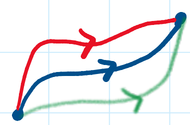
# Lecture 22 - Stokes' Theorem cont.; The Divergence Operator

- Read section 4.4 (on the divergence)
- HW7 is posted; due Thursday 11/18 at 11:59 pm
- I have OH right after lecture today at 9 am
- Midterm 2 is this upcoming Monday (11/15). I will review the practice midterm for Monday morning's lecture (remotely through Zoom; see Canvas for Zoom link)

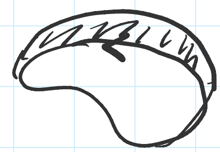
FTLI,

$$\int_C \nabla f \cdot dr = f\Big|_{\partial C}$$



Stokes' Theorem

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$



Recall $\nabla \times \nabla f = 0$ for any $f \in C^2$.

Let $S$ be any surface:

$$0 = \iint_S \underbrace{(\nabla \times \nabla f)}_{=0} \cdot d\vec{S}$$

$$\underset{\text{Stokes'}}{=} \int_{\partial S} \nabla f \cdot d\vec{r} \underset{\text{FTLI}}{=} f\Big|_{\partial\partial S} = 0$$

topological fact: $\partial\partial$ empty

"boundary of the boundary is empty"

$$\nabla \times \nabla f = 0 \iff \partial\partial S = \phi.$$

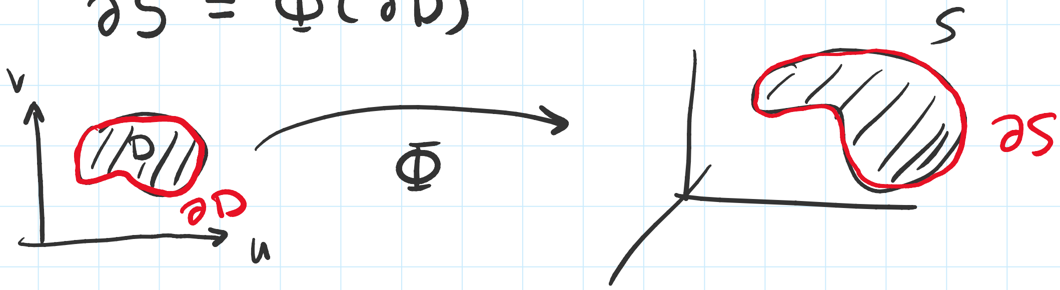
- Definition of $\partial S$?

Let $\Phi : D \to S$ be an injective parametrization of a surface $S = \Phi(D)$, where $D \subset \mathbb{R}^2$, $S \subset \mathbb{R}^3$

Then

of a surface $S = \Phi(D)$
Then,

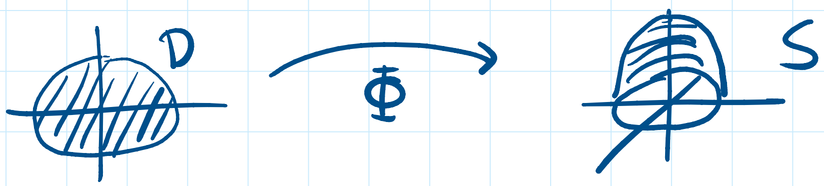
$$\partial S = \Phi(\partial D)$$

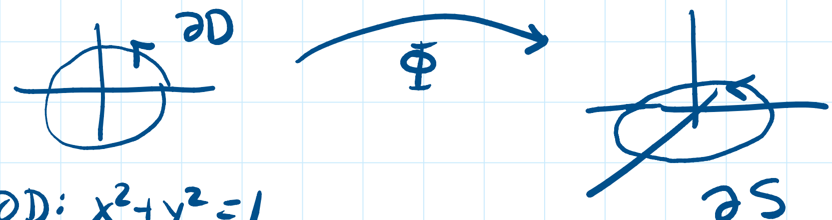



ex/ consider $S$ as the graph of $z = 1 - x^2 - y^2$ over $x^2 + y^2 \le 1$.

Determine $\partial S$.

