- (Re)read section 4.3 on vector fields
- Read section 7.1 on path integrals for next time
- OH today from 9 am to 10 and tomorrow 11 am - 12

Recall 3D C.O.V

Let $T: \underset{\cap\, \mathbb{R}^3}{W^*} \longrightarrow \underset{\cap\, \mathbb{R}}{W}$ be a cont. diff. bijection, then

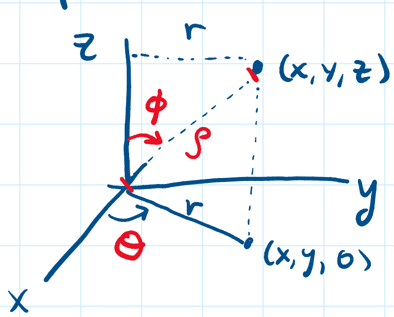
$$\iiint_W f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{W^*} f(T(u,v,w))\,\left|\det DT(u,v,w)\right|\,du\,dv\,dw$$

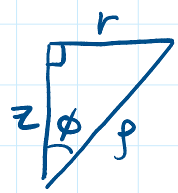
or, if $T(u,v,w) = \big(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\big)$

$$= \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Spherical Coordinates



(ρ, θ, ϕ)



$z = \rho\cos\phi$

$r = \rho\sin\phi$

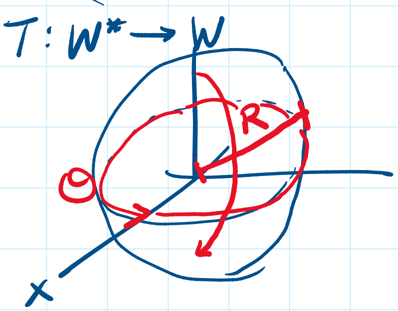
$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$$T: (\rho,\theta,\phi) \longmapsto \big(\underbrace{\rho\sin\phi\cos\theta}_{x(\rho,\theta,\phi)},\ \underbrace{\rho\sin\phi\sin\theta}_{y(\rho,\theta,\phi)},\ \underbrace{\rho\cos\phi}_{z(\rho,\theta,\phi)}\big)$$

Ball of radius R in xyz: $\{x^2+y^2+z^2 \le R^2\} = W$

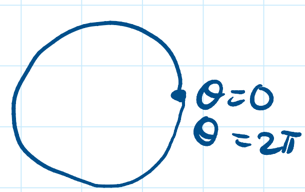
$T: W^* \to W$



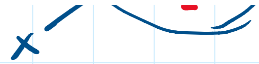
$0 \le \rho \le R$

$0 \le \theta < 2\pi$

$0 \le \phi \le \pi$



$W^* = [0, R] \times [0, 2\pi) \times [0, \pi]$ $\quad \int \ldots \quad \int^b$



$$W^* = [0,R] \times [0,2\pi) \times [0,\pi]$$

$\int_{[a,b]}$ $\int_a^b$

$\int_{(a,b)}$

Jacobian determinant

$$|\det DT(\rho,\theta,\phi)| = \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \rho^2 \sin\phi.$$

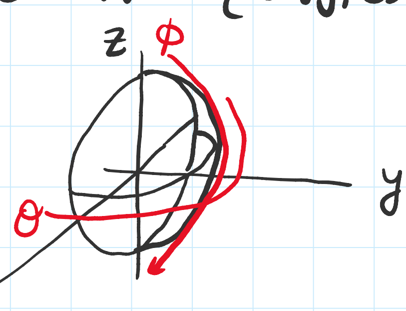
ex/ Volume of a ball of radius R $= W$

$$\begin{aligned} \mathrm{Vol}(W) &= \iiint_W 1 \, dx\,dy\,dz \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} 1 \cdot \overbrace{\rho^2 \sin\phi}^{\text{Jacobian}} \, d\rho\, d\theta\, d\phi \\ &= \underbrace{\int_0^{\pi} \sin\phi \, d\phi}_{} \; \underbrace{\int_0^{2\pi} d\theta}_{} \; \underbrace{\int_0^{R} \rho^2 \, d\rho}_{} \\ &= 2 \cdot 2\pi \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3. \end{aligned}$$

ex/ $\iiint_W \exp\left((x^2+y^2+z^2)^{3/2} \right) dV$

$y = 0 \Leftrightarrow xz$ plane

where $W = \{ (x,y,z) : \overline{x^2+y^2+z^2 \le 1} \text{ and } y \ge 0 \}$



$$\left\{ \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le \pi \\ 0 \le \phi \le \pi \end{array} \right\} = W^*$$

$$\iiint_W e^{(x^2+y^2+z^2)^{3/2}} \, dV$$

$$= \iiint e^{\big(\overbrace{x(\rho,\theta,\phi)^2 + y(\rho,\theta,\phi)^2 + z(\rho,\theta,\phi)^2}^{=\rho^2}\big)^{3/2}} \cdot |\det DT(\rho,\theta,\phi)| \, d\rho\, d\theta\, d\phi$$

