Lecture 16 - Surface Integrals of Vector Fields cont.

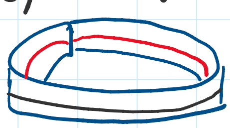
- Review section 7.6

- Last time:

Def:
An orientable surface is a surface s.t. at each point, there are two normal vectors $\hat{n}_1$ and $\hat{n}_2 = -\hat{n}_1$ (unit vector) and the vector $\hat{n}_1$ cannot be moved around the surface to coincide with $\hat{n}_2$.

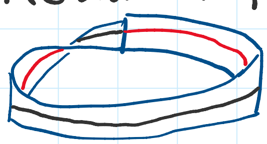
In other words, the surface has two distinct sides.

ex/ Cylinder:



Orientable ✓

Möbius strip:



"surface w/ one side"

Non-orientable surface ✗

An oriented surface is an orientable surface w/ a choice of "preferred" side/normal vector field ($\hat{n}_1$ or $\hat{n}_2$)

Let S be oriented (i.e., a choice of $\hat{n}(u,v)$) w/ a param. $\Phi : D \to \Phi(D) = S$, $\Phi(u,v)$

We call Φ orientation-preserving:

if $$\frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} = \hat{n}(u,v)$$

We call Φ orientation-reversing:

if $$\frac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} = -\hat{n}(u,v).$$

For an oriented surface S (chosen $\hat{n}$)

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, \underbrace{dS}_{\text{scalar surface element}}$$

Theorem

Let S be oriented and Φ be a (regular) param. Let $\vec{F}$ be a continuous vector field. Then,

- If Φ is orientation-preserving,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S,\Phi} \vec{F}\cdot d\vec{S}$$

- If Φ is orientation-reversing,

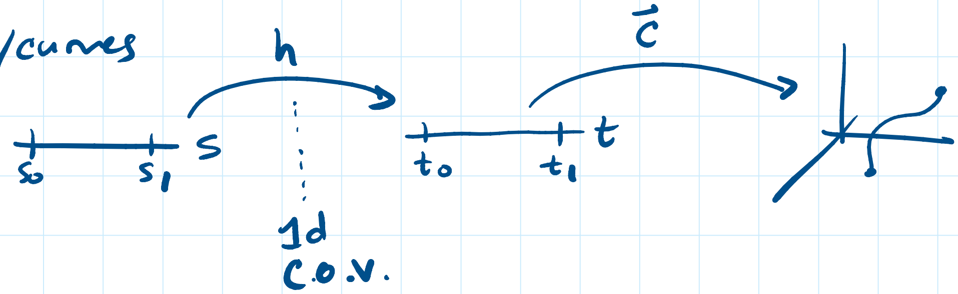
$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S,\Phi} \vec{F}\cdot d\vec{S}$$

Theorem Let Φ_1, Φ_2 be any two params of S,

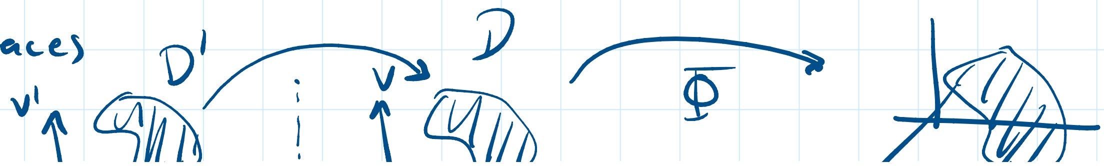
$$\iint_{S,\Phi_1} f\, dS = \iint_{S,\Phi_2} f\, dS$$

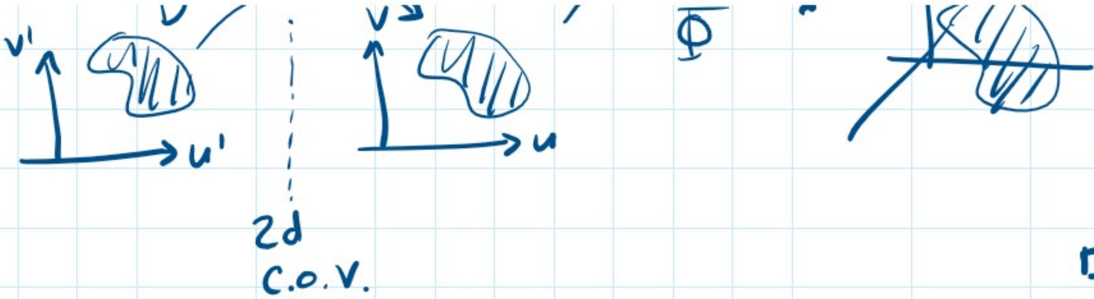
proof:

