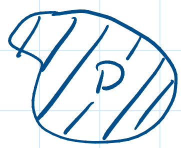
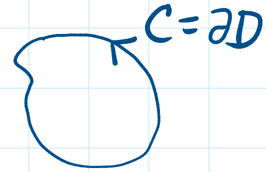
- Read section 8.1 on Green's Theorem
- Reminder: Lecture 19 (Friday 11/05) will be given remotely through Zoom. My OH tomorrow are cancelled; I'll reschedule it for sometime next week.
- I have OH right after lecture.
- MT1 grades will be posted tomorrow.
- MT2 on Week 8 Monday 11/15: Covers everything up to and including Green's theorem (which we will cover today and Friday). The midterm will be cumulative (i.e., requires knowledge of all content covered in the course), but the focus of the second midterm will be surfaces, scalar surface integrals, vector surface integrals, and Green's theorem (sections 7.4, 7.5, 7.6, 8.1) and homework 4, 5, 6. As in the first midterm, there will be four questions and one extra credit problem. I will post a practice midterm sometime next week.





Green's Theorem

Let $D \subset \mathbb{R}^2$ be a simple region (both x and y simple) and let $C = \partial D$ denote the boundary of D w/ the CCW orientation, C^+.

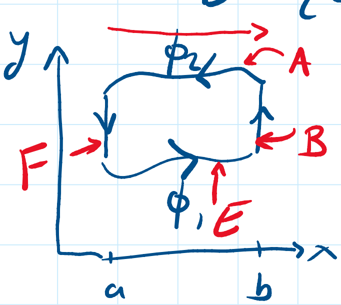
Let $(P, Q) : D \to \mathbb{R}^2$ be a C^1 vector field on D. Then,

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

proof:

Lemma: Let D be y-simple and let $C = \partial D$. For a C^1 function P on D, $\int_{C^+} P\,dx = \iint_D (-\partial P/\partial y)\,dx\,dy$

proof: $D = \{(x,y) : \phi_1(x) \le y \le \phi_2(x),\ a \le x \le b\}$



On B, F, $dx = 0$

$$\iint -\frac{\partial P}{\partial y}\,dx\,dy = \int_a^b \int^{\phi_2(x)} -\frac{\partial P}{}$$

$a \qquad b$

$$\iint_D -\frac{\partial P}{\partial y}\,dy\,dx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} -\frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_a^b \Big(-P\underbrace{(x,\phi_2(x))}_{C(x)=(x,\phi_2(x))} + P\underbrace{(x,\phi_1(x))}_{C(x)=(x,\phi_1(x))} \Big)\,dx$$

$$= \int_A P\,dx + \int_E P\,dx$$

$$= \int_A P\,dx + \int_B P\,dx + \int_E P\,dx + \int_F P\,dx$$

$$= \int_{C=\partial D} P\,dx.$$

Do similarly for $\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dx\,dy$.

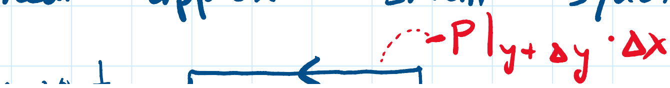
Add together for full result. $\square$

$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\substack{\text{circulation} \\ \text{infinitesimally}}} dx\,dy$$

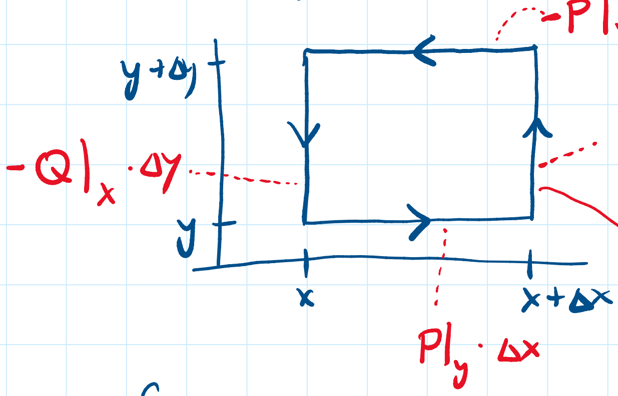
$$\left(\begin{array}{c} \vec{F} = (P,Q) \\ \int_C \vec{F}\cdot d\vec{r} \end{array} \right)$$

