Lecture 29 and 30 - Final Review

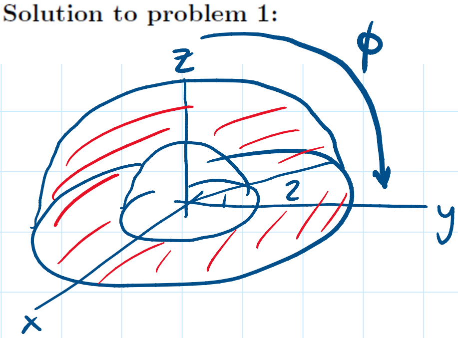
- You do not have to turn in homework 9 (the final review).
- Final Exam on Monday 12/6 at WLH 2001, from 8 to 11 am. You can bring two sheets of hand-written notes (four pages front and back); nothing else is allowed.
- Please fill out the CAPEs; thanks.

Problem 1 (20 points)

Let $W = \{(x, y, z) : 1 \leq x^2 + y^2 + z^2 \leq 4 \text{ and } z \geq 0\}$. Evaluate

$$\iiint_W z^2 \, dV.$$

Solution to problem 1:



Spherical Coordinates

$\rho \in [1,2]$

$\theta \in [0, 2\pi]$ $\qquad z = \rho\cos\phi$

$\phi \in [0, \pi/2]$

$$\iiint_W z^2 \, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_1^2 \rho^2\cos^2\phi \, \underbrace{\rho^2\sin\phi}_{\text{Jacobian}} \, d\rho \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \int_1^2 \rho^4 \, d\rho$$

$$= 2\pi \left(\frac{-\cos^3\phi}{3}\right)\Bigg|_0^{\pi/2} \frac{\rho^5}{5}\Bigg|_1^2$$

$$= 2\pi\left(\frac{1}{3}\right)\frac{2^5-1}{5} = \frac{2\pi(2^5-1)}{15}$$

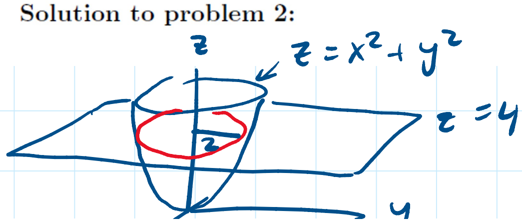
$\square$

Problem 2 (20 points)

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = y^2 + z$. Let C be the curve given by the intersection of the paraboloid $z = x^2 + y^2$ with the plane $z = 4$. Evaluate the path integral

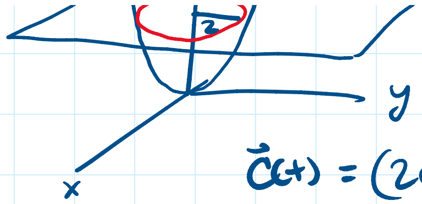
$$\int_C f \, ds.$$

Solution to problem 2:



$x^2 + y^2 = z = 4$

$C: \quad x^2 + y^2 = 4 \qquad z = 4$



$x^2+y^2 = z = 4$

$C: \quad x^2+y^2=4, \; z=4$

$\vec{c}(t) = (2\cos t, 2\sin t, 4) \quad t \in [0, 2\pi]$

$\vec{c}'(t) = (-2\sin t, 2\cos t, 0)$

$\|\vec{c}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$$\int_C f\,ds = \int_0^{2\pi} f(\vec{c}(t))\,\|\vec{c}'(t)\|\,dt$$

$$= \int_0^{2\pi} (4\sin^2 t + 4)\cdot 2\,dt$$

$$= 2\int_0^{2\pi} \left[4\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)\right) + 4\right] dt$$

$$= 8\int_0^{2\pi} \left[\tfrac{3}{2} - \tfrac{1}{2}\cos(2t)\right] dt$$

$$= 8\cdot\tfrac{3}{2}\cdot 2\pi = 24\pi$$

$\square$

Problem 3 (20 points)

Let the surface S be the graph of the function $z = g(x,y) = \sin(y)$ over $(x,y) \in [0,1]\times[0,\pi/2]$. Let $f:\mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = z\sqrt{1-z^2}$. Evaluate

$$\iint_S f\,dS.$$

Solution to problem 3:

