Lecture 24 - The Divergence Theorem

- I have my usual OH this week (right after this lecture and Thursday from 11 to 12)
- HW7 is due Thursday 11/18 at 11:59 pm
- Read section 8.4 (the divergence theorem)

Midterm 2 Solutions:

Problem 1 (20 points)

Let the surface S be the graph of the function $z = g(x, y) = 1 - x^2$ over $(x, y) \in [0, 1] \times [0, 1]$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xy$. Evaluate

$$\iint_S f\,dS.$$

$$\vec{T}_x = \frac{\partial \Phi}{\partial x} = (1, 0, \partial g/\partial x)$$
$$\vec{T}_y = \frac{\partial \Phi}{\partial y} = (0, 1, \partial g/\partial y)$$
$$\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

Solution. Parametrize S via the usual parametrization for a graph, $\Phi : D \to S$,

$$\Phi(x, y) = (x, y, g(x, y)) = (x, y, 1 - x^2), \ \ D = [0, 1] \times [0, 1].$$

We need $f(\Phi(x, y)) = xy$ and

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2} = \sqrt{1 + 4x^2}.$$

graphs

Then,

$$\iint_S f\,dS = \iint_D f(\Phi(x, y))\|\vec{T}_x \times \vec{T}_y\|\,dxdy = \int_0^1 \int_0^1 xy\sqrt{1 + 4x^2}\,dxdy$$
$$= \int_0^1 ydy \int_0^1 x\sqrt{1 + 4x^2}\,dx = \frac{1}{2}\int_0^1 x\sqrt{1 + 4x^2}\,dx$$
$$= \frac{1}{16}\int_1^5 u^{1/2}du = \frac{1}{16}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^5 = \frac{1}{24}(5^{3/2} - 1),$$

where we used the substitution $u = 1 + 4x^2, du = 8xdx$ to do the x-integral. □

Problem 2 (20 points)

Let S be the closed surface which is the union of two surfaces, $S = S_1 \cup S_2$, where S_1 is the upper half of the unit sphere,

$$S_1 = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\},$$

and S_2 is the unit disk in the plane $z = 0$,

$$S_2 = \{(x, y, z) : x^2 + y^2 \leq 1 \text{ and } z = 0\}.$$

Let S be oriented with the outward normal. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x, y, z) = (0, 0, z + 1)$. Evaluate

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Solution. Split the surface integral into the sum of the two pieces,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}.$$

Since S is oriented with the outward normal, S_1 is oriented with the upward normal (in the positive z direction) and S_2 is oriented with the downward normal (in the negative z direction).

Let's calculate the surface integral for S_1 first. Parametrize S_1 via the usual spherical coordinate surface parametrization,

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi),$$

where $(\theta, \phi) \in D = [0, 2\pi] \times [0, \pi/2]$ (here, ϕ ranges from 0 to $\pi/2$ since we only want the upper half of the sphere). The normal vector field is

$$\vec{T}_\phi \times \vec{T}_\theta = (\cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \sin\phi \cos\phi)$$

(we take $\vec{T}_\phi \times \vec{T}_\theta$ and not $\vec{T}_\theta \times \vec{T}_\phi$ because we want the upward pointing normal; see HW 6 Problem 1).

Solution. Split the surface integral into the sum of the two pieces,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}.$$

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$$\vec{T}_\phi \times \vec{T}_\theta = (\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi)$$

(we take $\vec{T}_\phi\times\vec{T}_\theta$ and not $\vec{T}_\theta\times\vec{T}_\phi$ because we want the upward pointing normal; see HW 6 Problem 1). The vector field $\vec{F}$ on S_1 is

$$\vec{F}(\Phi(\theta,\phi)) = (0,0,\cos\phi+1).$$

Hence,

$$\begin{aligned}\iint_{S_1}\vec{F}\cdot d\vec{S} &= \iint_D \vec{F}(\Phi(\theta,\phi))\cdot(\vec{T}_\phi\times\vec{T}_\theta)\,d\theta d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi}(\cos\phi+1)\sin\phi\cos\phi\,d\theta d\phi\\ &= 2\pi\int_0^{\pi/2}(\sin\phi\cos^2\phi + \sin\phi\cos\phi)d\phi\\ &= 2\pi\Big(\frac{-\cos^3\phi}{3} - \frac{\cos^2\phi}{2}\Big)\Big|_0^{\pi/2} = \frac{5}{3}\pi.\end{aligned}$$

Now, let's calculate the surface integral for S_2. You could parametrize it directly, but I will use the geometric formula for the surface integral. Observe that S_2 is contained in the plane $z=0$, so its normal vector is just $\hat{n} = -\hat{k} = (0,0,-1)$ (pointing downward because S is oriented with the outward normal). Then along the surface, $\vec{F}\cdot\hat{n} = -(z+1) = -1$, where we used that $z=0$ on S_2. Using the geometric formula for surface integrals,

