Lecture 4 - Change of Variables (in 2 dimensions)

- Read section 6.2
- Review section 1.3 (on matrices and determinants) and section 1.4 (on cylindrical/spherical coordinates)

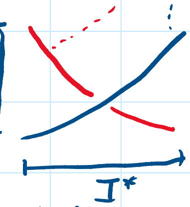
Motivation: change domain of integration and/or the integrand to be simpler.

Recall 1d c.o.v. or substitution
$$\int_a^b f(x)\,dx$$

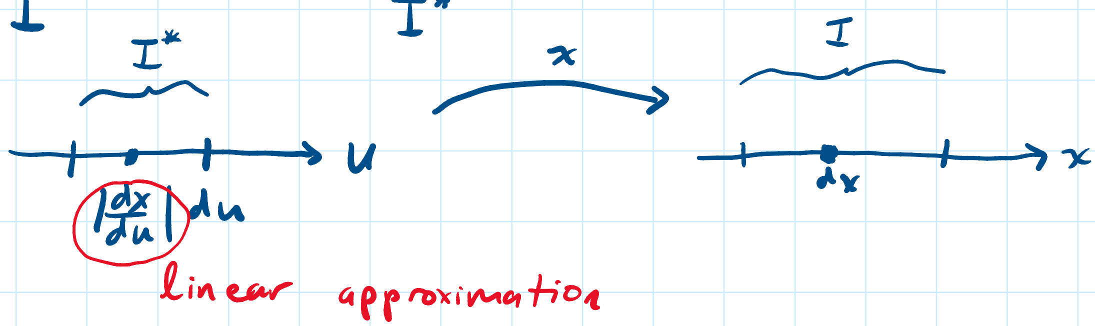
continuously diff. (C^1)

- Let $u \mapsto x(u)$ be a bijection mapping onto $[a,b] = I$ with domain $I^* = x^{-1}([a,b]) = [x^{-1}(a), x^{-1}(b)]$ or $[x^{-1}(b), x^{-1}(a)]$

$$\int_a^b f(x)\,dx = \int_{x^{-1}(a)}^{x^{-1}(b)} \left[f(x(u))\,\frac{dx}{du}\right] du$$



$$\int_I f(x)\,dx = \int_{I^*} f(x(u))\left|\frac{dx}{du}\right| du \leftarrow \text{1d c.o.v.}$$



Thm: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transf & invertible, then T maps parallelograms to parallelograms & vertices to vertices.

ex/ Consider $[T] = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad T(u,v) = [T]\begin{pmatrix} u \\ v \end{pmatrix}$

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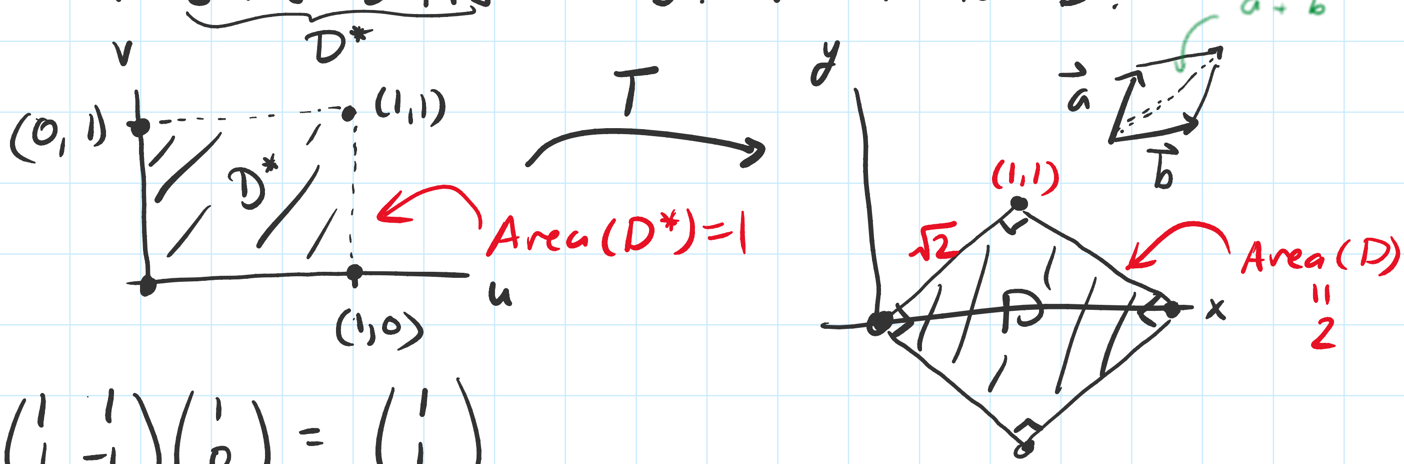
$T : \underbrace{[0,1] \times [0,1]}_{D^*} \longrightarrow D$. What is D?