paths/curves



surfaces





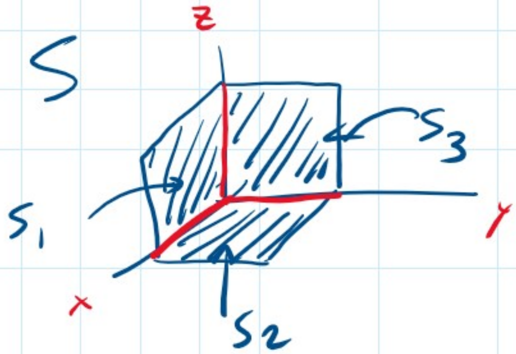
$\square$

Remark: If a surface S is a disjoint union (0 area) of surfaces S_i (except maybe intersecting along curves) $(i=1,\dots,N)$

$$\iint_S \vec{F}\cdot d\vec{S} = \sum_{i=1}^{N} \iint_{S_i} \vec{F}\cdot d\vec{S}. \qquad \begin{pmatrix}\text{see hw5} \\ 9\,4\,8\,9\end{pmatrix}$$

$$\iint_S f\, dS = \sum_{i=1}^{N} \iint_{S_i} f\, dS.$$

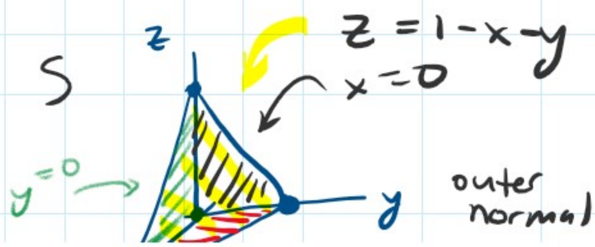
S

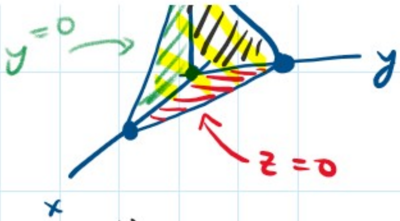


ex/ $\iint_S \vec{F}\cdot d\vec{S}$ $\quad \vec{F}(x,y,z) = (x,y,z)$

S tetrahedron with faces $x=0$, $y=0$, $z=0$, $z=1-x-y$ with outward unit normal.

S

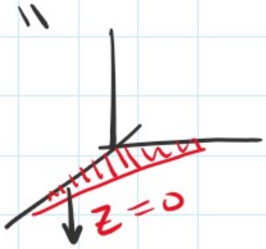




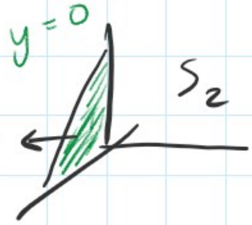
outer normal

=

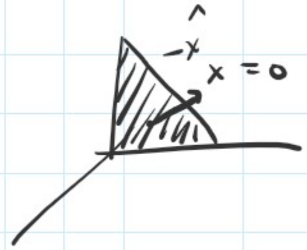
S_1



$\cup$ $-\hat{y}$

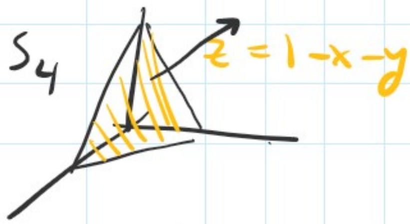


S_2 $\cup$



S_3

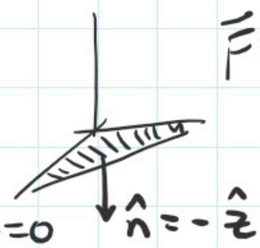
$\cup$ S_4



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$+\iint_{S_3} \vec{F}\cdot d\vec{S} + \iint_{S_4} \vec{F}\cdot d\vec{S}$$

S_1



$\vec{F}=(x,y,z)$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\hat{n}\,dS$$

$$= \iint_{S_1} (-\underset{\underset{0}{\shortparallel}}{z})\,dS = 0.$$

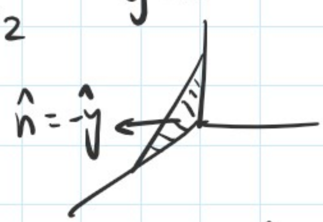
$z=0$

$\hat{n}\cdot(x,y,z)=0$

$(0,0,1)$

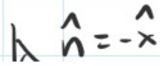
$(0,0,-1)$

S_2 $y=0$



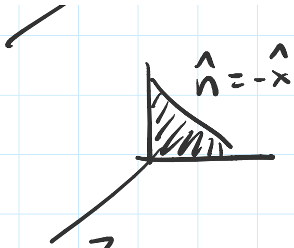
$\vec{F}(x,y,z)=(x,y,z)$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,dS = \iint_{S_2} (-\overset{\overset{0}{\shortparallel}}{y})\,dS = 0.$$



