial f}{\partial y}(x,y,z) = F_2(x,y,z) = \sin(x) + 2yz$$

$$\frac{\partial f}{\partial z}(x,y,z) = F_3(x,y,z) = y^2$$

$$\int \frac{\partial f}{\partial \underline{x}}\, dx = \int \cos(x) y\, dx$$

$$\Rightarrow f(x,y,z) = \underline{\sin(x)y} + C_1(y,z)$$

$$\Rightarrow f(x,y,z) = \sin(x)y + C_1(y,z)$$

$$\int \frac{\partial f}{\partial y}\,dy = \int (\sin(x) + 2yz)\,dy$$

$$\Rightarrow f(x,y,z) = \sin(x)y + y^2 z + C_2(x,z)$$

$$\int \frac{\partial f}{\partial z}\,dz = \int y^2\,dz$$

$$\Rightarrow f(x,y,z) = y^2 z + C_3(x,y)$$

$$\Rightarrow f(x,y,z) = \sin(x)y + y^2 z + \underline{\underline{C}}, \quad C \in \mathbb{R}$$

Check $\vec{F} = \nabla f$.

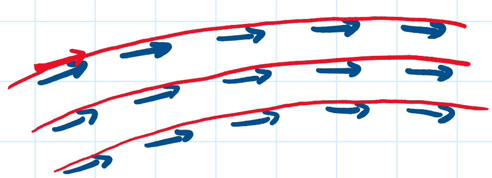
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \cos(x)y \\ \frac{\partial f}{\partial y} &= \sin(x) + 2yz \\ \frac{\partial f}{\partial z} &= y^2 \end{aligned}\right\} \vec{F}$$

---

## Flow Lines

- Let $\vec{F}$ be a vector field. A flow line along $\vec{F}$ is a parameterized curve $\vec{c}(t)$ such that

$$\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$$



$$\vec{c}(t) = (x(t), y(t), z(t)) \qquad \vec{F} = (F_1, F_2, F_3)$$

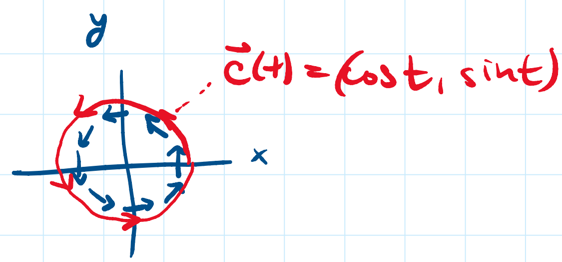
$$\begin{cases} x'(t) = F_1(x(t), y(t), z(t)) \\ y'(t) = F_2(x(t), y(t), z(t)) \\ z'(t) = F_3(x(t), y(t), z(t)) \end{cases}$$

System of ODEs

$y$

$z'(t) = F_3(x(t), y(t), z(t))$

ex/ $\vec{F}(x,y) = (-y, x)$




Check $\vec{c}(t) = (\cos t, \sin t)$ is a flow line.

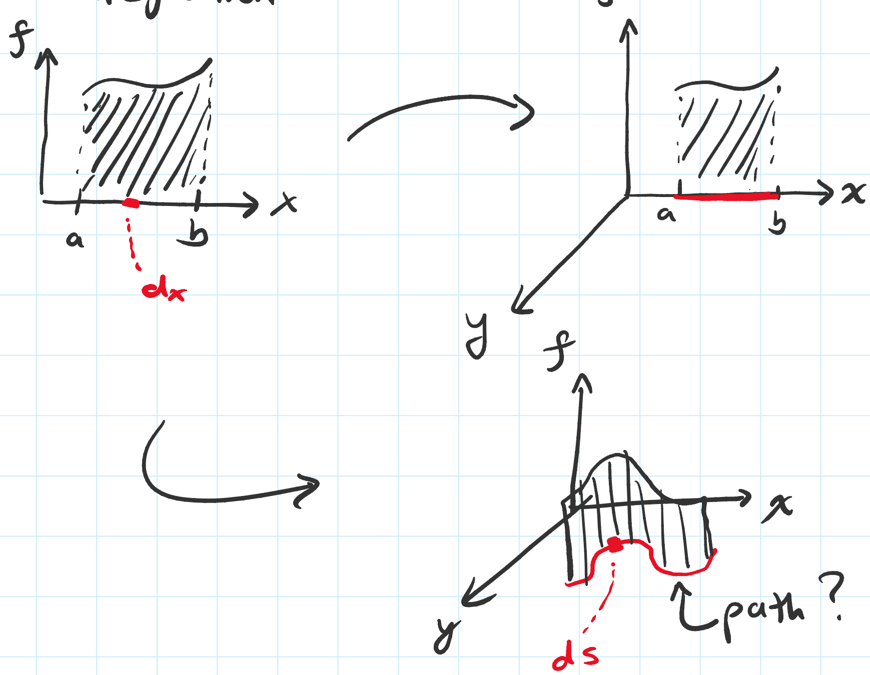
$\vec{c}\,'(t) = (-\sin t, \cos t)$

$\vec{F}(\vec{c}(t)) = (-y(t), x(t)) = (-\sin t, \cos t)$ $\Rightarrow$ $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$.