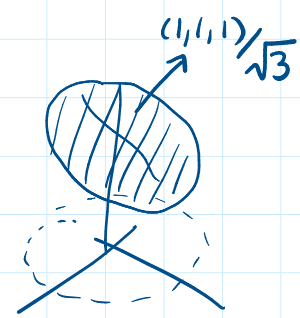
$$\iint_{S_2}\vec{F}\cdot d\vec{S} = \iint_{S_2}\vec{F}\cdot\hat{n}\,dS = \iint_{S_2}(-1)dS = -\text{Area}(S_2) = -\pi.$$

Adding these two results together,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1}\vec{F}\cdot d\vec{S} + \iint_{S_2}\vec{F}\cdot d\vec{S} = \frac{5}{3}\pi - \pi = \frac{2}{3}\pi.$$

In the coming lectures, we will see a much easier way to do this problem using the divergence theorem. □

Problem 3 (20 points)



Let S be the part of the plane $z = 2-x-y$ over $x^2+y^2\le 1$, equipped with the upward pointing normal (i.e., in the positive z direction). Let $\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ be given by $\vec{F}(x,y,z) = (x,2x,-3x)$. Evaluate

$$\iint_S \vec{F}\cdot d\vec{S}.$$

Hint: You can evaluate this directly by parametrizing S as the graph of a function of (x,y), but it is easier to use the geometric formula for the surface integral: what is the unit normal vector field to S and what is its dot product with $\vec{F}$?

$\Phi(x,y) = (x,y,2-x-y)$

$(x,y)\in D = \{x^2+y^2\le 1\}$

Solution. S is contained in the plane $z = 2-x-y$, which can be written as $x+y+z=2$. From the equation for a plane $\vec{n}\cdot(x,y,z) = c$, this means that a normal vector to S is $(1,1,1)$; so the unit normal vector field along S is $\hat{n} = (1,1,1)/\sqrt{3}$ (the $1/\sqrt{3}$ factor is just so $\hat{n}$ has unit length). Observe that

$$\vec{F}\cdot\hat{n} = (x,2x,-3x)\cdot\frac{(1,1,1)}{\sqrt{3}} = \frac{(x+2x-3x)}{\sqrt{3}} = 0.$$

Hence, using the geometric formula for the surface integral,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}dS = \iint_S 0\,dS = 0.$$

□

$\Phi(x,y) = (x, y, 2-x-y)$
$(x,y) \in D = \{x^2+y^2 \leq 1\}$

Solution. S is contained in the plane $z = 2 - x - y$, which can be written as $\underline{x + y + z} = 2$. From the equation for a plane $\vec{n} \cdot (x, y, z) = c$, this means that a normal vector to S is $(1, 1, 1)$; so the unit normal vector field along S is $\hat{n} = (1, 1, 1)/\sqrt{3}$ (the $1/\sqrt{3}$ factor is just so $\hat{n}$ has unit length). Observe that

$$\vec{F} \cdot \hat{n} = (x, 2x, -3x) \cdot \frac{(1,1,1)}{\sqrt{3}} = \frac{(x + 2x - 3x)}{\sqrt{3}} = 0.$$

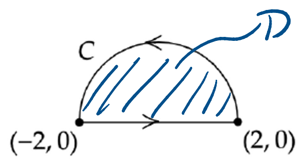
Hence, using the geometric formula for the surface integral,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS = \iint_S 0\, dS = 0.$$

□

Problem 4 (20 points)

Let C be the following closed curve in the xy plane: C connects the point $(2, 0)$ to the point $(-2, 0)$ along the upper half of the circle $x^2 + y^2 = 4$, and then travels from $(-2, 0)$ back to $(2, 0)$ along the x-axis. This is shown in the figure below.



Let $P(x, y) = y + e^{x^2}, Q(x, y) = \sin(y^2) - x$. Evaluate the line integral

$$\int_C P dx + Q dy.$$

Hint: Use Green's theorem.