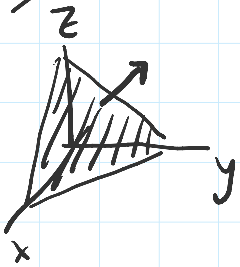
$$\iint \vec{F}\cdot\hat{n}\,dS = \iint (-x)\,dS = 0.$$

S_3



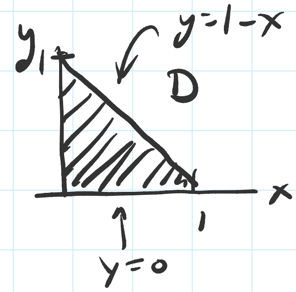
$$\iint_{S_3} \vec{F}\cdot\hat{n}\,dS = \iint_{S_3} (-x)\,dS = 0.$$

S_4



$$z = \underbrace{1-x-y}_{g(x,y)} \qquad z+x+y=1$$

$$\vec{n} = (1,1,1) \quad \text{or} \quad \vec{n} = (-1,-1,-1)$$



$$\Phi(x,y) = (x,y,1-x-y) \qquad \Phi : D \to S_4$$

$$\vec{T}_x = \partial\Phi/\partial x = (1,0,-1)$$

$$\vec{T}_y = \partial\Phi/\partial y = (0,1,-1)$$

$$\vec{T}_x \times \vec{T}_y = (1,1,1)$$

$$\iint_{S_4} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(x,y))\cdot(\vec{T}_x\times\vec{T}_y)\,dx\,dy$$

$$= \iint_D (x,y,1-x-y)\cdot(1,1,1)\,dx\,dy$$

$$= \iint_D 1\,dx\,dy \quad \left(= A(D) = 1/2\right)$$

$$= \int_0^1\int_0^{1-x} dy\,dx = \int_0^1 (1-x)\,dx = x - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

$$\iint_S \vec{F}\cdot d\vec{S} = \cancel{\iint_{S_1} \vec{F}\cdot d\vec{S}} + \cancel{\iint_{S_2} \vec{F}\cdot d\vec{S}} + \cancel{\iint_{S_3} \vec{F}\cdot d\vec{S}} + \iint_{S_4} \vec{F}\cdot d\vec{S}$$

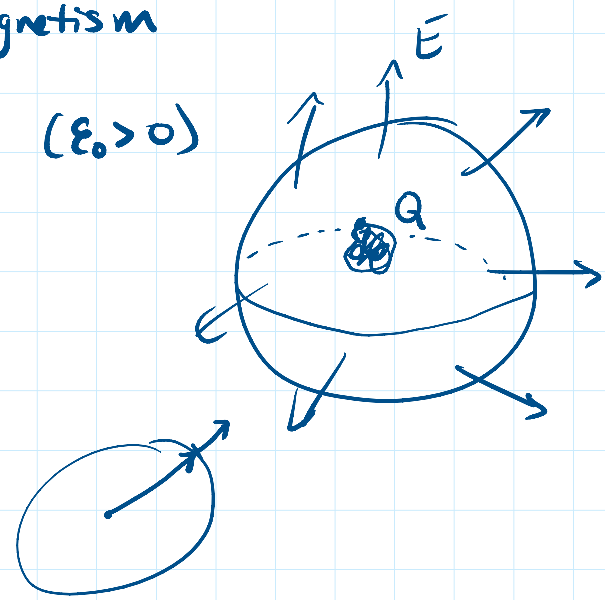
$$= 1/2.$$

$\square$

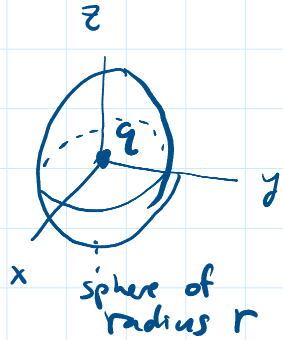
ex Gauss' Law for Electromagnetism

$$\oiint_S \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0} \qquad (\varepsilon_0 > 0)$$

(outward)



Point charge



$\hat{n} = \hat{r}$

By symmetry, $\vec{E}(x,y,z) = E(r)\hat{r}$

$$\frac{q}{\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0} = \oiint_S \vec{E}\cdot d\vec{S} = \oiint_S \vec{E}\cdot\hat{n}\, dS$$

$$= \oiint_S E(r)\, dS = E(r)\oiint_S dS = E(r)\, 4\pi r^2$$

$$\Rightarrow E(r) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}$$

$$\Rightarrow \vec{E}(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$ electric field produced by a point charge $\square$