Lecture 3 - Maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

- Read section 6.1
- Screen flickering issue on podcast
- Course Piazza for discussing homework (clarification) / course material
- See weekly schedule on page 2 of the course schedule file
- My OH today from 9 am to 10 am (see Zoom LTI PRO on Canvas for link)

domain codomain/range

- A function $f : A \to B$ is a map from a set A to a set B, assigning to each element x in A (denote $x \in A$) an element $f(x) \in B$.

- We call $f(A) = \{ f(x) : x \in A \}$ the image of f. $f(A) \subseteq B$

ex/ $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. Image

$$\{ e^x : x \in \mathbb{R} \} = (0, \infty) \subsetneq \mathbb{R}.$$

In this course, A is usually a subset of $\mathbb{R}^n$ $(n = 1, 2, 3)$ and B is usually a subset of $\mathbb{R}^m$ $(m = 1, 2, 3)$

ex/ $f : \mathbb{R}^n \to \mathbb{R}$ real-valued/scalar function

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ vector field

$\vec{c} : \mathbb{R} \to \mathbb{R}^n$ curve

$\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ surfaces

Properties

$$\begin{array}{ccc} \mathbb{R}^n & & \mathbb{R}^m \\ \cup & & \cup \end{array}$$

Continuity: A function $f : A \to B$ is continuous at $x_0 \in A$ if: $\lim_{x \to x_0} f(x) = f(x_0)$.

$\mathbb{R}^n$ $\mathbb{R}^m$

... at $x_0 \in A$ is: $\lim_{x \to x_0} f(x) = f(x_0)$.

Differentiability: A function $f : \underset{\mathbb{R}^n}{\overset{\cup}{A}} \to \underset{\mathbb{R}^m}{\overset{\cup}{B}}$,

$f = (f_1, f_2, \dots, f_{m-1}, f_m)$, is differentiable at $\vec{x}_0 \in A$ if:

$$\frac{\partial f_j}{\partial x_i} \text{ exists at } \vec{x}_0 \text{ for } \begin{array}{l} j = 1, \dots, m \\ i = 1, \dots, n. \end{array}$$

We define its derivative matrix at $\vec{x}_0 \in A$

$$Df(\vec{x}_0) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{\vec{x}_0} \\ \vdots & & \\ \left.\frac{\partial f_m}{\partial x_1}\right|_{\vec{x}_0} & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_{\vec{x}_0} \end{pmatrix}$$

$n = 2$, $m = 1$

$Df(\vec{x}_0)$

This is an $m \times n$ matrix,

$Df(\vec{x}_0)$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

$$[Df(\vec{x}_0)]_{ij} = \frac{\partial f_i}{\partial x_j}(\vec{x}_0), \quad \begin{array}{l} i = 1, \dots, m \\ j = 1, \dots, n. \end{array}$$

[ex/ $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x$ not invertible "lose information"]

Def:

Say $f : A \to B$ is injective (or one-to-one) if:

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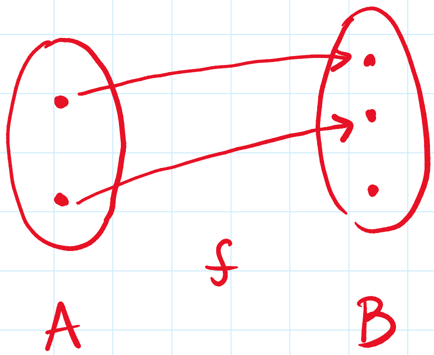
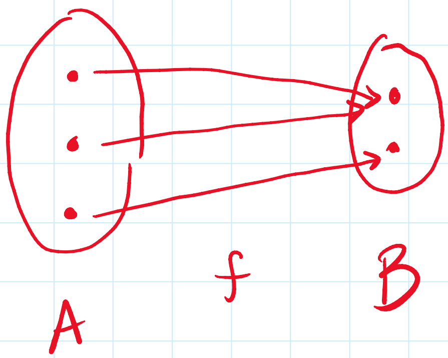
For every $u, v \in A$ such that $u \neq v$, then $f(u) \neq f(v)$

(Equivalently, $f(u) = f(v) \Longrightarrow u = v$)

implies



injective



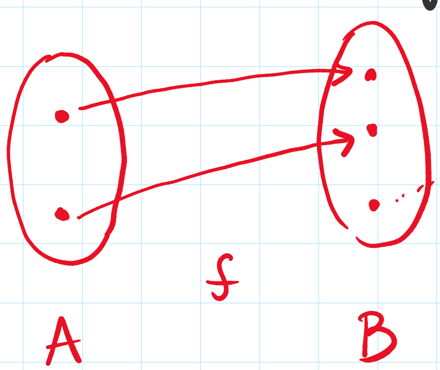
not injective

Def:

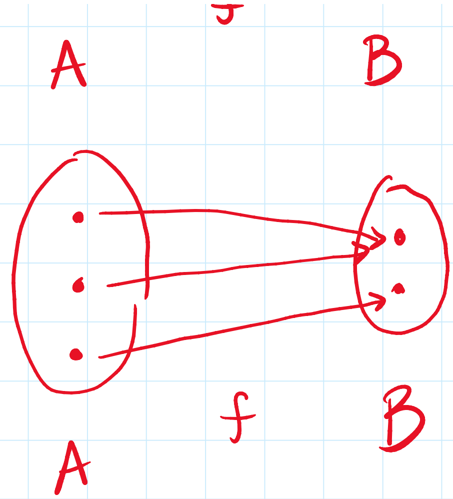
A function $f: A \to B$ is surjective (or onto)

if: the image equals codomain, $f(A) = B$.

