Lecture 17 - Surface Integrals of Vector Felds, cont..

Sunday, October 31, 2021 8:34 PM

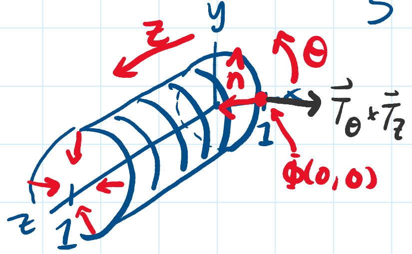
- Friday's lecture (11/05) will be given remotely via Zoom. Link on Canvas - Zoom LTI PRO.
- I won't have OH this Thursday.

ex/ $\iint_S \vec{F}\cdot d\vec{S}$ $\quad \vec{F} = (1,1,z(x^2+y^2)^2)$

$S = \{(x,y,z) : x^2+y^2=1,\ z\in[0,1]\}$

w/ orientation pointing in towards z-axis

$\hat{n}(u,v)$



Param: $\Phi(\theta,z) = (\cos\theta, \sin\theta, z)$ $\quad \theta\in[0,2\pi]$, $z\in[0,1]$

$\vec{T}_\theta = (-\sin\theta, \cos\theta, 0)$

$\vec{T}_z = (0, 0, 1)$

$\Rightarrow \underbrace{\vec{T}_\theta\times\vec{T}_z = (\cos\theta, \sin\theta, 0)}_{\text{is } \Phi \text{ orientation-preserving?}}$

Consider point $(x,y,z) = (1,0,0)\in S$. $\theta=0, z=0$

$= \Phi(0,0)$

$\hat{n}(0,0) = -\hat{x}$ but $\vec{T}_\theta\times\vec{T}_z\big|_{(0,0)} = (1,0,0) = \hat{x}$

Instead, use $\vec{T}_z\times\vec{T}_\theta = -\vec{T}_\theta\times\vec{T}_z = (-\cos\theta, -\sin\theta, 0)$

$\vec{F}(\Phi(\theta,z)) = (1,1,z(\cos^2\theta+\sin^2\theta)^2) = (1,1,z)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(\theta,z))\cdot(\vec{T}_z\times\vec{T}_\theta)\,dz\,d\theta$$

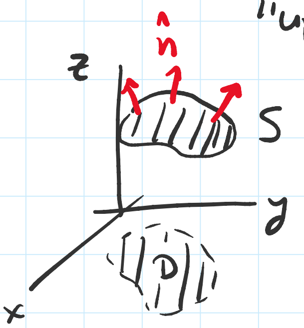
$$= \int_0^{2\pi}\int_0^1 (-\cos\theta-\sin\theta)\,dz\,d\theta$$

$$= \int_0^{2\pi}(-\cos\theta-\sin\theta)\,d\theta = 0 \quad \square$$

Surface Integral of a Vector Field over a Graph

S: graph of $z = g(x,y)$ over some domain $D \subset \mathbb{R}^2$

"upward" pointing normal $(\hat{n}\cdot\hat{z} > 0)$



$$\Phi(x,y) = (x, y, g(x,y)) \qquad \Phi: D \to S$$

$$\vec{T}_x = (1, 0, \partial g/\partial x)$$

$$\vec{T}_y = (0, 1, \partial g/\partial y)$$

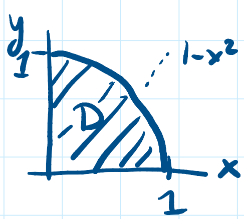
$$\vec{T}_x \times \vec{T}_y = (-\partial g/\partial x\, ,\ -\partial g/\partial y\, ,\ 1) \quad \text{upward pointing } \checkmark$$

$$\vec{F} = (F_1, F_2, F_3)$$

$$\iint_S \vec{F}\cdot d\vec{S} \overset{\vec{F}\cdot(\vec{T}_x\times\vec{T}_y)}{=} \iint_D \left(-F_1\frac{\partial g}{\partial x} - F_2\frac{\partial g}{\partial y} + F_3\right)\Bigg|_{z=g(x,y)} dx\,dy$$

ex/ Let $z = \underbrace{ye^x}_{g(x,y)}$ for $\left\{\begin{matrix} 0 \le y \le 1-x^2 \\ x \in [0,1] \end{matrix}\right\} = D$ be the surface S, w/ upward normal.