$S: \quad z = \sin(y) \quad \text{over} \quad (x,y) \in [0,1]\times[0,\pi/2]$

$\Phi(x,y) = (x, y, \sin(y))$

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}$$

$$= \sqrt{1+\cos^2(y)}$$

$$f(\Phi(x,y)) = \sin y\sqrt{1-\sin^2 y}$$

$$= \sin y\sqrt{\cos^2 y} \qquad y \in [0, \pi/2]$$

$$= \sin y\,|\cos y| = \sin y\cos y$$

$$= \sin y \, |\cos y| \doteq \sin y \cos y$$

$$\iint_S f \, dS = \int_0^1 \int_0^{\pi/2} f(\Phi(x,y)) \, \|\vec{T}_x \times \vec{T}_y\| \, dy \, dx$$

$$= \int_0^1 \int_0^{\pi/2} \sin y \cos y \sqrt{1+\cos^2 y} \, dy \, dx$$

$$= \int_0^{\pi/2} \sin y \cos y \sqrt{1+\cos^2 y} \, dy$$

$u = 1+\cos^2 y$

$du = -2\cos y \sin y \, dy$

$$= \int_2^1 \frac{u^{1/2} \, du}{-2}$$

$$= \frac{1}{2} \int_1^2 u^{1/2} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^2$$

$$= \frac{1}{3}\left(2^{3/2} - 1\right)$$

$\square$

Problem 4 (20 points)

Let the surface $S = \{(x,y,z) : x^2+y^2+z^2 = 1 \text{ and } z \geq 0\}$ with upward normal (in the positive z direction). Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

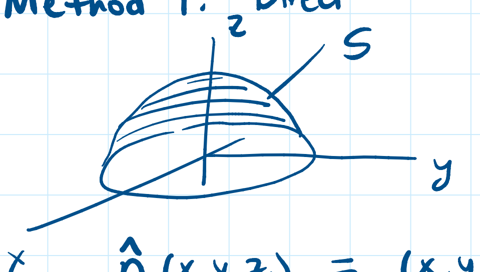
$$\vec{F}(x,y,z) = \Big(x+1,\ y - z\sin(z^2),\ z + y\sin(z^2)\Big).$$

Evaluate the flux of $\vec{F}$ through S; i.e.,

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Solution to problem 4:

Method 1: Direct



$$\hat{n}(x,y,z) = (x,y,\overset{\substack{0 \\ \wedge\mid\mid}}{z})$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS$$

$$= \iint_S \left(x^2 + x + y^2 - \cancel{yz\sin(z^2)} + z^2 + \cancel{yz\sin(z^2)}\right) dS$$

$$= \iint_S \left(x^2 + x + y^2 - \cancel{yz\sin(z^2)} + z^2 + \cancel{yz\sin(z^2)}\right) dS$$

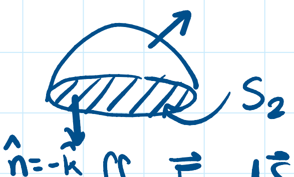
$$= \iint_S (\underbrace{x^2 + y^2 + z^2}_{=\rho^2 = 1} + x)\, dS$$

$$= \iint_S (1 + \cancel{x})\, dS \quad \text{integrates to } 0$$

$$= \iint_S 1\, dS = \text{Area}(S) = 2\pi. \qquad \square$$

Method 2: Divergence Theorem

S is not closed



$S \cup S_2$ is closed, bounds some volume W

$$\iint_S \vec{F}\cdot d\vec{S} = \underbrace{\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}} - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$= \iint_{S\cup S_2} \vec{F}\cdot d\vec{S} - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

div. thm

$$= \iiint_W (\nabla\cdot\vec{F})\, dV - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$= \iiint_W (3 + \cancel{2zy\sin(z^2)})\, dV + \cancel{\iint_{S_2} (\overset{0}{z} + y\sin(\overset{0}{z^2}))\, dS}^{\,0}$$

odd, integrates to 0

$$= 3\,\text{Vol}(W) = 3\cdot\tfrac{2}{3}\pi = 2\pi \qquad \square$$

Problem 5 (20 points)

Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x,y,z) = (2xe^{x^2+y^2}, 2ye^{x^2+y^2}, -2z\cos(z^2))$. Show that $\vec{F}$ is a gradient vector field by computing that $\nabla \times \vec{F} = 0$. Subsequently, find a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

Finally, compute the line integral

$$\int_C \vec{F}\cdot d\vec{r},$$

where C is some curve starting at $(0,0,\sqrt{\pi/2})$ and ending at $(1,0,0)$.