$\Phi(x,y) = (x, y, 1 - x^2 - y^2)$, $(x,y) \in D = \{x^2 + y^2 \le 1\}$



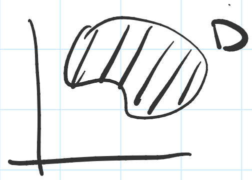

$\partial D: x^2 + y^2 = 1$

## Green's theorem as a special case of Stokes' Theorem:

Recall Green's theorem:

$(P, Q): \mathbb{R}^2 \to \mathbb{R}^2$, domain $D \subset \mathbb{R}^2$

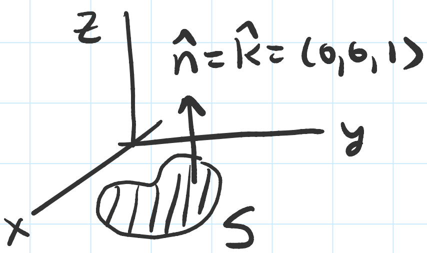
$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P\,dx + Q\,dy$$

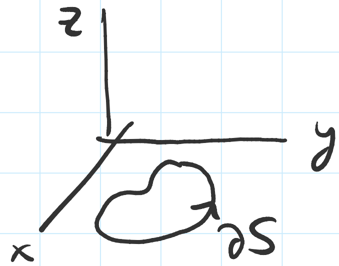




Embed this problem into $\mathbb{R}^3$

Think of $S$ as $D$ inside $z=0$




Embed our vector field:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$

$\vec{F}(x,y,z) = (P(x,y), Q(x,y), 0)$

Stokes'

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix}$$

$$= \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\partial S} (P, Q, 0) \cdot (dx, dy, dz) = \int_{\partial D} P\,dx + Q\,dy$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS \qquad \hat{n} = \hat{k}$$

flat surface

$dS \to dA$

parametrize $S$:

$$= \iint_S (\nabla \times F) \cdot n \, dS$$

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$dS \to dA$

parametrize $S$:

$$\Phi(x,y) = (x,y,0)$$
$$(x,y) \in D$$

$$\vec{T}_x = (1,0,0) = \hat{\imath}$$
$$\vec{T}_y = (0,1,0) = \hat{\jmath}$$
$$\|\vec{T}_x \times \vec{T}_y\| = \|\hat{\imath} \times \hat{\jmath}\| = \|\hat{k}\| = 1$$
$$\Rightarrow dS = \|\vec{T}_x \times \vec{T}_y\| \, dx\, dy$$
$$= dx\, dy$$

$$\iint_S \underset{\text{vector field}}{\vec{G}} \cdot d\vec{S} \underset{\text{geometric formula}}{=} \iint_S \underbrace{\vec{G} \cdot \hat{n}}_{\text{scalar function}} \, dS$$

$$\iint_S \vec{G} \cdot d\vec{S} \underset{\text{analytical formula}}{=} \iint_D \vec{G}(\Phi(u,v)) \cdot (\vec{T}_u \times \vec{T}_v) \, du\, dv$$

**Claim:** Let $S$ be a regular surface, with parametrization $\Phi(u,v)$ s.t. $\vec{T}_u \times \vec{T}_v \neq 0$.

Then, $d\vec{S} = \hat{n} \, dS$

**Proof:**

$dS = \|\vec{T}_u \times \vec{T}_v\| \, du\, dv$

$$dS = \|\vec{T}_u \times \vec{T}_v\|\, du\, dv$$

$$\hat{n} = \frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|}$$

$$\hat{n}\, dS = \frac{\vec{T}_u \times \vec{T}_v}{\cancel{\|\vec{T}_u \times \vec{T}_v\|}} \cancel{\|\vec{T}_u \times \vec{T}_v\|}\, du\, dv$$

$$= (\vec{T}_u \times \vec{T}_v)\, du\, dv = d\vec{S}.$$

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## Divergence Operator

**Def:** Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a diff. vector field. We define the divergence of $\vec{F} = (F_1, F_2, F_3)$

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= (\partial_x, \partial_y, \partial_z) \cdot (F_1, F_2, F_3)$$

note: $\nabla \cdot \vec{F} : \mathbb{R}^3 \to \mathbb{R}$. We think of $\nabla \cdot$ as a differential operator

