Lecture 23 - Midterm 2 Review

- Midterm 2 will be available today at 12 noon until 11:59 pm. 90 minute timer once viewed; view before 10:29 pm for full time.

Problem 1 (20 points)

Let the surface S be the graph of the function $z = g(x, y) = y^3/3 + 1$ over $(x, y) \in [-1, 1] \times [0, 1]$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = xy^2$. Evaluate

$$\iint_S f\, dS.$$

Solution:

Parametrize graph

$$\Phi(x,y) = (x, y, g(x,y)) = (x, y, \tfrac{y^3}{3} + 1), \quad D \in [-1,1] \times [0,1]$$

$$\vec{T}_x = (1, 0, \partial g/\partial x) = (1,0,0) \qquad \vec{T}_x \times \vec{T}_y = (0, -y^2, 1)$$

$$\vec{T}_y = (0, 1, \partial g/\partial y) = (0, 1, y^2)$$

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + y^4}$$

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}$$

dxdy if D x-simple

dydx if D y-simple

$$\iint_S f\, dS = \iint_D f(\Phi(x,y)) \|\vec{T}_x \times \vec{T}_y\|\, dA$$

$$= \int_{-1}^{1} \int_0^1 xy^2 \sqrt{1 + y^4}\, dy\, dx$$

$$= \underbrace{\int_{-1}^{1} x\, dx}\int_0^1 y^2 \sqrt{1+y^4}\, dy$$

$$= 0$$

□

Problem 2 (20 points)

Let S be the closed surface which is the union of three surfaces, $S = S_1 \cup S_2 \cup S_3$, where S_1 is the curved part of the cylinder,

$$S_1 = \{(x, y, z) : x^2 + y^2 = 1 \text{ and } 0 \le z \le 1\},$$

S_2 is the bottom "lid" of the cylinder,

$$S_2 = \{(x, y, z) : x^2 + y^2 \le 1 \text{ and } z = 0\},$$

and S_3 is the top "lid" of the cylinder,

$$S_3 = \{(x, y, z) : x^2 + y^2 \le 1 \text{ and } z = 1\}.$$

Let S be oriented with the outward normal. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x, y, z) = (x, y, z)$. Evaluate

$$\iint_S \vec{F} \cdot d\vec{S}.$$

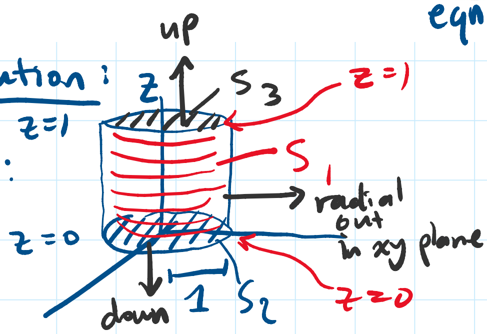
eqn $\vec{n} \cdot (x, y, z) = c$

$\Rightarrow \vec{n} = \pm(0, 0, 1)$

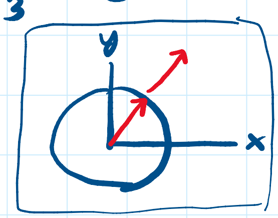
$\vec{F}(x,y,z) = (x,y,z)$

Solution:

S :



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F}_3 \cdot d\vec{S}$$



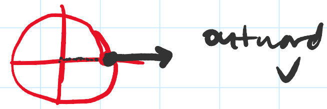
S_1 : $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$ $\quad D = [0, 2\pi] \times [0, 1]$

$\vec{T}_\theta = (-\sin\theta, \cos\theta, 0)$ $\quad \vec{T}_\theta \times \vec{T}_z = (\cos\theta, \sin\theta, 0)$ ✓

$\vec{T}_z = (0, 0, 1)$

$\theta = 0$

$T_\theta \times T_z = (1, 0, 0)$



$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D (\cos\theta, \sin\theta, z) \cdot (\cos\theta, \sin\theta, 0)\, d\theta\, dz$$

$$= \int_0^1 \int_0^{2\pi} d\theta\, dz = 2\pi.$$

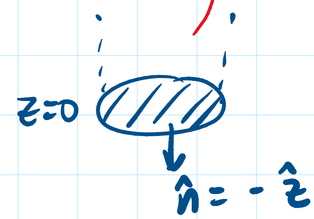
geometric way $\quad \hat{n}(x,y,z) = \dfrac{(x,y,0)}{\sqrt{x^2+y^2}} = (x, y, 0)$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} (x,y,z) \cdot (x,y,0)\, dS$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} (x,y,z)\cdot(x,y,0)\, dS$$