- For every $y \in B$, there exists $x \in A$ such that $f(x) = y$.



not surjective



surjective

(aside: domain/range are part of the definition of a function
$f(x)=e^x$, $f:\mathbb{R}\to\mathbb{R}$ not surj.
$f:\mathbb{R}\to(0,\infty)$ surj.

Say a function is bijective if it is both injective & surjective.

Theorem (Invertibility)

If $f: A\to B$ is bijective, then there exists $f^{-1}: B\to A$ s.t. (such that)

$$\left.\begin{array}{ll} f^{-1}(f(x)) = x & \text{for all } x\in A, \\ f(f^{-1}(y)) = y & \text{for all } y\in B. \end{array}\right\} (*)$$

Proof:

Injectivity: there is at most one solution $x\in A$ to the equation $f(x)=y$, for $y\in B$.

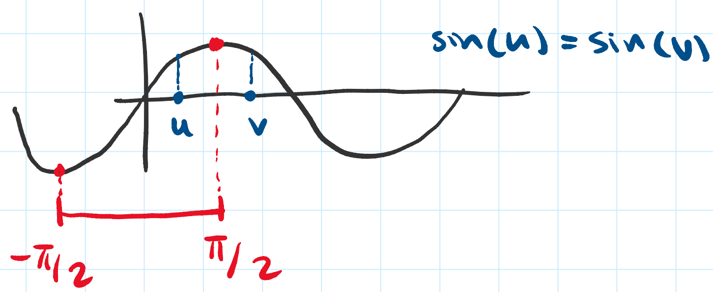
Surjectivity: there is at least one solution $x\in A$ to the eqn $f(x)=y$, for $y\in B$.

$\Rightarrow$ there is exactly one solution $x \in A$ to $f(x) = y$ for each $y \in B$.

We can define $f^{-1}(y) = x$.

Check (*) $\square$

ex/ $\sin(x)$ $\quad \sin : \mathbb{R} \to [-1,1]$



$\sin : [-\pi/2, \pi/2] \longrightarrow [-1,1]$ is a bijection.

$\Rightarrow$ Can define $\sin^{-1}$ or $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$.

ex/ Linear Transformations

- Let T be a linear transf. $T : \mathbb{R}^n \to \mathbb{R}^m$, viewed as an $m \times n$ matrix

$$T = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mn} \end{pmatrix}$$

- Recall two of the fundamental subspaces

$$\ker(T) = \{ x \in \mathbb{R}^n : T(x) = 0 \} \leftarrow \text{injectivity}$$

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$$\text{col}(T) = \text{span}\left\{ \begin{pmatrix} T_{11} \\ \vdots \\ T_{m1} \end{pmatrix}, \dots, \begin{pmatrix} T_{1n} \\ \vdots \\ T_{mn} \end{pmatrix} \right\} \leftarrow \text{surjectivity}$$

<u>claim</u>: T is injective $\overset{\text{if \& only if}}{\iff}$ $\ker(T) = \{0\}$.

<u>proof:</u>

($\Rightarrow$) Assume T is injective,

$$T(x) = T(y) \Rightarrow x = y.$$

Want to show $\ker(T) = \{0\}$.

Let $x \in \ker(T)$.

$$T(x) = \underset{\mathbb{R}^m}{0} = T(\underset{\mathbb{R}^n}{0}) \Rightarrow x = 0.$$

$$\Rightarrow \ker(T) = \{0\}.$$

($\Leftarrow$) $\ker(T) = \{0\}$.

Assume $T(x) = T(y)$

$$\Rightarrow T(x-y) = 0$$

$$\Rightarrow x - y \in \ker(T) = \{0\}$$

$$\Rightarrow x - y = 0 \Rightarrow x = y \Rightarrow T \text{ injective} \quad \square$$

<u>claim</u>: T is surjective $\iff$ $\text{col}(T) = \mathbb{R}^m$

<u>proof:</u> Observe $\text{col}(T) = T(\mathbb{R}^n)$ $\square$

· Consider square $n = m$ $T : \mathbb{R}^n \to \mathbb{R}^n$ linear transf.

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Theorem: A square matrix is injective $\Leftrightarrow$ it's surjective.

proof: Rank-Nullity Theorem

$$\underbrace{\overbrace{\dim(\mathrm{Col}(T))}^{\text{rank}}}_{=n} + \underbrace{\overbrace{\dim(\ker(T))}^{\text{nullity}}}_{=0} = n$$

Recall: a square matrix T is invertible when $\det(T) \neq 0$.