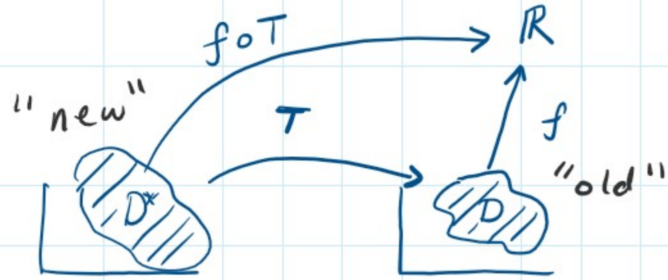
Change of Variables (Examples and in 3 dimensions)

Monday, October 4, 2021 7:56 AM

- Finish reviewing section 6.2
- Read section 4.3 for Wednesday's lecture

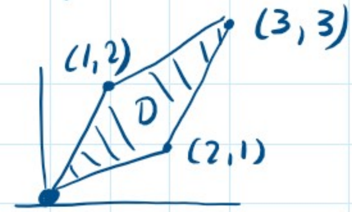


Examples of 2D c.o.v.

· Recall $T: D^* \to D$ C^1 bijection

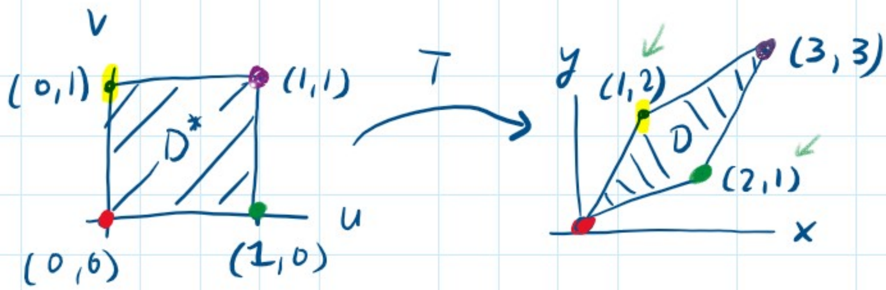
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \underline{f(T(u,v))}\,|\det DT(u,v)|\,du\,dv$$

ex/ Let D be the parallelogram



$$\iint_D x^2\,dx\,dy.$$

· Idea: Find $T: D^* \to D$ s.t. D^* is easier to integrate over. Choose D^* as a square



$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \Rightarrow \begin{matrix} a = 2 \\ c = 1 \end{matrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \Rightarrow \begin{matrix} b = 1 \\ d = 2 \end{matrix}$$

$$\Rightarrow [T] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad [T]\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

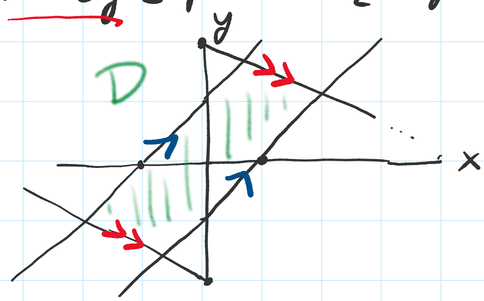
$$T(u,v) = [T]\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u+v \\ u+2v \end{pmatrix} \begin{matrix} \leftarrow x(u,v) \\ \leftarrow y(u,v) \end{matrix}$$

$$\iint_D x^2\,dx\,dy = \underbrace{\int_0^1\int_0^1}_{D^*} (2u+v)^2 \underbrace{|\det DT(u,v)|}_{\substack{=|\det[T]| \\ = 3}} du\,dv \quad \left(\begin{array}{l}\text{see hw1} \\ \text{problem 11}\end{array}\right)$$

$$= 3\int_0^1\int_0^1 (4u^2 + 4uv + v^2)\,du\,dv \;\ldots$$

ex/ Consider D : $-1 \le x-y \le 1$ $-1+y \le x \le 1+y$
$-4 \le x+2y \le 4$ $-2-\frac{x}{2} \le y \le 2-\frac{x}{2}$

Evaluate $\iint_D 9xy\,dx\,dy$.



"new" "old"

$$\begin{cases} u = x-y \\ v = x+2y \end{cases} \qquad D^* \begin{cases} -1 \le u \le 1 \\ -4 \le v \le 4 \end{cases}$$

$$\hookrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

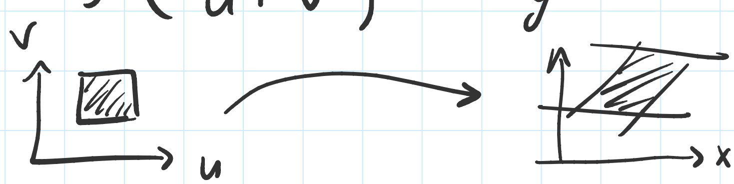
$\det A = 2-(-1) = 3 \ne 0$
$\Rightarrow A$ invertible

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \qquad \det(A^{-1}) = (\det(A))^{-1}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{A^{-1}}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2u+v \\ -u+v \end{pmatrix} \leftarrow \begin{array}{l} x(u,v) \\ y(u,v) \end{array}$$

$[T] = A^{-1}$:



$$\iint_D 9xy\,dx\,dy$$
$$= \int^{4}\int^{1} 9\left(\frac{2u+v}{3}\right)\left(\frac{v-u}{3}\right) \underbrace{|\det(A^{-1})|}_{=1/3}\, du\,dv$$

$$= \int_{-4}^{4}\int_{-1}^{1} 9\left(\frac{2u+v}{3}\right)\left(\frac{v-u}{3}\right)\left|\det(A^{-1})\right|\, du\, dv$$

...