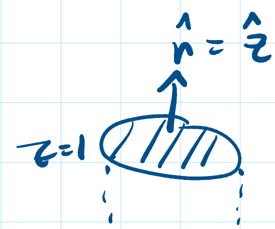
$$= \iint_{S_1} \underbrace{(x^2+y^2)}_{=1 \text{ on } S_1} dS = \text{Area}(S_1) = 2\pi .$$

$$S_2 = \{(x,y,z) : x^2+y^2 \le 1 \quad \text{and} \quad z=0\}$$



$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot(-\hat{z})\, dS$$

$$= \iint_{S_2} (-\underbrace{z}_{\overset{\shortparallel}{0} \text{ on } S_2})\, dS = 0.$$



$$S_3 = \{(x,y,z) : x^2+y^2 \le 1 \quad \text{and} \quad z=1\}$$

$$\left[\begin{array}{l} \Phi(r,\theta) = (r\cos\theta, r\sin\theta, 1) \\ 0 < r \le 1, \quad 0 \le \theta \le 2\pi \end{array}\right]$$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{S_3} \vec{F}\cdot\hat{z}\, dS$$

$$= \iint_{S_3} \underbrace{z}_{\overset{\shortparallel}{1}}\, dS = \text{Area}(S_3) = \pi .$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \sum_{j=1}^{3} \iint_{S_j} \vec{F}\cdot d\vec{S} = 2\pi + 0 + \pi \qquad \square$$

[divergence theorem

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\, dV = 3 \iiint_W dV$$

$S = \partial W$

$$= 3\,\text{Vol}(W) = 3\pi$$
]

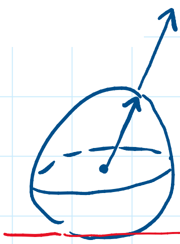
Problem 3 (20 points)

Let $\mathbb{S}^2$ be the surface of the unit sphere, $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, oriented with the outward normal. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\vec{F}(x, y, z) = (y^2, z - xy, -y)$. Evaluate

$$\iint_{\mathbb{S}^2} \vec{F} \cdot d\vec{S}.$$

Hint: While you can in principle evaluate this directly by parametrizing S (e.g., with spherical coordinates), it is easier to use the geometric formula for the surface integral: what is the unit normal vector field to $\mathbb{S}^2$ and what is its dot product with $\vec{F}$?

Solution:



$$\iint_{\mathbb{S}^2} \vec{F}\cdot d\vec{S} = \iint_{\mathbb{S}^2} \vec{F}\cdot\hat{n}\, dS$$

$$\hat{n}(x,y,z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = (x,y,z)$$

$$\boxed{\overbrace{\hat{n}(x,y,z)}^{\in S_1} = (x,y,z) \quad \text{HW6 problem 1}}$$

$$= \iint_{\mathbb{S}^2} (y^2, z-xy, -y)\cdot(x,y,z)\, dS$$

$$= \iint_{\mathbb{S}^2} (\cancel{xy^2} + \cancel{yz} - \cancel{xy^2} - \cancel{yz})\, dS = 0$$

spherical parametrization

$$dS = \|T_\phi \times T_\theta\|\, d\phi\, d\theta$$

analytical way

$$\underbrace{\vec{F}(\Phi(\theta,\phi))\cdot(\vec{T}_\phi \times \vec{T}_\theta)}_{=0}$$

$\square$

Problem 4 (20 points)

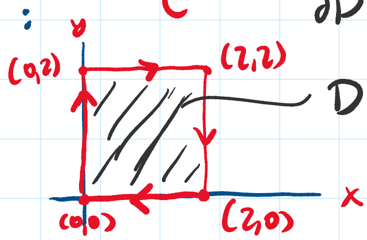
Let C be the following closed curve in the xy plane: C goes from $(0, 0)$ to $(0, 2)$ along a straight line, from $(0, 2)$ to $(2, 2)$ along a straight line, from $(2, 2)$ to $(2, 0)$ along a straight line, and from $(2, 0)$ back to $(0, 0)$ along a straight line.