Solution. Let D be the region contained inside the closed curve C (i.e., $\partial D = C$). Since C is oriented counterclockwise, Green's theorem gives

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy = \iint_D (-1 - 1) dx dy = -2 \iint_D dx dy = -2\text{Area}(D) = -4\pi,$$

where we used that the area of a half-disk of radius R is $\pi R^2/2$; here, the radius $R = 2$ so the area is $\text{Area}(D) = \pi(2)^2/2 = 2\pi$. □

Problem 5 (Extra Credit: 10 points)

Let S be an oriented (regular) surface with unit normal vector field $\hat{n}$. Let $\vec{F}$ be a (continuous) vector field defined over S. Prove the following two statements:

(a) If $\vec{F}$ is orthogonal to $\hat{n}$ along S, then

$$\iint_S \vec{F} \cdot d\vec{S} = 0.$$

Solution. $\vec{F}$ is orthogonal to $\hat{n}$ along S; i.e., $\vec{F} \cdot \hat{n} = 0$ at all points on S. Hence, by the geometric formula for surface integrals,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \overset{0}{\overset{\|}{\underline{\vec{F} \cdot \hat{n}}}}\, dS = \iint_S 0\, dS = 0.$$

An example of where this theorem applies is Problem 3 above. □

(b) If $\vec{F}$ is parallel to $\hat{n}$ along S (that is, they point in the same direction), then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \|\vec{F}\|\, dS.$$

Solution. $\vec{F}$ is parallel to $\hat{n}$ along S; i.e., the angle θ between them is zero at all points on S. Hence

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \|\vec{F}\|\, dS.$$

Solution. $\vec{F}$ is parallel to $\hat{n}$ along S; i.e., the angle θ between them is zero at all points on S. Hence, along S, we have

$$\vec{F}\cdot\hat{n} \overset{\text{in general}}{=} \|\vec{F}\|\|\hat{n}\|\cos\theta \overset{\text{by our assumptions}}{=} \|\vec{F}\|$$

(since the unit normal vector field has magnitude one $\|\hat{n}\| = 1$ and $\cos\theta = \cos(0) = 1$). Then, by the geometric formula for surface integrals,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \|\vec{F}\|\, dS.$$

□

Remark. You proved a similar result for line integrals in HW3 Problem 9, where instead of the normal vector field which you need for surface integrals, you consider the velocity vector of the curve $\vec{c}\,'$.

Properties of the divergence:

(i) Linearity:

$$\nabla\cdot(a\vec{F} + b\vec{G}) = a\,\nabla\cdot\vec{F} + b\,\nabla\cdot\vec{G}$$

$a, b \in \mathbb{R}$
$\vec{F}, \vec{G}$ differentiable vector fields

(ii) Product rule

dot prod.

$$\nabla\cdot(f\vec{F}) = f\,\nabla\cdot\vec{F} + (\nabla f)\cdot\vec{F}$$

f diff. scalar function
$\vec{F}$ diff. vector field

proof: HW8

(iii) Curls have zero divergence

$$\nabla\cdot(\nabla\times\vec{F}) = 0$$ for any C^2 vector field $\vec{F}$.

proof:

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla\cdot(\nabla\times\vec{F}) = \frac{\partial}{\partial x}(\quad) + \frac{\partial}{\partial y}(\quad) + \frac{\partial}{\partial z}(\quad)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x\partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y\partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y\partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z\partial y}}$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}}$$

$$= 0.$$

$\square$

Thm

If $\vec{G}$ is a C^1 vector field, then

$$\vec{G} = \nabla \times \vec{F} \iff \nabla \cdot \vec{G} = 0$$

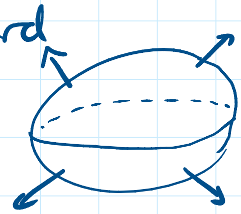
$$\begin{tikzcd}
\text{scalar functions} \arrow[r, "\nabla"] \arrow[d, dash, dotted] & \text{vector fields} \arrow[r, "\nabla\times"] \arrow[d, dash, dotted] & \text{vector fields} \arrow[r, "\nabla\cdot"] \arrow[d, dash, dotted] & \text{scalar functions} \arrow[d, dash, dotted] \\
\text{points} & \text{curve} \arrow[l, "\partial"'] & \text{surface} \arrow[l, "\partial"'] & \text{volumes} \arrow[l, "\partial"']
\end{tikzcd}$$

[The Divergence Theorem]

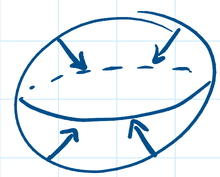
- A closed surface S is a surface whose boundary is empty, $\partial S = \underbrace{\phi}_{\text{empty set}}$

  Given a closed surface, there are 2 orientations:

  outward

  e.g. 

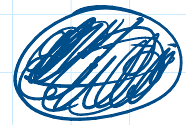
  inward

  e.g. 

