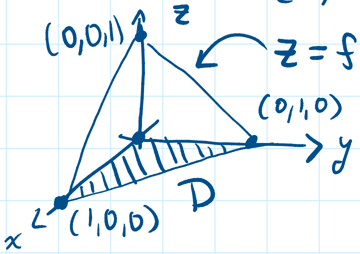
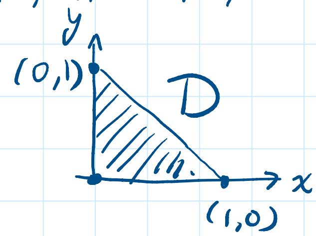
- Read sections 5.2, 5.3, 5.4, 5.5

ex/ Find the volume of the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$



$z = f(x,y)$

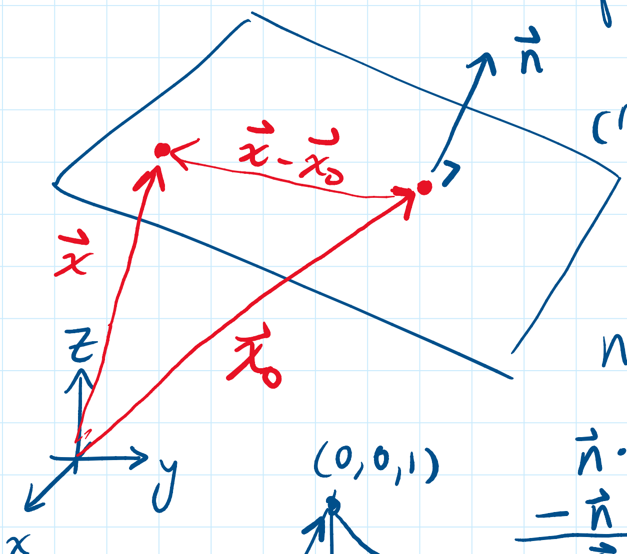


$0 \le y \le 1-x$
$0 \le x \le 1$ y-simple

$\left(\begin{array}{l} \text{left} \quad \text{right} \\ 0 \le x \le 1-y \\ 0 \le y \le 1 \end{array} \quad x\text{-simple}\right)$

$$\text{Volume} = \iint_D f(x,y)\, dA$$

Equation of a plane:

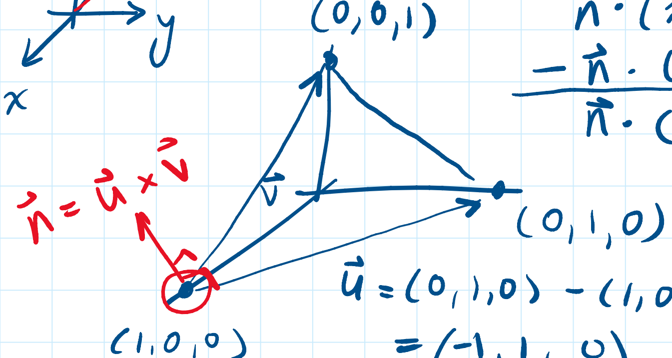


(n_x, n_y, n_z) (x,y,z) (x_0, y_0, z_0)

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$n_x(x-x_0) + n_y(y-y_0) + n_z(z-z_0) = 0$$

$$\begin{array}{r} \vec{n}\cdot(\vec{x}-\vec{x}_1) \\ -\,\vec{n}\cdot(\vec{x}-\vec{x}_0) \\ \hline \vec{n}\cdot(\vec{x}_0-\vec{x}_1) = 0 \end{array}$$