Solution to problem 5:

Compute $\nabla \times \vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xe^{x^2+y^2} & 2ye^{x^2+y^2} & -2z\cos(z^2) \end{vmatrix}$$

$$= \left(0-0, -(0-0), \frac{\partial}{\partial x}(2y\, e^{x^2+y^2}) - \frac{\partial}{\partial y}(2x\, e^{x^2+y^2})\right)$$

$$= (0, 0, 4xye^{x^2+y^2} - 4xye^{x^2+y^2})$$

$$= (0,0,0)$$

$\Rightarrow \nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$ for some f.

$$(F_1, F_2, F_3) = \vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$f(x,y,z) = \int F_1(x,y,z)\, dx = \int 2xe^{x^2+y^2} dx = \underline{e^{x^2+y^2}} + C_1(y,z)$$

$$f(x,y,z) = \int F_2(x,y,z)\, dy = \int 2ye^{x^2+y^2} dy = \underline{e^{x^2+y^2}} + C_2(x,z)$$

$$f(x,y,z) = \int F_3(x,y,z)\, dz = \int -2z\cos(z^2) dz = \underline{\underline{-\sin(z^2)}} + C_3(x,y)$$

$$\Rightarrow f(x,y,z) = e^{x^2+y^2} - \sin(z^2)$$

C starts at $(0,0,\sqrt{\pi/2})$
ends at $(1,0,0)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$\overset{\text{FTLI}}{=} f \Big|_{(1,0,0)}^{(0,0,\sqrt{\pi/2})}$$

$$= f(0,0,\sqrt{\pi/2}) - f(1,0,0)$$

$$= e^0 - \sin(\pi/2) - (e^1 - \cancel{\sin(0)})$$

$$= 1 - 1 - e = -e$$

$\square$

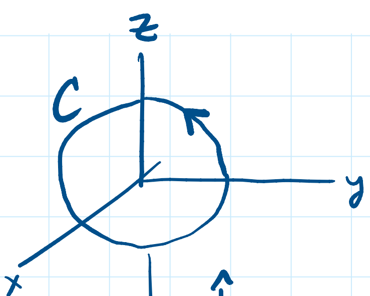
Problem 6 (20 points)

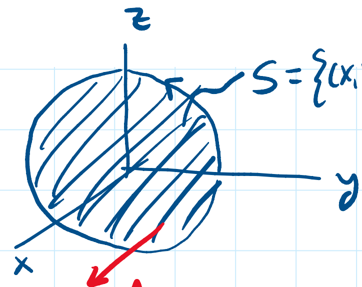
Let C be the closed curve in $\mathbb{R}^3$ given by $C = \{(x,y,z) : y^2+z^2 = 1 \text{ and } x = 0\}$, oriented counterclockwise in the yz plane. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x,y,z) = (\sin(x^4) + z, e^{y^2} - x^2, z^6 + y^3)$.

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Hint: Apply Stokes' theorem; find a surface S whose boundary is C (be careful with the orientation).

Solution to problem 6:





$S = \{(x,y,z) : y^2+z^2 \le 1 \text{ and } x=0\}$

$\partial S = C$

$\hat{n} = +\hat{\imath}$ (positive x direction)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \sin(x^4)+z & e^{y} - x^2 & z^6+y^3 \end{vmatrix}$$

$$= (3y^2 - 0, -(0-1), -2x - 0)$$

$$= (3y^2, 1, -2x)$$

$$\int_C \vec{F}\cdot d\vec{r} \overset{\text{Stokes' Theorem}}{=} \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_S (\nabla\times\vec{F})\cdot \hat{n}\, dS$$

$$= \iint_S (3y^2, 1, -2x)\cdot(1,0,0)\, dS$$

$$= \iint_S 3y^2\, dS$$

$r \in [0,1]$, $\theta \in [0, 2\pi]$

$\Phi(r,\theta) = (0, \underbrace{r\cos\theta}_{y(r,\theta)}, r\sin\theta)$

$\|\vec{T}_r \times \vec{T}_\theta\| = r$

$$= \int_0^{2\pi}\int_0^1 3r^2\cos^2\theta \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^1 3r^3\, dr$$

$$= \int_0^{2\pi} \Big(\frac{1}{2} + \underbrace{\cancel{\frac{1}{2}\cos(2\theta)}}_{\text{integrates to } 0}\Big)\, d\theta \int_0^1 3r^3\, dr$$

integrates to 0

$$= \pi \cdot \frac{3}{4} = \frac{3\pi}{4}$$

$\square$

Problem 7 (20 points)

Let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ be an oriented surface which is the union of oriented surfaces given as follows.