$\nabla \cdot$ : differentiable vector fields on $\mathbb{R}^3$ $\longrightarrow$ scalar functions on $\mathbb{R}^3$

$$\left( \vec{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \nabla \cdot \vec{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x^i} \right)$$

$F_1$ $\quad$ $F_2$ $\quad$ $F_3$

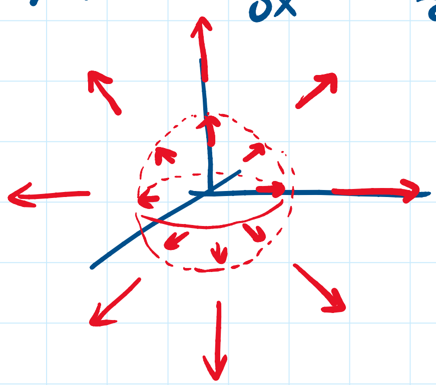
ex/ let $\vec{F}(x,y,z) = (\overbrace{x^2+y}^{F_1}, \overbrace{yx+z}^{F_2}, \overbrace{\sin(z)x}^{F_3})$

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2x + x + \cos(z)x.$$

ex/ let $\vec{F}(x,y,z) = (x, y, z)$

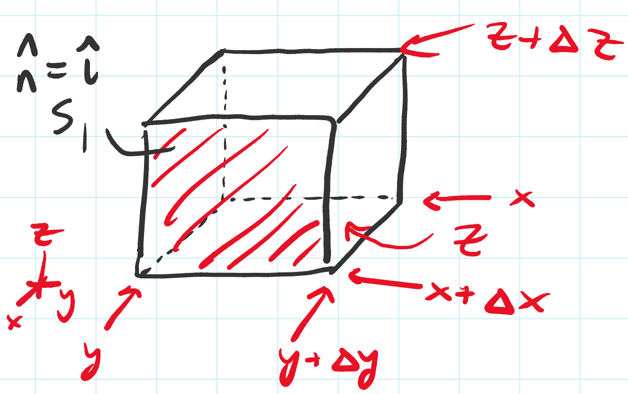
$$\nabla\cdot\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 > 0$$



* The divergence $\nabla\cdot\vec{F}(x,y,z)$ measures the infinitesimal flux out a surface enclosing $(x,y,z)$, per unit volume

$\nabla\cdot\vec{F}(x,y,z) > 0$, $\vec{F}$ is a source at $(x,y,z)$

$\nabla\cdot\vec{F}(x,y,z) < 0$, $\vec{F}$ is a sink at $(x,y,z)$



$y \qquad y+\Delta y$

flux of $\vec{F}$ out of cube's boundary $\partial C$
(to linear order)

$$\iint_{\partial C} \vec{F} \cdot d\vec{S} = \sum_{i=1}^{6} \iint_{S_i} \vec{F} \cdot d\vec{S}$$

$$= \left(F_1\big|_{x+\Delta x} \Delta y \Delta z - F_1\big|_x \Delta y \Delta z\right)$$

$$+ \left(F_2\big|_{y+\Delta y} \Delta x \Delta z - F_2\big|_y \Delta x \Delta z\right)$$

$$+ \left(F_3\big|_{z+\Delta z} \Delta x \Delta y - F_3\big|_z \Delta x \Delta y\right)$$

divide by volume $\Delta x \Delta y \Delta z$

$$\frac{\iint_{C} \vec{F} \cdot d\vec{S}}{\Delta x \Delta y \Delta z} = \frac{\left(F_1\big|_{x+\Delta x} - F_1\big|_x\right)}{\Delta x} + \frac{\left(F_2\big|_{y+\Delta y} - F_2\big|_y\right)}{\Delta y} + \frac{\left(F_3\big|_{z+\Delta z} - F_3\big|_z\right)}{\Delta z}$$

$$\overset{\text{to linear order}}{\simeq} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)_{(x,y,z)} = \nabla \cdot \vec{F}(x,y,z)$$

proved (to linear order)

If $W$ is some volume, and $\partial W$ is the boundary surface w/ outward normal, the boundary surface w/ outward normal,

$$\iiint_W (\nabla \cdot \vec{F})\, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$$