Lecture 28 - Differential Forms cont.

- This is the final in-person lecture. The next two lectures will be asynchronous. Find the video posted in the Media Gallery on Canvas and the lecture notes under the Lecture Notes module.
  - I already recorded the final review lecture; it is posted as a single video.
- The final is next Monday (12/6) in-person from 8 am to 11 am, WLH 2001. You can bring two sheets of hand-written notes (four pages front and back).

$$
\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, dash, dotted] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, dash, dotted] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, dash, dotted] & \Omega^3(\mathbb{R}^3) \arrow[d, dash, dotted] \\
\text{points} & \text{curves} \arrow[l, "\partial"'] & \text{surfaces} \arrow[l, "\partial"'] & \text{volumes} \arrow[l, "\partial"']
\end{tikzcd}
$$

The exterior derivative

$$d : \Omega^k(\mathbb{R}^3) \longrightarrow \Omega^{k+1}(\mathbb{R}^3)$$

For 0-forms, $\Omega^0(\mathbb{R}^3) \ni f : \mathbb{R}^3 \to \mathbb{R}$,
df is defined to be the differential of $f(x,y,z)$,

$$df = \underline{\frac{\partial f}{\partial x}}\, dx + \underline{\frac{\partial f}{\partial y}}\, dy + \underline{\frac{\partial f}{\partial z}}\, dz \in \Omega^1(\mathbb{R}^3)$$

$df \sim \nabla f$

For k-forms, when we expand the k-form in the coordinate basis with functions as the expansion coefficients, then the exterior derivative of that k-form is given by taking the exterior derivative of those functions and wedge it into the associated basis.

For 1-forms,

$$\alpha = \alpha_1\, dx + \alpha_2\, dy + \alpha_3\, dz \in \Omega^1(\mathbb{R}^3)$$
$$d\alpha = d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz \in \Omega^2(\mathbb{R}^3)$$

For 2-forms,

$d\alpha = d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz \in \Omega^2(\mathbb{R}^3)$

For 2-forms,

$$\beta = \beta_1\, dy\wedge dz + \beta_2\, dz\wedge dx + \beta_3\, dx\wedge dy \in \Omega^2(\mathbb{R}^3)$$

$$d\beta = d\beta_1\wedge dy\wedge dz + d\beta_2\wedge dz\wedge dx + d\beta_3\wedge dx\wedge dy \in \Omega^3(\mathbb{R}^3)$$

More explicitly,

$\alpha \in \Omega^1(\mathbb{R}^3)$

$$\begin{aligned}
d\alpha &= d\alpha_1\wedge dx + d\alpha_2\wedge dy + d\alpha_3\wedge dz \\
&= \left(\frac{\partial\alpha_1}{\partial x}dx + \frac{\partial\alpha_1}{\partial y}dy + \frac{\partial\alpha_1}{\partial z}dz\right)\wedge dx \\
&\quad + \left(\frac{\partial\alpha_2}{\partial x}dx + \frac{\partial\alpha_2}{\partial y}dy + \frac{\partial\alpha_2}{\partial z}dz\right)\wedge dy \\
&\quad + \left(\frac{\partial\alpha_3}{\partial x}dx + \frac{\partial\alpha_3}{\partial y}dy + \frac{\partial\alpha_3}{\partial z}dz\right)\wedge dz \\
&= \cancel{\frac{\partial\alpha_1}{\partial x}dx\wedge dx} + \frac{\partial\alpha_1}{\partial y}\,{}^{3}\underbrace{dy\wedge dx}_{-dx\wedge dy} + \frac{\partial\alpha_1}{\partial z}\,{}^{2}\underline{dz\wedge dx} \\
&\quad + \frac{\partial\alpha_2}{\partial x}\,{}^{3}\underline{dx\wedge dy} + \cancel{\frac{\partial\alpha_2}{\partial y}dy\wedge dy} + \frac{\partial\alpha_2}{\partial z}\,{}^{1}\underline{dz\wedge dy} = -dy\wedge dz \\
&\quad + \frac{\partial\alpha_3}{\partial x}\,{}^{2}\underbrace{dx\wedge dz}_{-dz\wedge dx} + \frac{\partial\alpha_3}{\partial y}\,{}^{1}\underline{dy\wedge dz} + \cancel{\frac{\partial\alpha_3}{\partial z}dz\wedge dz} \\
&= {}^{1}\left(\frac{\partial\alpha_3}{\partial y} - \frac{\partial\alpha_2}{\partial z}\right)dy\wedge dz \\
&\quad {}^{2}+ \left(\frac{\partial\alpha_1}{\partial z} - \frac{\partial\alpha_3}{\partial x}\right)dz\wedge dx \\
&\quad {}^{3}+ \left(\frac{\partial\alpha_2}{\partial x} - \frac{\partial\alpha_1}{\partial y}\right)dx\wedge dy
\end{aligned}$$