Let $P(x, y) = 2xy + \sin(x^4)$, $Q(x, y) = \sin(y^3) + x$. Evaluate the line integral

$$\int_C P dx + Q dy.$$

Hint: Use Green's theorem.

Solution:



$\partial D = C$

Green's theorem

C is clockwise

$$\int P(x,y)\,dx + Q(x,y)\,dy = -\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= -\iint_D (\underbrace{1-2x}_{\neq 1})\,dA$$

$$= -\int_0^2\int_0^2 (1-2x)\,dy\,dx$$

$$= -2\int_0^2 (1-2x)\,dx = -2\,(x-x^2)\Big|_0^2$$

$$= -2(2-4) = 4. \qquad \square$$

Problem 5 (Extra Credit: 10 points)

Prove the following statement:

Let the surface S be the graph of a differentiable function $z = g(x,y)$ over the domain $(x,y) \in [-a,a] \times [b,c]$ (where a,b,c are fixed constants satisfying $a > 0$ and $c > b$). Furthermore, assume that g only depends on y; that is, it can be expressed $g(x,y) = h(y)$ for some differentiable function of one variable h. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a (continuous) function that is odd with respect to the x variable; i.e., $f(-x,y,z) = -f(x,y,z)$ for all (x,y,z). Then,

$$\iint_S f\,dS = 0.$$

Hint: Parametrize the surface using the usual parametrization of a graph,

$$\Phi(x,y) = (x,y,g(x,y)) = (x,y,h(y)).$$

The domain of Φ is $D = [-a,a] \times [b,c]$. Use the definition of the surface integral to write $\iint_S f\,dS$ as a double integral over D. Split the domain into two pieces, $D_+ = [0,a] \times [b,c]$ and $D_- = [-a,0] \times [b,c]$ so that $D = D_+ \cup D_-$. Then, split the double integral over D into two double integral over the two pieces, D_+ and D_-. For the D_- double integral, make a change of variable $x \to -x$, and you will see that the D_+ double integral and the D_- double integral exactly cancel each other, using the fact that f is odd with respect to x.

example: Problem 1

Solution:

$$\Phi(x,y) = (x,y,h(y)) \qquad D = [-a,a] \times [b,c]$$

$$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + \cancel{(\partial h/\partial x)^2} + (\partial h/\partial y)^2} = \sqrt{1 + h'(y)^2}$$

$$\iint_S f\,dS = \iint_D f(x,y,h(y))\sqrt{1+h'(y)^2}\,dx\,dy$$

$S \qquad D$

$$= \int_b^c \int_{-a}^{a} f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy$$

$$= \int_b^c \int_0^a f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy + \int_b^c \int_{-a}^{0} f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy$$

$$\left\{ \begin{array}{l} s_{upper} = -(0) \\ s_{lower} = -(-a) \end{array} \right.$$

$$\boxed{s = -x}$$

$x \to -x$

$dx \to \underline{-dx}$

$$= \int_b^c \int_0^a f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy - \int_b^c \int_{a}^{0} f(-x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy$$

$$\int_a^0 \to -\int_0^a$$

$$= \int_b^c \int_0^a f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy + \int_b^c \int_0^a f(-x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy$$

f odd w.r.t. x

$f(-x,y,z) = -f(x,y,z)$

$$= \cancel{\int_b^c \int_0^a f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy} - \cancel{\int_b^c \int_0^a f(x,y,h(y)) \sqrt{1+h'(y)^2}\, dx\, dy}$$

