Lecture 25 - The Divergence Theorem, cont.

- The Final will take place on Monday of Finals Week (12/6) from 8 am to 11 am in this lecture hall, WLH 2001.
  - It will cover ALL of the material discussed in the course (excluding the final lectures on differential forms), the relevant sections in the textbook (see the course schedule), and HW1-9.
  - There will be 7 or 8 normal problems (20 points each) and 1 extra credit problem (20 bonus points). The breakdown in terms of content will be roughly 2 problems on the material in MT1, 2 problems on the material in MT2, and 3 to 4 problems on the material that came after the midterms.
  - I will post a practice final sometime at the end of week 9
  - The final is in-person. You may bring two 8.5 x 11 sheets of hand-written notes (so 4 pages front and back). You may not use any other resources; e.g., calculators, textbook, phones, etc.
- The course and professor evaluations (CAPEs) begin next Monday and are available to be filled out for 2 weeks (ends on Monday of Finals week). Please fill these out; the feedback really helps me as I'm a relatively new instructor and I want to improve as much as possible! Be honest on your CAPEs; I don't get to view them until after final grades are submitted and they are anonymous anyway.
  - The only thing that I can view beforehand is the percent of students that have completed CAPEs for this course. If at least 60% of the class fills out CAPEs, I'll give the whole class 1% extra credit (again, any extra credit in this class is applied after any curve)
- HW8 is now posted, due Wednesday 11/24 at 11:59 pm.

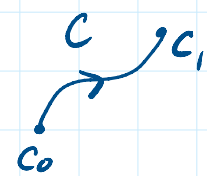
Fundamental Theorems of Calculus

FTC I

$$\int_{[a,b]} \frac{df}{dx}\,dx = f\Big|_{\partial([a,b])} \equiv f(b) - f(a)$$

FTLI

$$\int_C \nabla f \cdot d\vec{r} = f\Big|_{\partial C} \equiv f(C_1) - f(C_0)$$



Stokes' Thm

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

Divergence Thm

$$\iiint_W (\nabla \cdot \vec{F})\,dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$$

$$\text{scalar functions} \xrightarrow{\nabla} \text{vector fields} \xrightarrow{\nabla \times} \text{vector fields} \xrightarrow{\nabla \cdot} \text{scalar functions}$$

functions $\longrightarrow$ fields $\longrightarrow$ fields $\longrightarrow$ functions

$$\text{points} \xleftarrow{\partial} \text{curves} \xleftarrow{\partial} \text{surfaces} \xleftarrow{\partial} \text{volumes}$$

$$f\big|_{\partial C} = \int_C \nabla f \cdot d\vec{r} \qquad \int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot d\vec{S} \qquad \iint_{\partial W} \vec{G}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{G})\, dV$$

<u>de Rham complex on $\mathbb{R}^3$</u>

ex/ HW5 Problem 8 (Ex. 7.6.13)

$$\vec{V}(x,y,z) = (3xy^2,\ 3x^2y,\ z^3)$$

$$\iint_{S^2} \vec{V}\cdot d\vec{S}, \quad S^2 = \text{unit sphere outward normal}$$

$$B = \{x^2+y^2+z^2 \le 1\} \qquad \partial B = S^2.$$

Apply divergence theorem

$$\nabla\cdot\vec{V} = 3y^2 + 3x^2 + 3z^2$$

$$\iint_S \vec{V}\cdot d\vec{S} = \iiint_{B \leftarrow \text{ball of radius } 1} (\nabla\cdot\vec{V})\, dV = \iiint_B 3(x^2+y^2+z^2)\, dV$$

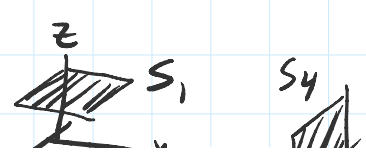
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 3\rho^2 \cdot \underbrace{\rho^2 \sin\phi}_{\text{Jacobian}}\, d\rho\, d\phi\, d\theta$$