$S_1 = \{(x,y,z) : x^2+y^2 = 1 \text{ and } 0 \le z \le 1\}$, oriented with the radially inward normal (where radial is meant with respect to the xy plane).

$S_2 = \{(x,y,z) : x^2+y^2 = 4 \text{ and } 0 \le z \le 1\}$, oriented with the radially outward normal (where radial is meant with respect to the xy plane).

$S_3 = \{(x,y,z) : 1 \le x^2+y^2 \le 4 \text{ and } z = 1\}$, oriented with the upward normal (in the positive z direction).

$S_4 = \{(x,y,z) : 1 \le x^2+y^2 \le 4 \text{ and } z = 0\}$, oriented with the downward normal (in the negative z direction).

Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

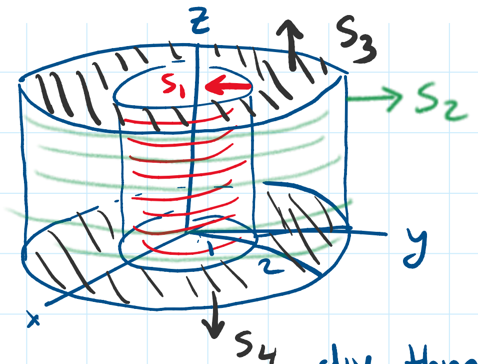
$$\vec{F}(x,y,z) = \left(xy^2 + \cos(z^2)e^{y^2},\ 3z^2e^{x^4} + e^{xz},\ zx^2 + e^{x^4+y^4}\right).$$

Evaluate $\iint_S \vec{F} \cdot d\vec{S}$.

$$\nabla \cdot \vec{F} = y^2 + 0 + x^2 = x^2 + y^2$$

Hint: Apply the divergence theorem; what is the region W whose boundary is S?

Solution to problem 7:



$S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a closed surface, w/ outward normal. Bounds some region W.

$$W = \left\{(x,y,z) : \begin{array}{l} 1 \le x^2+y^2 \le 4 \\ 0 \le z \le 1 \end{array}\right\}$$

In cylindrical coordinates,
$r \in [1,2]$
$\theta \in [0, 2\pi]$
$z \in [0,1]$

$$\iint_S \vec{F} \cdot d\vec{S} \overset{\text{div. thm}}{=} \iiint_W (\nabla \cdot \vec{F})\, dV$$

$$= \iiint_W (x^2+y^2)\, dV$$

$$\overset{\text{c.o.v.}}{=} \int_0^1 \int_0^{2\pi} \int_1^2 (r^2)\, \underbrace{r}_{\text{Jacobian}}\, dr\, d\theta\, dz$$

$$= 2\pi \int_1^2 r^3\, dr$$

$$= 2\pi \left.\frac{r^4}{4}\right|_1^2 = 2\pi\left(\frac{16-1}{4}\right) = \frac{30\pi}{4}$$

$$= \frac{15\pi}{2}$$

$= \frac{15\pi}{2}$

$\square$

Problem 8 (Extra Credit 20 points)

Prove the following:

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable scalar function and let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field. Let W be a region in $\mathbb{R}^3$ and let ∂W be its boundary (with the induced orientation). Then,

$$\iint_{\partial W} (\vec{F} \times \nabla f) \cdot d\vec{S} = \iiint_W \nabla f \cdot (\nabla \times \vec{F})\, dV$$

Hint: Start with the left hand side of the above equation and apply the divergence theorem. You may use the following product rule for the divergence of a cross product of two vector fields $\vec{G}$ and $\vec{H}$:

$$\nabla \cdot (\vec{G} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{G}) - \vec{G} \cdot (\nabla \times \vec{H}).$$

Solution to problem 8 extra credit:

$$\iint_{\partial W} (\vec{F} \times \nabla f) \cdot d\vec{S} \qquad \vec{G} \quad \vec{H}$$

$$\overset{\text{divergence theorem}}{=} \iiint_W \left(\nabla \cdot (\vec{F} \times \nabla f)\right) dV$$

$$\overset{\text{product rule}}{=} \iiint_W \left(\nabla f \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\underbrace{\nabla \times \nabla f}_{=0})\right) dV$$

$$= \iiint_W \left(\nabla f \cdot (\nabla \times \vec{F})\right) dV$$

$\square$