$\alpha \sim \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$

under i.d. of 2-forms $\sim$ v.f.,

$d\alpha \sim \nabla\times\vec{\alpha}$

$\beta \in \Omega^2(\mathbb{R}^3)$

$$\beta = \beta_1\, dy\wedge dz + \beta_2\, dz\wedge dx + \beta_3\, dx\wedge dy$$

$$d\beta = d\beta_1\wedge dy\wedge dz + d\beta_2\wedge dz\wedge dx + d\beta_3\wedge dx\wedge dy$$

$$= \left(\frac{\partial\beta_1}{\partial x}dx + \frac{\partial\beta_1}{\partial y}\cancel{dy} + \frac{\partial\beta_1}{\partial z}\cancel{dz}\right)\wedge dy\wedge dz \qquad dy\wedge dy = 0$$

$$= \left(\frac{\partial \beta_1}{\partial x} dx + \frac{\partial \beta_1}{\partial y} \cancel{dy} + \frac{\partial \beta_1}{\partial z} \cancel{dz}\right) \wedge dy \wedge dz$$

$$+ \left(\frac{\partial \beta_2}{\partial x} \cancel{dx} + \frac{\partial \beta_2}{\partial y} dy + \frac{\partial \beta_2}{\partial z} \cancel{dz}\right) \wedge dz \wedge dx$$

$$+ \left(\frac{\partial \beta_3}{\partial x} \cancel{dx} + \frac{\partial \beta_3}{\partial y} \cancel{dy} + \frac{\partial \beta_3}{\partial z} dz\right) \wedge dx \wedge dy$$

$$= \frac{\partial \beta_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial \beta_2}{\partial y} dy \wedge dz \wedge dx$$

$$+ \frac{\partial \beta_3}{\partial z} dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_2}{\partial y} + \frac{\partial \beta_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$\beta \sim \vec{\beta} = (\beta_1, \beta_2, \beta_3) \qquad d\beta \sim \nabla \cdot \vec{\beta}$$

Recall:

Given a 0-dim. region P (points). and a 0-form f,

$$\int_P f = f(P)$$

$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$

Given a 1-dim region C (curve) and a 1-form α,

$$\int_C \alpha = \int_C \vec{\alpha} \cdot d\vec{r}$$

$\vec{\beta} = (\beta_1, \beta_2, \beta_3)$

Given a 2-dim region S (surface) and a 2-form β,

$$\int_S \beta = \iint_S \vec{\beta} \cdot d\vec{S}$$

Given a 3-dim region W (volume) and a 3-form $f\, dx \wedge dy \wedge dz$

$$\int_W f\, dx \wedge dy \wedge dz = \iiint_W f\, dx\, dy\, dz$$

THE FUNDAMENTAL THEOREM OF CALCULUS
(The Generalized Stokes' Theorem)

- Given a k-form α and a $(k+1)$-dimensional region M, then

exterior derivative

$$\boxed{\int_M d\alpha = \int_{\partial M} \alpha}$$

Boundary operator

(in $\mathbb{R}^3$)

case $k=0 \Rightarrow$ FTLI

case $k=1 \Rightarrow$ Stokes' Theorem

case $k=2 \Rightarrow$ Divergence Theorem