C.O.V in 3D

Let $T: \underset{\mathbb{R}^3}{W^*} \longrightarrow \underset{\mathbb{R}^3}{W}$ be a cont. diff. bijection, then

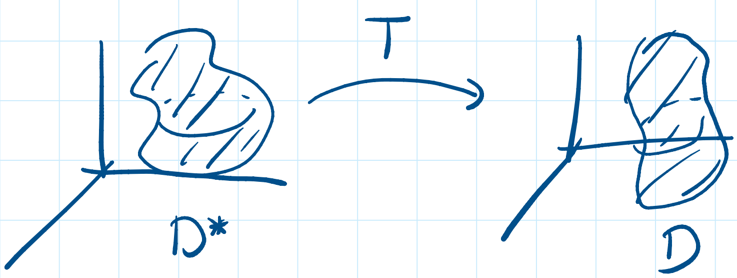
$$\iiint_W f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_{W^*} f(T(u,v,w))\,\left|\det DT(u,v,w)\right|\,du\,dv\,dw$$

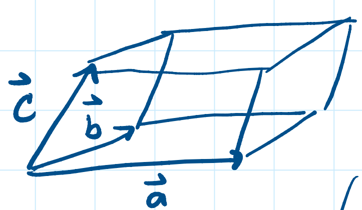
or, if $T(u,v,w) = (x(u,v,w),\ y(u,v,w),\ z(u,v,w))$

$$= \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw$$

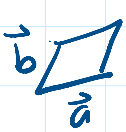
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det DT(u,v,w) = \det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}$$



Proof: Volume of a parallelepiped is



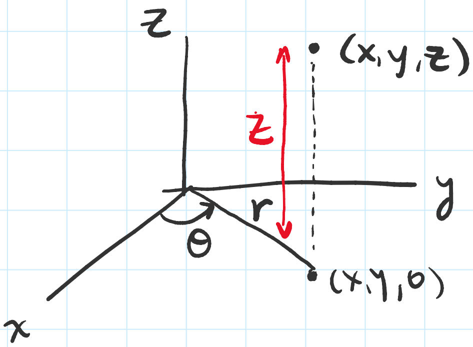
$$Vol = \left|\det(\vec{a}\ \ \vec{b}\ \ \vec{c})\right|$$

(in 2d, 

Area $= |\det(\vec{a}\ \ \vec{b})|$)

· The rest analogous to 2d. $\square$

Cylindrical Coordinates



$$(r, \theta, z) \longmapsto (x(r,\theta,z),\ y(r,\theta,z),\ z(r,\theta,z))$$

$$x(r,\theta,z) = r\cos\theta$$
$$y(r,\theta,z) = r\sin\theta$$
$$z(r,\theta,z) = z.$$

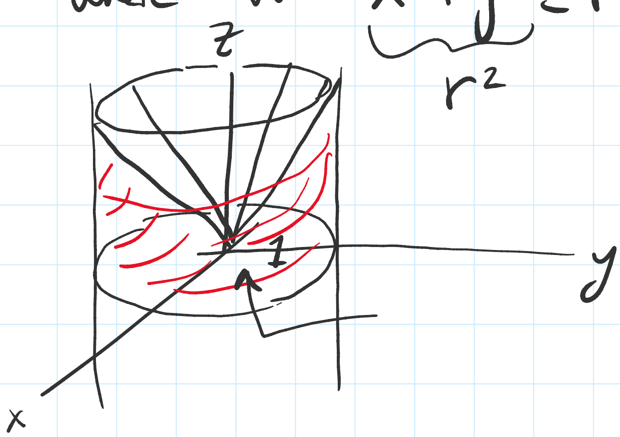
$$\det DT(r,\theta,z) = \det\begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \text{---} & \text{"}y\text{"} & \text{---} \\ \text{---} & \text{"}z\text{"} & \text{---} \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$

ex/ Evaluate $\iiint_W z\,dx\,dy\,dz$

where W: $\underbrace{x^2+y^2}_{r^2} \le 1$, $\quad 0 \le z \le \underbrace{(x^2+y^2)^{1/2}}_{=\left\|\binom{x}{y}\right\|}$ cone



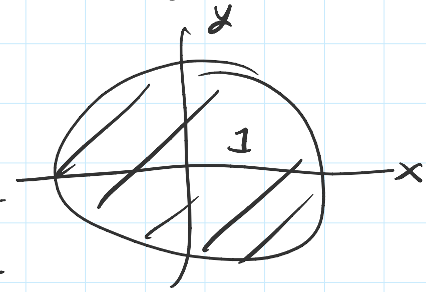
zx plane
$z = |x|$

zy plane
$z = |y|$

$z = |y|$

In cylindrical coordinates,

$$\left.\begin{array}{l} 0 \le z \le r \\ 0 \le r \le 1 \\ 0 \le \theta < 2\pi \end{array}\right]$$



$$\iiint_W \overset{f}{z}\, dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_0^r \overset{f}{z}\, \overset{\text{Jacobian}}{r}\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left.\frac{z^2}{2}\right|_0^r \cdot r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 r^3\,dr$$

$$= \pi \cdot \left.\frac{r^4}{4}\right|_0^1 = \pi/4.$$