Let $\vec{F}(x,y,z) = (0,1,x)$. Compute $\iint_S \vec{F}\cdot d\vec{S}$.



$$g(x,y) = ye^x$$

$$\partial g/\partial x = ye^x, \quad \partial g/\partial y = e^x$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-0\cdot ye^x - 1\cdot e^x + x)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x^2} (x - e^x)\,dy\,dx$$

$$= \int_0^1 (1-x^2)(x-e^x)\,dx$$

IBP

$$\int_a^b \frac{d}{dx}(uv)\,dx = \int_a^b \left(\frac{du}{dx}v + u\frac{dv}{dx}\right)dx$$

$$uv\Big|_a^b \Rightarrow \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$= \int_0^1 (1-x^2)(x - e^x)\,dx$$

$$= \int_0^1 (x - e^x - x^3 + x^2 e^x)\,dx$$

$$= \frac{1}{2} - e + 1 - \frac{1}{4} + \int_0^1 x^2 e^x\,dx$$

$$x^2e^x\Big|_0^1 - \int_0^1 2x e^x\,dx$$

$$= x^2e^x\Big|_0^1 - \left(2xe^x\Big|_0^1 - \int_0^1 2e^x\,dx\right)$$

$$= \frac{1}{2} - \frac{1}{4} - 1 \qquad = \#\ldots$$

$$\boxed{uv\Big|_a^b \Rightarrow \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

$u = x^2$ $\quad dv = e^x dx$

$du = 2x\,dx$ $\quad v = e^x$

$u = 2x$

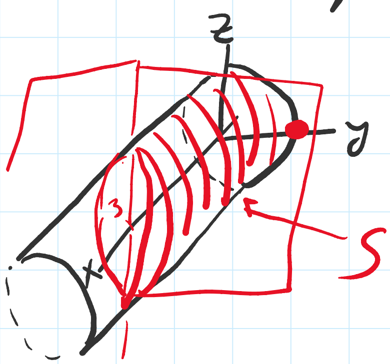
$du = 2dx$

$dv = e^x dx$

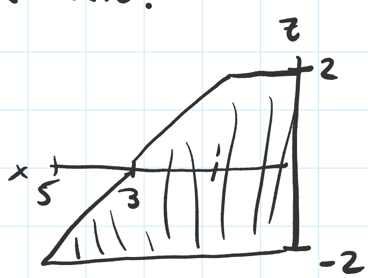
$v = e^x$

ex/ Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ $\quad \vec{F}(x,y,z) = (x^2, 2z, -3y)$

over $S = \{(x,y,z) : y^2+z^2=4,\ 0 \le x \le 3-z\}$, oriented away from the x-axis.



xz:



$$\Phi(x,\theta) = (x, 2\cos\theta, 2\sin\theta)$$

$$D = \{(x,\theta) : 0 \le x \le \underbrace{3 - 2\sin\theta}_{3-z},\ 0 \le \theta \le 2\pi\}$$

$\vec{T}_x = (1,0,0)$

$\vec{T}_\theta = (0, -2\sin\theta, 2\cos\theta)$

$$\vec{T}_x \times \vec{T}_\theta = (0, -2\cos\theta, -2\sin\theta)$$

wrong orientation $\Rightarrow$ use $\vec{T}_\theta \times \vec{T}_x = (0, 2\cos\theta, 2\sin\theta)$

$$\iint \vec{F}\cdot d\vec{S} = \iint \vec{F}(\Phi(x,\theta)) \cdot (\vec{T}_\theta \times \vec{T}_x)\,dx\,d\theta$$

$\underline{\vec{F} = (x^2, 2z, -3y)}$

$\vec{F} = (x^2, 2z, -3y)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(x,\theta))\cdot(\vec{T}_\theta \times \vec{T}_x)\,dx\,d\theta$$

$$= \iint_D (x^2, 4\sin\theta, -6\cos\theta)\cdot(0, 2\cos\theta, 2\sin\theta)\,dx\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3-2\sin\theta} (-4\sin\theta\cos\theta)\,dx\,d\theta$$

$$= -4\int_0^{2\pi} (3-2\sin\theta)\sin\theta\cos\theta\,d\theta$$

$$= -4\int_0^{2\pi}\left[3\sin\theta\cos\theta - 2\sin^2\theta\cos\theta\right]d\theta$$

$$= -4\int_0^{2\pi} \frac{d}{d\theta}\left(\frac{3}{2}\sin^2\theta - \frac{2}{3}\sin^3\theta\right)d\theta$$

$$= 0 \qquad \square$$