$\vec{v} = (0,0,1) - (1,0,0)$
$= (-1,0,1)$

$\vec{u} = (0,1,0) - (1,0,0)$
$= (-1,1,0)$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\vec{n} = \begin{array}{c} \begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,1) \end{array} \\ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \end{array} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = 1\hat{i} - (-1)\hat{j} + 1\hat{k}$$
$$= (1,1,1)$$

$$0 = \vec{n}\cdot(\vec{x}-\vec{x}_0) = (1,1,1)\cdot(x-1,\, y-0,\, z-0)$$

$$0 = \vec{n} \cdot (\vec{x} - \vec{x}_0) = (1,1,1) \cdot (x - 1, y - 0, z - 0)$$

$$= x - 1 + y + z \qquad (\vec{a})_1 \quad (\vec{a})_2 \quad (\vec{a})_3$$

$$\Rightarrow \boxed{z = 1 - x - y.}$$

$$(\vec{a} \times \vec{b})_k = \sum_{i,j=1}^{3} \varepsilon_{ijk} a_i b_j$$

$\varepsilon_{123} = 1$

even permutation $= 1$

odd permutation $= -1$

not permutation $= 0.$

Volume ~ z

$$\iint_D f \, dA$$

$$= \int_0^1 \int_0^{1-x} (1 - x - y) \, dy \, dx \qquad (-a)^2 = a^2$$

$$= \int_0^1 y - yx - \frac{y^2}{2} \Big|_0^{1-x} dx \qquad (1-x)^2 = (x-1)^2$$

$$= \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2} \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 \, dx = \frac{1}{2} \int_0^1 (x-1)^2 \, dx$$

$$= \frac{1}{6} (x-1)^3 \Big|_0^1 = -\frac{(-1)^3}{6} = 1/6.$$

$\boxed{5.5}$ Triple Integrals

$$\iiint_W f\, dV, \quad \iiint_W f(x,y,z)\, dz\, dy\, dx$$

$$f : W \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$$

- If f represents some kind of density (mass density, charge density, energy density),

  $\underline{\underline{\iiint_W f\, dV}}$ = total quantity in W.

- Analogous properties (linearity, monotone, domain additive)

- Analogous Fubini's theorem.

ex) $W = \underbrace{[0,1]}_{x} \times \underbrace{[0,2]}_{y} \times \underbrace{[0,3]}_{z}$

$$\iiint_W (x+y+2z)^3\, dV$$

$$= \int_0^3 \int_0^2 \int_0^1 (x+y+2z)^3\, dx\, dy\, dz$$

$$\int (x+a)^3\, dx = \frac{1}{4}(x+a)^4 + C$$

$$= \int_0^3 \int_0^2 \frac{1}{4}(x+y+2z)^4 \Big|_0^1\, dy\, dz$$

$$\int^3 \int^2 \Big[(\ldots)^4 - (\ldots)^4\Big]$$

$$= \frac{1}{4}\int_0^3\int_0^2 \left[(1+y+2z)^4 - (y+2z)^4\right] dy\, dz$$

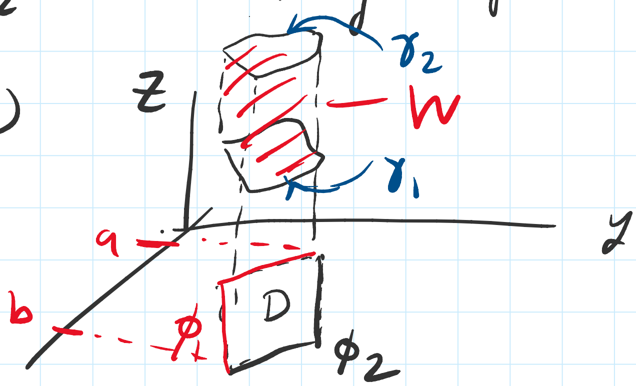
$$= \frac{1}{4}\int_0^3 \left[\frac{(1+y+2z)^5}{5} - \frac{(y+2z)^5}{5}\right]\Bigg|_0^2 dz \quad \ldots$$