$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

pt (x,y) $\quad$ vector $\begin{pmatrix} x \\ y \end{pmatrix}$

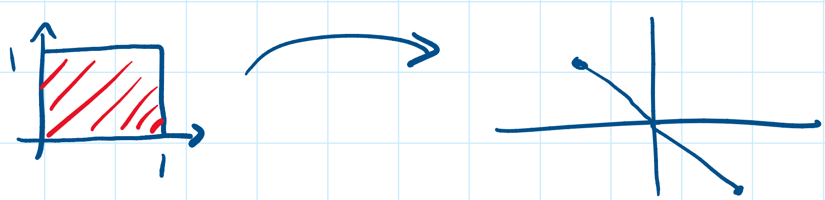
$$\left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = |-1-1| = 2$$

For $T : D^* \longrightarrow D$ linear, (parallelograms)

$$\text{Area}(D) = |\det(T)| \, \text{Area}(D^*)$$

What about non-invertible T?

$[T] = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad \det[T] = 1 - (-1)(-1) = 0.$

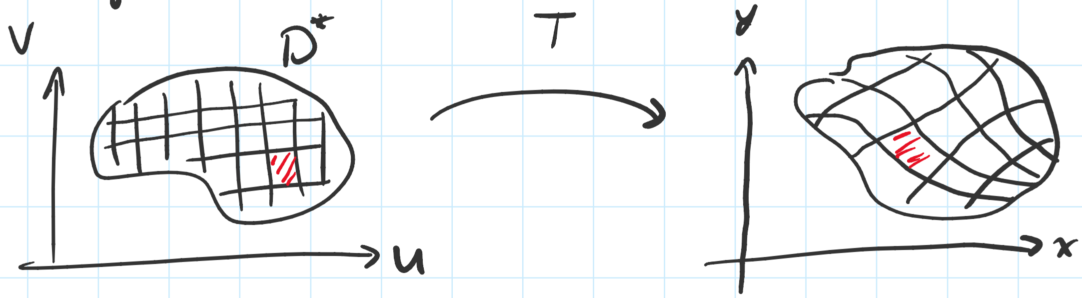


$[T] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad [T]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\quad \ker[T] = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

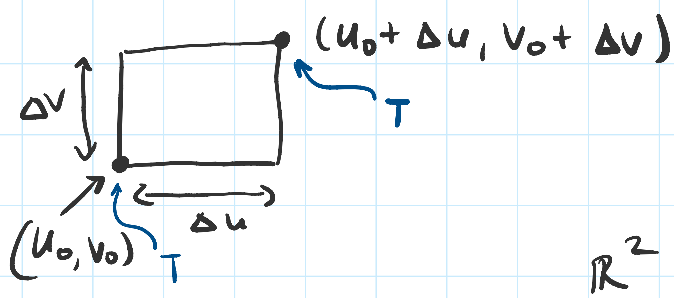
$L[\;\;] = \begin{pmatrix} \cdot & \cdot \\ 0 & 0 \end{pmatrix}$ $\quad L[\;\;]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\quad \ker[L] = \operatorname{span}\{(\ldots)\}$

consider $T : \underset{\mathbb{R}^2}{D^*} \longrightarrow \underset{\mathbb{R}^2}{D}$ continuously differentiable bijection

(where $D^* \subseteq \mathbb{R}^2$, $D \subseteq \mathbb{R}^2$)



consider a rectangle



Taylor's Theorem

$$T(u_0+\Delta u, v_0+\Delta v) = \underbrace{T(u_0,v_0)}_{\in\,\mathbb{R}^2} + \underbrace{DT(u_0,v_0)}_{2\times 2\text{ matrix}}\underbrace{\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}}_{\in\,\mathbb{R}^2}$$