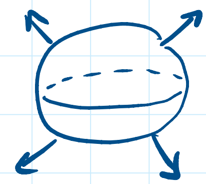
- Let $W \subset \mathbb{R}^3$ be some volume (in 3d space). Then, ∂W is a closed surface, equipped with the outward normal (induced orientation).

with the outward normal (induced orientation).

$$B_1 = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$$



$$S_1 = \partial B_1 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$



<u>Theorem</u> (Gauss' Divergence Theorem)

Let $W \subset \mathbb{R}^3$ be a 3d-region and let ∂W denote its boundary w/ the induced orientation (outward normal).
Let $\vec{F}$ be a C^1 vector field on W, $\vec{F} : W \to \mathbb{R}^3$.
Then,

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W (\nabla \cdot \vec{F})\, dV$$

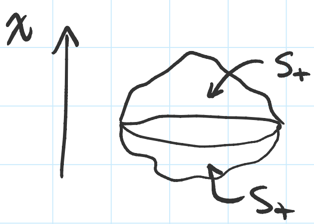
<u>proof sketch</u>: (see textbook)

· start w/ elementary region (x & y & z-simple)

$$\iiint_W (\nabla \cdot \vec{F})\, dV$$

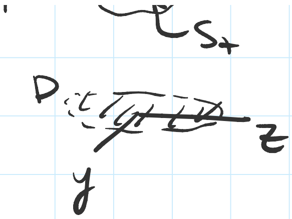
$$= \iiint_W \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

$$= \iiint_W \underbrace{\frac{\partial F_1}{\partial x}\, dx}_{\text{FTC I}}\, dy\, dz + \underset{dy \text{ first}}{\cdots} + \underset{dz \text{ first}}{\cdots}$$



$$\iint_D \left(F_1(S_+) - F_1(S_-) \right) dy\, dz$$

$$= \iint \vec{F}_1 \cdot d\vec{S} \qquad \vec{F}_1 = (F_1, 0, 0)$$



D

$$\cdots = \iint_{\partial W} \vec{F}_1 \cdot d\vec{S} \qquad \vec{F}_1 = (F_1, 0, 0)$$

$\square$

ex/ **Problem 2** (20 points)

Let S be the closed surface which is the union of two surfaces, $S = S_1 \cup S_2$, where S_1 is the upper half of the unit sphere,

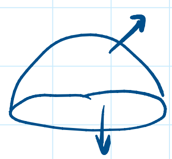
$$S_1 = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\},$$

and S_2 is the unit disk in the plane $z = 0$,

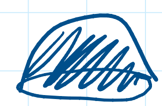
$$S_2 = \{(x, y, z) : x^2 + y^2 \leq 1 \text{ and } z = 0\}.$$

Let S be oriented with the outward normal. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x, y, z) = (0, 0, z + 1)$. Evaluate

$$\iint_S \vec{F} \cdot d\vec{S}. \qquad \nabla \cdot \vec{F} = \frac{\partial}{\partial z}(z+1) = 1$$



S is closed

W 

$W: \quad (x,y,z) \quad x^2+y^2 \leq 1 \quad \text{and} \quad 0 \leq z \leq \sqrt{1-x^2-y^2}$

$\partial W = S$

Divergence theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W (\nabla \cdot \vec{F})\, dV$$

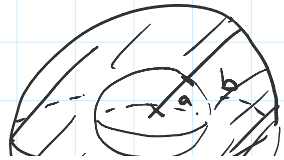
$$= \iiint_W 1\, dV = \text{Vol}(W) = \frac{2}{3}\pi. \qquad \square$$

ex/ Consider

$$W = \{(x,y,z) : a^2 \leq x^2+y^2+z^2 \leq b^2\} \qquad 0<a<b$$

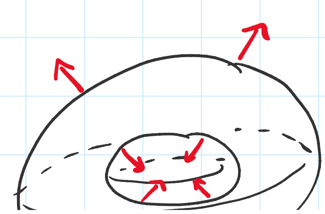
- Let $\vec{F}(x,y,z) = \left(x^3/3,\ y^3/3,\ z^3/3\right)$.
- Evaluate

$$\iint \vec{F} \cdot d\vec{S}.$$

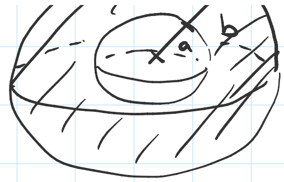


W

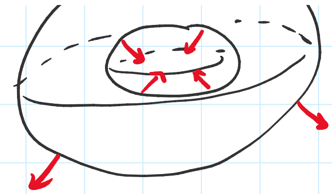
∂W



$\iint_{\partial W} \vec{F}\cdot d\vec{S}.$



∂W



"outward" ~ pointing away from W

$\iiint_W = \iiint_{B_b} - \iiint_{B_a}$

$=$

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\, dV$$

dxdydz

$$= \iiint_W \underbrace{(x^2+y^2+z^2)}\, dV$$

$a \le \rho \le b$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

spherical

$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho^2 \cdot \underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \underbrace{\int_0^{\pi}\sin\phi\, d\phi}\int_a^b \rho^4\, d\rho$$

$$= 2\pi \cdot 2 \cdot \frac{b^5-a^5}{5} = \frac{4\pi}{5}(b^5-a^5)$$

$\square$