Elementary Region

$$\underbrace{\gamma_1(x,y) \le z \le \gamma_2(x,y)}$$

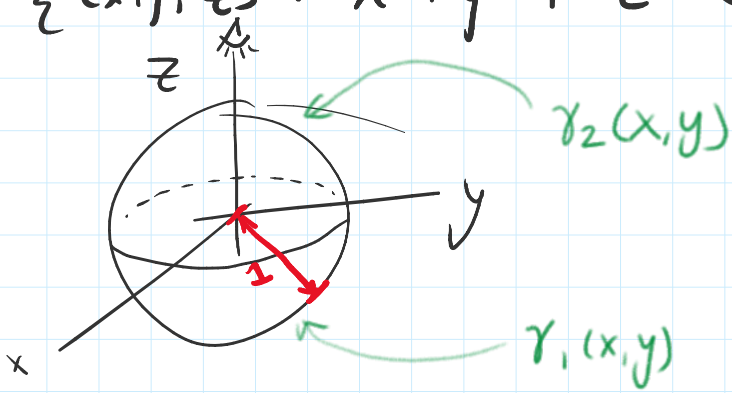
domain of γ_1, γ_2 is x or y simple

$$W \begin{cases} \gamma_1(x,y) \le z \le \gamma_2(x,y) \\ \phi_1(x) \le y \le \phi_2(x) \\ a \le x \le b \end{cases}$$



$$\iiint_W f\, dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Unit Ball $B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$

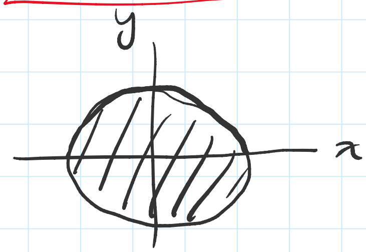


southern hemi. $\le z \le$ northern hemisphere

$$x^2+y^2+z^2=1$$

$$z=\pm\sqrt{1-x^2-y^2}$$

$$-\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}$$



$$-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$$

$$-1\le x\le 1$$

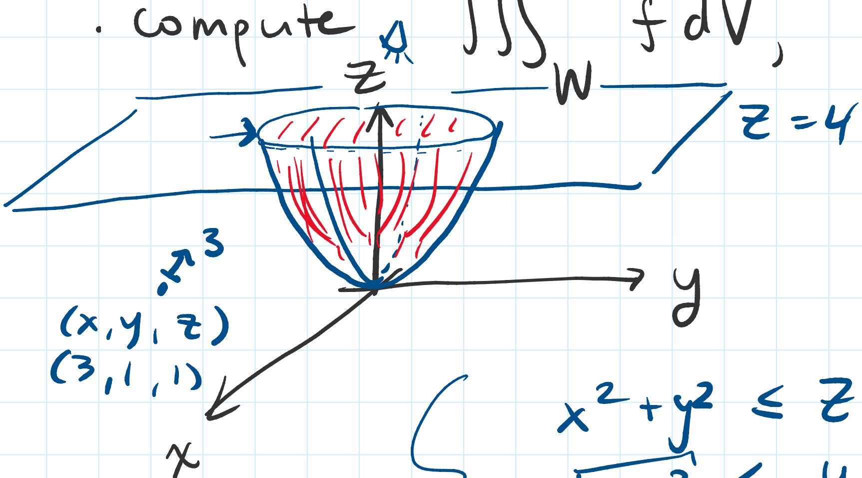
textbook $\iiint_B 1\,dV = \text{Volume}(B)$

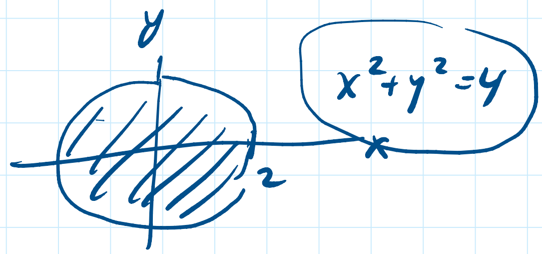
$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dz=\frac{4}{3}\pi.$$

ex/ Consider the region W bounded by

$z=x^2+y^2$ and $z=4$

. compute $\iiint_W f\,dV$, $f(x,y,z)=x$.





$$W\begin{cases} x^2+y^2\le z\le 4\\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\\ -2\le x\le 2\end{cases}$$

$$W \begin{cases} -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \\ -2 \le x \le 2 \end{cases}$$

$$\iiint_W x \, dV$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x \, dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x(4-x^2-y^2) \, dy \, dx$$

$$= 0.$$