$$+\, O(\Delta u^2, \Delta u \Delta v, \Delta v^2)$$

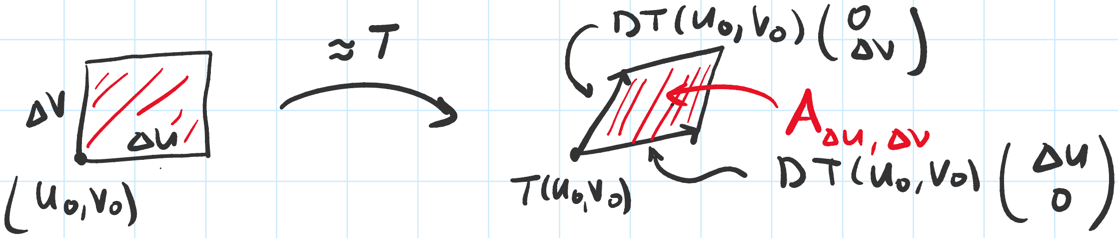
big O-notation

$$T(u_0+\Delta u, v_0+\Delta v) \approx T(u_0,v_0) + DT(u_0,v_0)\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$$

Write $T(u,v) = (x(u,v), y(u,v))$

$$DT(u_0,v_0) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial \cdot} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$\left.\begin{pmatrix} \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}\right|_{(u_0,v_0)}$$

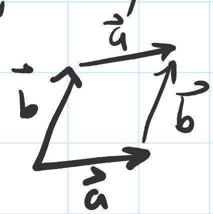


Parallelogram has edge vectors

$$DT(u_0,v_0)\begin{pmatrix}\Delta u\\ 0\end{pmatrix} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}\begin{pmatrix}\Delta u\\ 0\end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u}\Delta u \\ \frac{\partial y}{\partial u}\Delta u \end{pmatrix}$$

$$DT(u_0,v_0)\begin{pmatrix}0\\ \Delta v\end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial v}\Delta v \end{pmatrix}$$

Area of a parallelogram



$$A = \left|\det\begin{pmatrix}\vec{a} & \vec{b}\end{pmatrix}\right|$$

$$A_{\Delta u,\Delta v} = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{pmatrix}\right|$$

$$= \Bigg|\underbrace{\det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}}_{\text{Jacobian determinant}}\Bigg|\,\Delta u\,\Delta v \qquad \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \left|\det DT(u,v)\right|\,\Delta u\,\Delta v$$

$$= |\det DT(u,v)|\, \Delta u\, \Delta v$$

$$\Delta u \quad \Delta v \longrightarrow 0$$

$$\iint_D dx\,dy = \iint_{D^*} |\det DT(u,v)|\, du\, dv$$

<u>Theorem</u> (C.O.V. in 2d)

Let $T: D^* \to D$ be continuously diff & a bijection. Denote $T(u,v) = (x(u,v), y(u,v))$.

Let $f: D \to \mathbb{R}$ be integrable. Then,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \underbrace{f(\underline{x(u,v)}, \underline{y(u,v)}) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|}_{T?} du\,dv$$

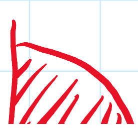
Equivalently,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \underbrace{(f \circ T)(u,v)}_{= f(T(u,v))} |\det DT(u,V)|\, du\,dv$$

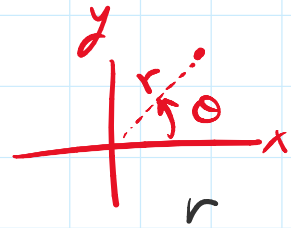
$$\begin{tikzcd} D^* \arrow[r, "T"] \arrow[rr, bend right] & D \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$f \circ T : D^* \to \mathbb{R}$$

ex/ $\displaystyle\iint_D (x^2+y^2)^3\, dy\, dx,$

$D =$ 

$D =$ 



Polar coord.

$$(r,\theta) \longmapsto \Big(\underbrace{x(r,\theta)}_{\substack{\parallel \\ r\cos\theta}},\ \underbrace{y(r,\theta)}_{\substack{\parallel \\ r\sin\theta}}\Big)$$

$$\overbrace{(0,\infty)}^{r} \times [0,2\pi) \;\Big\downarrow\; \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

- $$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$\iint_D (x^2+y^2)^3\, dy\, dx$$

$0 \le r \le 1$
$0 \le \theta \le \pi/2$

$$D^* = \overbrace{[0,1]}^{r} \times \overbrace{[0, \pi/2]}^{\theta}$$

$$= \int_0^{\pi/2}\int_0^1 (r^2\cos^2\theta + r^2\sin^2\theta)^3\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^7\, dr\, d\theta$$

$$= \int_0^{\pi/2} d\theta\ \left.\frac{r^8}{8}\right|_0^1 = \pi/16.$$