## Path Integrals

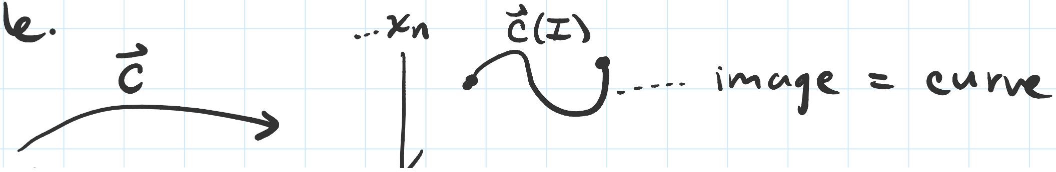
Idea:

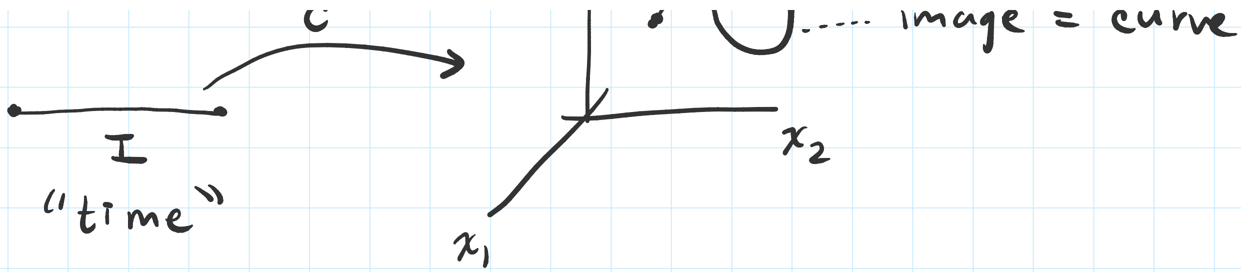
1d integration




**Def:** A path in $\mathbb{R}^n$ is a map $\vec{c} : I \to \mathbb{R}^n$. (interval $\subseteq \mathbb{R}$)

We'll assume paths are (piecewise) continuously differentiable.






Denote

$$\vec{c}(t) = (x_1(t), \dots, x_n(t))$$
$$\vec{c}\,'(t) = (x_1'(t), \dots, x_n'(t))$$

Given $f : \mathbb{R}^n \to \mathbb{R}$, how do we make sense of

$$\int_{\vec{c}} f \, ds \qquad s \text{ "arclength element"}$$

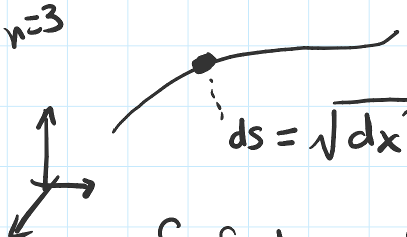
Path Integral is defined: $\vec{c} : [a, b] \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$

$$\int_{\vec{c}} f \, ds = \int_a^b f(\vec{c}(t)) \, \|\vec{c}\,'(t)\| \, dt$$

In $\mathbb{R}^3$, $\vec{c}(t) = (x(t), y(t), z(t))$

$$\int_{\vec{c}} f ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

$n=3$



$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$$\Uparrow$$

$$\int_c f \, ds = \int_a^b f \, \frac{ds}{dt} \, dt$$

$$\frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

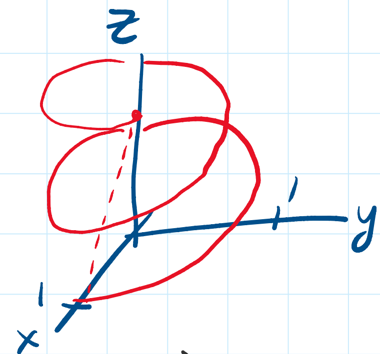
Why does it make sense?

$$\int_{\vec{c}} 1\, ds = \int_a^b \|\vec{c}\,'(t)\|\, dt = \text{Arclength}(\vec{c})$$

$$\int_C f\, ds = \int_a^b f(\vec{c}(t))\, \underbrace{\|\vec{c}\,'(t)\|}\, dt$$

show that $\int_C f\, ds$ is independent of parameterization

ex/ Consider the helix




$\vec{c} : [0, 4\pi] \to \mathbb{R}^3,$

$$\vec{c}(t) = (\underset{x(t)}{\cos t}, \underset{y(t)}{\sin t}, \underset{z(t)}{t})$$

( reparameterization:
$\vec{c}_1 : [0, 2\pi] \to \mathbb{R}^3 \qquad \vec{c}_1(t) = (\cos(2t), \sin(2t), 2t)$ )

Let $f(x, y, z) = z e^{x^2 + y^2}$

Evaluate $\int_{\vec{c}} f\, ds.$

$$\int_{\vec{c}} f\, ds = \int_0^{4\pi} \underbrace{f(\vec{c}(t))}_{?}\, \|\underbrace{\vec{c}\,'(t)}_{?}\|\, dt$$

$$f(\vec{c}(t)) = z(t) e^{x(t)^2 + y(t)^2}$$
$$= t e^{\cos^2 t + \sin^2 t} = te$$

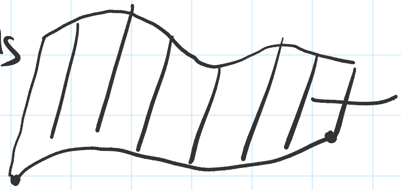
$$\vec{c}\,'(t) = (-\sin t, \cos t, 1)$$
$$\|\vec{c}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\|\vec{c}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

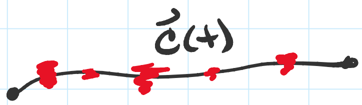
$$\Rightarrow \int_{\vec{c}} f\,ds = \int_0^{4\pi} t e \sqrt{2}\,dt = e\sqrt{2} \int_0^{4\pi} t\,dt$$

$$= e\sqrt{2}\,\frac{(4\pi)^2}{2}.$$

$\int_C f\,ds$



Area of the fence




$\int_C f\,ds$ = total mass of string.

$f(\vec{c}(t))$ linear mass density