Lecture 20 - The Curl Operator and Stokes' Theorem

- Read section 4.4 (the part on the curl; not the divergence) and section 8.2
- HW3 grades posted
- Midterm 2 next Monday (11/15), covering everything through lecture 19 : focus on surfaces, surfaces integrals, and Green's theorem; homework 4, 5, 6.
- I will have OH this week at the usual times (Wed after lecture and Thursday at 11 am).
  - Additionally, I will have an extra OH on Wednesday at 11 am.

Line Integral

$$\int_C \vec{F}\cdot d\vec{r}$$

Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} \underline{P\,dx + Q\,dy}$$

$$= \int_{\partial D} \vec{F}\cdot d\vec{r}$$

∂D has + orientation

$\vec{F}(x,y) = (P(x,y), Q(x,y))$

$d\vec{r} = (dx, dy)$

∂D param by $\vec{c}(t) = (x(t), y(t))$, $t \in [a,b]$

$$= \int_a^b \vec{F}(\vec{c}(t))\cdot \vec{c}\,'(t)\,dt$$

$$= \int_a^b \left[P(x(t),y(t))\frac{dx}{dt} + Q(x(t),y(t))\frac{dy}{dt}\right] dt$$

In Green's theorem, we see $\boxed{\underbrace{\frac{\partial Q}{\partial x}}_{F_2} - \underbrace{\frac{\partial P}{\partial y}}_{F_1}}$ ← measures inf. rotation

__Def__: Let $\vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ be a differentiable vector field on $\mathbb{R}^3$. The curl of $\vec{F}$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Note: $\nabla \times \vec{F}$ is a vector field. We think of the curl as a diff. operator

$$\text{curl}: \begin{matrix}\text{differentiable} \\ \text{vector} \\ \text{fields} \\ \text{on } \mathbb{R}^3\end{matrix} \longrightarrow \begin{matrix}\text{vector} \\ \text{fields} \\ \text{on } \mathbb{R}^3\end{matrix}$$

ex/ let $\vec{F}(x,y,z) = (x^2y + z, e^{xy} - z, \sin(xy))$
Compute $\nabla \times \vec{F}$

$$(\nabla \times \vec{F})(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y+z & e^{xy}-z & \sin(xy) \end{vmatrix}$$

$$= \Big(\frac{\partial}{\partial y}\sin(xy) - \frac{\partial}{\partial z}(e^{xy} - z), \frac{\partial}{\partial z}(x^2y + z) - \frac{\partial}{\partial x}\sin(xy), \ldots$$

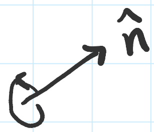
$$\ldots \frac{\partial}{\partial x}(e^{xy} - z) - \frac{\partial}{\partial y}(x^2y + z) \Big)$$

$$= (\cos(xy)x + 1, 1 - \cos(xy)y, ye^{xy} - x^2).$$

The curl of a v.f. measures the rotation of a v.f. if $\hat{n}$ is some unit vector

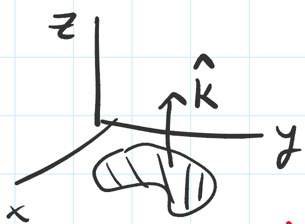
The curl of a v.f. measures the rotation of a v.f.; if $\hat{n}$ is some unit vector,

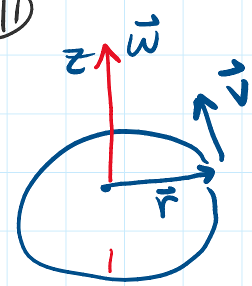
$(\nabla \times \vec{F}(x,y,z)) \cdot \hat{n}$ gives the inf. rotation of $\vec{F}$ at (x,y,z) about the axis $\hat{n}$.



what if axis $\hat{n} = \hat{k} = (0,0,1)$

$$(\nabla \times \vec{F}) \cdot \hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$





ex/ Angular velocity

$$\underset{\sim m/s}{\vec{v}} = \underset{\sim s^{-1}}{\vec{\omega}} \times \underset{\sim m}{\vec{r}}$$

$\vec{\omega} = (0, 0, \omega)$ $\quad \omega > 0$

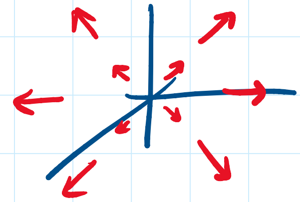
$\vec{r} = (x, y, z)$

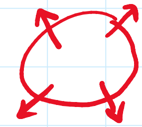
$\nabla \times \vec{v}$? $\quad \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-\omega y, \omega x, 0)$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\omega y & \omega x & 0 \end{vmatrix} \qquad \partial_x = \frac{\partial}{\partial x}$$

$$= (0, 0, \omega + \omega) = (0, 0, 2\omega) = 2\vec{\omega}$$

ex/ $\vec{F}(x,y,z) = (x,y,z)$





$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix}$$

$$= (0,0,0) \quad \Rightarrow \quad \nabla \times \vec{F} = \underset{0 \in \mathbb{R}^3}{0}$$

This is an example of an irrotational vector field (i.e., a v.f. s.t. its curl is zero)

<u>Theorem</u>:

Let $\vec{F}$ be a C^1 vector field on $\mathbb{R}^3$. Then:

$$\vec{F} = \nabla f \iff \nabla \times \vec{F} = 0.$$

In words, gradient v.f.'s = irrotational v.f.'s

<u>Proof</u>: One direction $\Rightarrow$

Assume $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} = \Big(\underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{\substack{=0 \\ \text{mixed partials}}}, 0, 0\Big)$$

$\Leftarrow$ direction later

$\square$

Properties of curl:

(i) Linearity

$$\nabla \times (a\vec{F} + b\vec{G}) = a\nabla\times\vec{F} + b\nabla\times\vec{G}$$ $a, b \in \mathbb{R}$, $\vec{F}, \vec{G}$ diff v.f.

(ii) Gradients are irrotational

$$\nabla \times \nabla f = 0$$

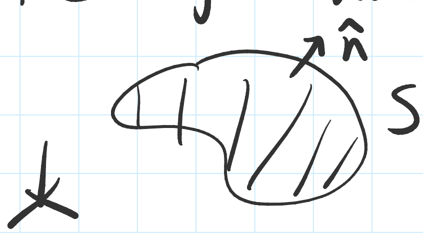
(iii) Product Rule: Let $f: \mathbb{R}^3 \to \mathbb{R}$, $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ (both diff.)

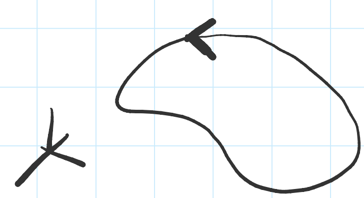
$$\nabla \times (f\vec{F}) = (\nabla f)\times\vec{F} + f(\nabla\times\vec{F})$$

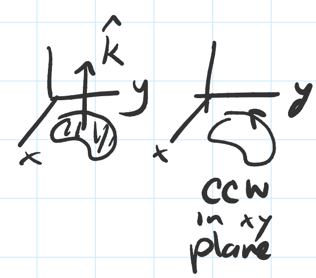
proof: hw 7.

Stokes' Theorem:

Let S be an oriented surface in $\mathbb{R}^3$ and let ∂S denote its oriented boundary; the orientation of the boundary is given by the right hand rule:







Let $\vec{F}$ be a C^1 vector field. Then,

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$$

remarks:

Remarks:

- Similar form to FTC I
- If S has no boundary, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$

  e.g. a sphere 
- What is the boundary of a surface?

  If you're an ant living on S, then ∂S is the set of points you'd leave the surface.