$$= \iiint_{W^*} e^{(x(\rho,\theta,\phi)^2 + y(\rho,\theta,\phi)^2 + z(\rho,\theta,\phi)^2)^{3/2}} \cdot |\det DT(\rho,\theta,\phi)| \, d\rho \, d\theta \, d\phi$$

$$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2$$

$$= \rho^2\sin^2\phi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) + \rho^2\cos^2\phi = \rho^2$$

$$= \int_0^{\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \cdot \rho^2\sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \underbrace{\int_0^{\pi}\sin\phi\,d\phi}_{2}\ \underbrace{\int_0^{\pi}d\theta}_{\pi}\ \int_0^{1}\underbrace{e^{\rho^3}\rho^2}_{=\frac{d}{d\rho}\left(\frac{1}{3}e^{\rho^3}\right)}d\rho$$

$$= 2\pi \, \frac{1}{3}e^{\rho^3}\Big|_0^1 = \frac{2\pi}{3}(e-1).$$

Change of Variables

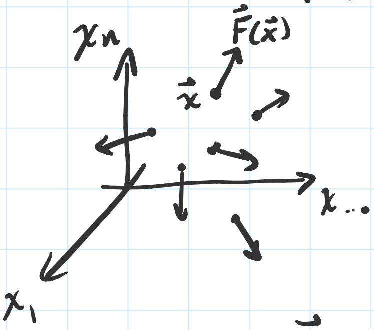
2D Linear Transformations (Bijection / Invertible)
Polar Coordinates

3D Cylindrical Coordinates
Spherical Coordinates

Vector Fields (read section 4.3)

Def: A vector field on (a subset of) $\mathbb{R}^n$ is a map $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$; assigning to each $\vec{x} \in \mathbb{R}^n$ a vector $\vec{F}(\vec{x})$.

$F : \mathbb{R} \to \mathbb{R}$; assigning to each $x \in \mathbb{R}$ a vector $\vec{F}(\vec{x})$.



describe for ex,
velocity of a fluid
electric field

In components, $\vec{F}(\vec{x}) = \left(F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x})\right)$

$n=2 \quad \vec{F}(x,y) = \left(F_1(x,y), F_2(x,y)\right)$

$n=3 \quad \vec{F}(x,y,z) = \left(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\right)$.

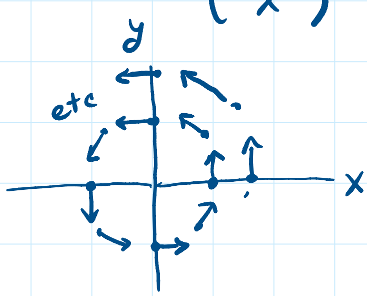
(If the components are cont., diff., etc, then we say $\vec{F}$ is as well. Generally, we'll assume continuously differentiable (C^1) vector fields)

ex/ $\vec{F}(x,y) = \underline{(-y, x)} = -y\hat{\imath} + x\hat{\jmath}$

$= \begin{pmatrix} -y \\ x \end{pmatrix}$

$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\vec{F}(1,0) = (0,1)$

Important class of vector fields

Gradient Vector Fields

Recall for a differentiable function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$

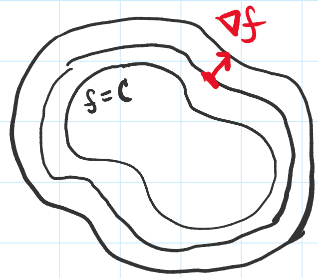
$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x^1}(\vec{x}), \dots, \frac{\partial f}{\partial x^n}(\vec{x}) \right)$$

$$n=3 \quad \nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right)$$

$n=3 \quad \nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z),\ \frac{\partial f}{\partial y}(x,y,z),\ \frac{\partial f}{\partial z}(x,y,z) \right)$

Important?

· Optimization



gradient points in steepest ascent.

· Conservation of Energy

In physics, a force is called conservative if $\vec{F} = -\nabla V$ scalar function: potential energy

Newton's 2nd law force = mass × accel.

$$m\vec{x}''(t) = \vec{F}(\vec{x}(t)) = -\nabla V(\vec{x}(t))$$

claim:

The energy

$$E = \overbrace{\frac{1}{2} m\, \vec{x}'(t)\cdot\vec{x}'(t)}^{\text{Kinetic}} + \overbrace{V(\vec{x}(t))}^{\text{potential}}$$

is conserved

pf:

$$\frac{d}{dt}E = m\vec{x}'(t)\cdot\vec{x}''(t) + \nabla V(\vec{x}(t))\cdot\vec{x}'(t)$$

$$= \vec{x}'(t)\cdot\Big(\underbrace{m\vec{x}''(t) + \nabla V(\vec{x}(t))}_{=0}\Big) = 0$$