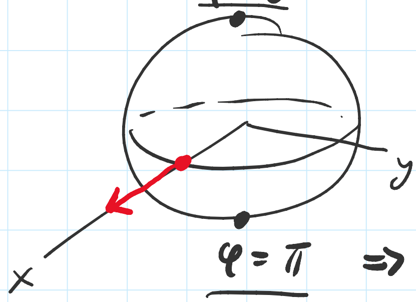
$$= 0 \qquad \square$$

$= 0$ $\square$

scratch

HW6 problem 1

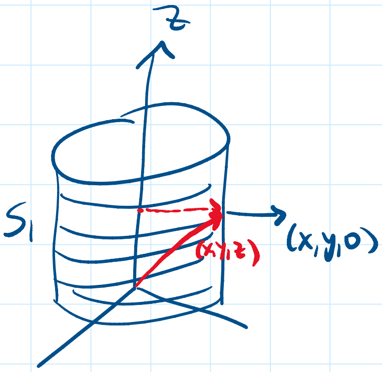
$$\vec{T}_\varphi \times \vec{T}_\theta = \sin\varphi \, \Phi(\theta, \varphi)$$



$\varphi = 0 \Rightarrow \vec{T}_\varphi \times \vec{T}_\theta = 0$

$\begin{bmatrix} \varphi = \pi/2 \\ \theta = 0 \end{bmatrix}$

$\varphi = \pi \Rightarrow \vec{T}_\varphi \times \vec{T}_\theta = 0$



$$\hat{n}(\overbrace{x,y,z}^{\in S_1}) = (x, y, 0)$$

Parametrize S_1:

$$\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$$

$$\vec{T}_\theta \times \vec{T}_z = (\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y}, 0)$$

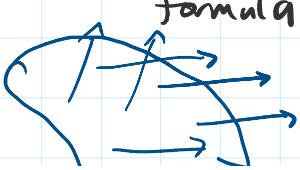
$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \, \|\vec{T}_u \times \vec{T}_v\| \, du\,dv$$

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{(\pm)}_{\text{orientation}} \iint_D \vec{F}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v) \, du\,dv$$

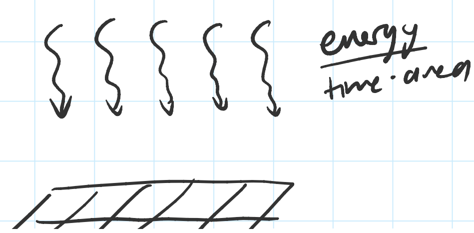
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} \, dS$$

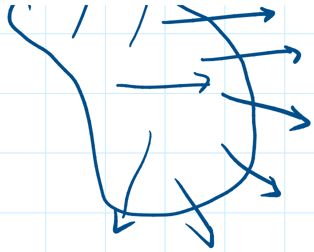
geometric formula

scalar surface differential



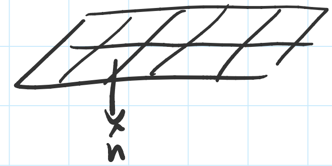
$\vec{F}$ $\iint \vec{F} \cdot d\vec{S}$





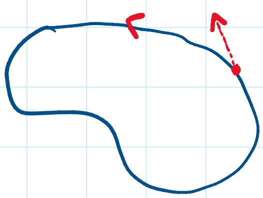
$$\iint_S \vec{F}\cdot d\vec{S}$$

flux

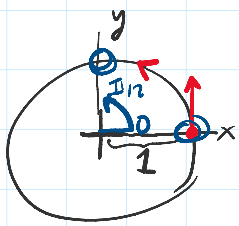


$$\text{flux} = \frac{\text{energy}}{\text{time}} = \text{power.}$$

$$\int_C \vec{F}\cdot d\vec{r}$$



ex

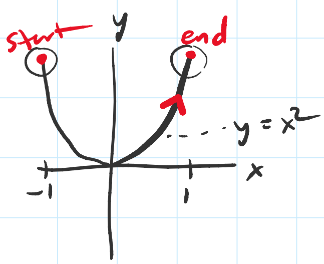


$$\vec{c}(t) = (x(t), y(t))$$
$$= (\cos t, \sin t) \qquad t \in [0, 2\pi]$$
$$\vec{c}\,'(t) = (x'(t), y'(t))$$
$$= (-\sin t, \cos t)$$

$t=0,\quad \vec{c}(0) = (1,0)$

$\vec{c}\,'(0) = (0,1)$

ex



$\vec{c}(x) = (x, x^2) \qquad x \in [-1,1]$

$\vec{c}(-1) = (-1,1) =$ start ✓

$\vec{c}(1) = (1,1) =$ end