$$= 4\pi \int_0^1 3\rho^4\, d\rho = \frac{12\pi}{5}.$$

ex/

Let $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$

$S_1 = [0,1]\times[0,1]\times\{z=1\}$



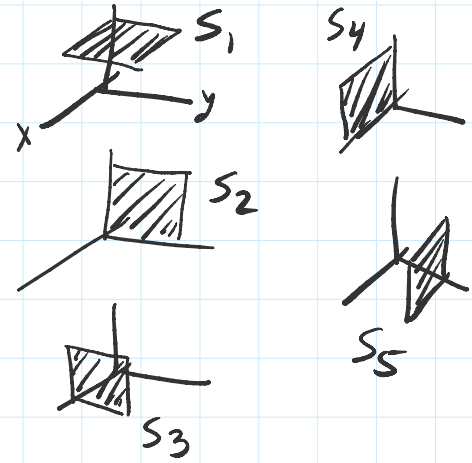
$S_1 = [0,1]\times[0,1]\times\{z=1\}$

$S_2 = \{x=0\}\times[0,1]\times[0,1]$

$S_3 = \{x=1\}\times[0,1]\times[0,1]$

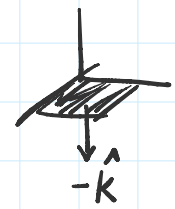
$S_4 = [0,1]\times\{y=0\}\times[0,1]$

$S_5 = [0,1]\times\{y=1\}\times[0,1]$



$\vec{F}(x,y,z) = (x,y,z)$

Let $S_6 = [0,1]\times[0,1]\times\{z=0\}$



$$\iint_S \vec{F}\cdot d\vec{S}$$

$$= \underbrace{\iint_S \vec{F}\cdot d\vec{S} + \iint_{S_6} \vec{F}\cdot d\vec{S}} - \iint_{S_6} \vec{F}\cdot d\vec{S}$$

$W = [0,1]\times[0,1]\times[0,1]$

$$= \iint_{S\cup S_6} \vec{F}\cdot d\vec{S} - \iint_{S_6} \vec{F}\cdot d\vec{S}$$

$$= \iint_{\partial W} \vec{F}\cdot d\vec{S} - \iint_{S_6} \vec{F}\cdot d\vec{S}$$

$$= \iiint_W (\nabla\cdot\vec{F})\,dV - \iint_{S_6} \vec{F}\cdot d\vec{S} \qquad \vec{F}\cdot\hat{n} = -z = 0$$

div. thm.

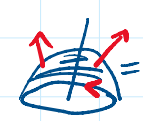
$$= 3\,\mathrm{Vol}(W) - 0$$

$$= 3.$$

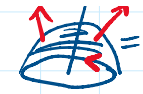
<u>Theorem</u>:

Let $\vec{F}$ be a C^1 vector field on $\mathbb{R}^3$ s.t. $\nabla\cdot\vec{F} = 0$. Then, for any two surfaces S_1 and S_2 s.t. $\underline{\partial S_1 = \partial S_2}$,

$$\iint \vec{F}\cdot d\vec{S} - \iint \vec{F}\cdot d\vec{S}$$

/ e.g. northern hemi  $= S_1$)

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot d\vec{S}$$

(e.g. northern hemi  $= S_1$

unit disk in $z=0$  $= S_2$)

proof:

$S_1 \cup S_2^-$ is a closed surface, bounds a region W s.t. $\partial W = S_1 \cup S_2^-$

$$\iint_{S_1 \cup S_2^-} \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\, dV = 0$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2^-} \vec{F}\cdot d\vec{S}$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$\square$

$$\left(\begin{array}{l} \nabla\cdot\vec{F} = 0 \Rightarrow \vec{F} = \nabla\times\vec{G} \\ \iint_{S_1} (\nabla\times\vec{G})\cdot d\vec{S} = \int_{\partial S_1} \vec{G}\cdot d\vec{r} \end{array} \right)$$

Application:

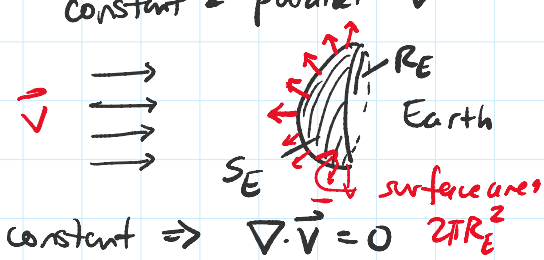
$\frac{\text{energy}}{\text{time}}$ aka power radiated onto Earth's surface

$$\iint_{S_E} \vec{V}\cdot d\vec{S}$$

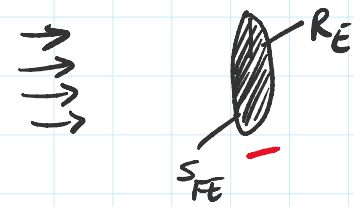
$$= \iint \vec{V}\cdot d\vec{S} = \|\vec{V}\|\, \pi R_E^2 .$$

$\frac{\text{energy}}{\text{time}\cdot\text{area}}$

constant & parallel $\vec{V}$



$\vec{V}$ constant $\Rightarrow \nabla\cdot\vec{V} = 0$



$$= \iint_{S_{FE}} \vec{v} \cdot d\vec{S} = \|\vec{v}\| \, \pi R_E^2 .$$