FTC $\displaystyle\int_{[a,b]} \frac{d}{dx}[f(x)]\,dx = f(x)\Big|_a^b = \int_{\partial[a,b]} f$

linear approx. small square edges $\Delta x, \Delta y$



linear approx. small square edges $\Delta x, \Delta y$



$$s \in [y, y+\Delta y]$$

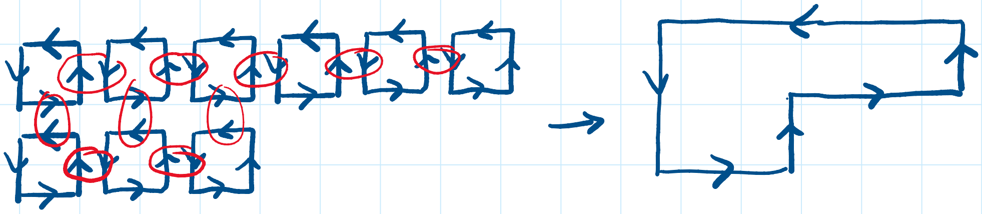
$$\int_C Q\,dy \qquad C(s) = (x+\Delta x, s)$$

$$= \int_y^{y+\Delta y} Q(x+\Delta x, s)\,ds$$

$$\int P\,dx + Q\,dy$$

$$\approx Q|_{x+\Delta x}\,\Delta y - Q|_x\,\Delta y - P|_{y+\Delta y}\cdot\Delta x + P|_y\cdot\Delta x$$

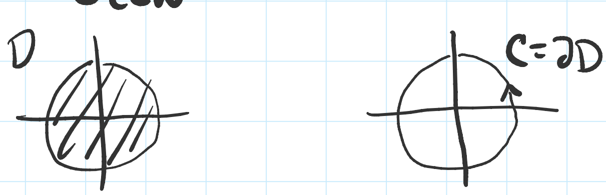
$$= \left(\frac{Q|_{x+\Delta x} - Q|_x}{\Delta x} - \frac{P|_{y+\Delta y} - P|_y}{\Delta y}\right)\Delta x\Delta y$$



ex/ Verify Green's theorem for $(P,Q) = (x, xy)$ where $D = \{(x,y) : x^2+y^2 \le 1\}$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

∂D ccw



$$\partial(x^2+y^2 \le 1) : x^2+y^2 = 1$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$\iint_D \left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right)$$

$$= \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x) \right) dx\,dy = \iint_D y\, dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} y\, dy\, dx = 0$$

$$\vec{c}(\theta) = (\cos\theta, \sin\theta) \qquad \theta \in [0, 2\pi]$$

$$\vec{c}\,'(\theta) = (-\sin\theta, \cos\theta)$$

$$\int_C P\,dx + Q\,dy = \int_C (\overset{x}{P}, \overset{xy}{Q}) \cdot \overbrace{d\vec{r}}^{\vec{c}\,'(\theta)\,d\theta}$$

$$= \int_0^{2\pi} (\cos\theta, \cos\theta\sin\theta) \cdot (-\sin\theta, \cos\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(-\sin\theta\cos\theta + \cos^2\theta\sin\theta \right) d\theta$$

$$= -\frac{\sin^2\theta}{2}\Big|_0^{2\pi} - \frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0.$$

ex/ Computing areas:

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)}_{\text{choose } P \text{ and } Q \text{ s.t. this } = 1.} dx\,dy$$

For example, $Q = \frac{1}{2}x$, $P = -\frac{1}{2}y$

$$A(D) = \iint_D dx\,dy = \iint_D \left(\frac{\partial}{\partial x}\Big(\underbrace{\tfrac{1}{2}x}_{Q}\Big) - \frac{\partial}{\partial y}\Big(\underbrace{-\tfrac{1}{2}y}_{P}\Big) \right) dx\,dy$$

$$= \int -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

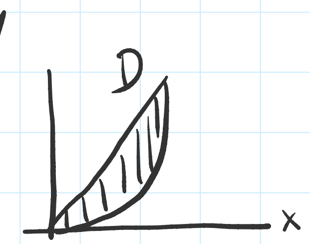
$$= \int_{\partial D} -\frac{1}{2} y \, dx + \frac{1}{2} x \, dy$$

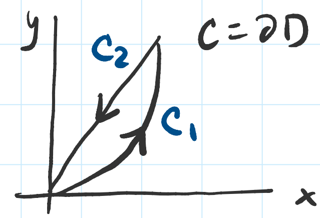
see MTI problem 3.

ex/ let $\vec{F} = (\underbrace{xy^2}_{P}, \underbrace{x+y}_{Q})$

let D be region $x^2 \leq y \leq x$

$0 \leq x \leq 1$





Verify Green's theorem

$C_1(x) = (x, x^2)$, $x \in [0,1]$ $\quad C_1'(x) = (1, 2x)$

$C_2(s) = (1-s, 1-s)$, $s \in [0,1]$ $\quad C_2'(s) = (-1,-1)$

$$\int_C P\,dx + Q\,dy = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \left(x(x^2)^2, x + x^2\right) \cdot (1, 2x)\, dx$$

$$+ \int_0^1 \left((1-s)(1-s)^2, (1-s) + (1-s)\right) \cdot (-1,-1)\, ds$$

$\square$