Lecture 15 - Surface Integrals of Vector Fields

- Read section 7.6
- Homework 4 due tomorrow at 11:59 pm (extended by one day due to midterm).
- Homework 5 is posted, due Wed Nov 3rd at 11:59 pm. There are 9 problems.
- My OH: today after lecture and tomorrow at 11 am.

Can integrate:

Scalar functions over intervals ($\int_a^b f(x)\,dx$, 1d calc.)

Scalar functions over areas $\subseteq \mathbb{R}^2$ ($\iint_D f(x,y)\,dx\,dy$, double int.)

" over volumes $\subseteq \mathbb{R}^3$ ($\iiint_W f(x,y,z)\,dV$, triple int.)

" over paths $\subseteq \mathbb{R}^n$ ($\int_C f\,ds$, path int.)

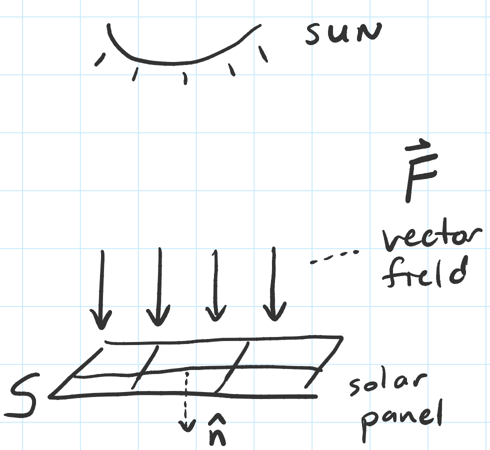
Vector fields over paths $\subseteq \mathbb{R}^n$ ($\int_{\vec{c}} \vec{F}\cdot d\vec{r}$, line int.)

Scalar functions over surfaces $S \subseteq \mathbb{R}^3$ ($\iint_S f\,dS$, scalar surface integral)

Last one:

→ Vector fields over surfaces $S \subseteq \mathbb{R}^3$ ($\iint_S \vec{F}\cdot d\vec{S}$, (vector) surface int.)

Physical motivation:



"energy flux"

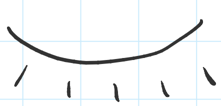
$$\frac{\text{energy}}{(\text{unit area})(\text{unit time})}$$

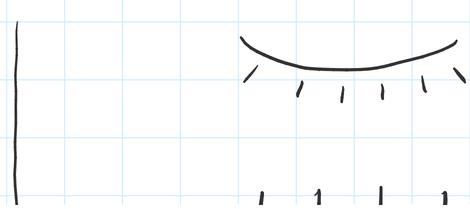
$\|\vec{F}\| = \text{const.}$

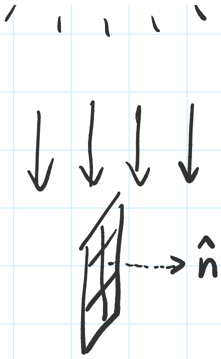
How much power = energy/unit time does the solar panel draw?

$$\text{power} = \iint_S \vec{F}\cdot d\vec{S}$$

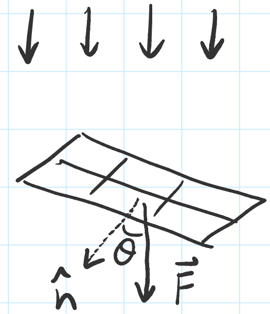
$$\text{power} = \|\vec{F}\|\,A(S)$$







power = 0



S angled

$$\text{power} = \vec{F}\cdot\hat{n}\, A(S) = \|\vec{F}\|\cos\theta\, A(S)$$

Def:

Let $\vec{F}$ be a cont. vector field, $\Phi : \underset{\mathbb{R}^2}{\overset{\cup}{D}} \longrightarrow \underset{\Phi(D)}{\overset{\shortparallel}{S}}$ be a param. of a surface S. The surface integral of $\vec{F}$ over S

$$\iint_{S,\Phi} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v))\cdot \underbrace{(\vec{T}_u \times \vec{T}_v)}_{\text{normal vector } \vec{n}(u,v)}\, du\,dv \quad \text{"analytic"}$$

let's assume S has a unit normal $\hat{n}(u,v)$

$$\iint_{S,\Phi} \vec{F}\cdot d\vec{S} = \iint_{S,\Phi} \vec{F}\cdot\hat{n}\, dS \quad \text{"geometric"}$$

↖ scalar surface area element $dS = \|\vec{T}_u \times \vec{T}_v\|\, du\,dv$

$$= \iint_D \vec{F}(\Phi(u,v))\cdot \underbrace{\hat{n}(u,v)\,\|\vec{T}_u \times \vec{T}_v\|}_{=\vec{n}(u,v)}\, du\,dv$$

ex/ Let $\Phi(\theta,\phi) = (\overbrace{\cos\theta\sin\phi}^{x}, \overbrace{\sin\theta\sin\phi}^{y}, \overbrace{\cos\phi}^{z})$ $\quad \theta \in [0, 2\pi)$, $\phi \in [0,\pi]$

unit sphere

$S = \{(x,y,z) : x^2+y^2+z^2 = 1\}$

Let $\vec{F}(x,y,z) = (x,y,z)$. $\quad \iint_{S,\Phi} \vec{F}\cdot d\vec{S}$ $\quad$ "flux of $\vec{F}$ over S"

$$\vec{T}_\theta = \frac{\partial\Phi}{\partial\theta} = (-\sin\theta\sin\phi,\ \cos\theta\sin\phi,\ 0)$$

$$\vec{T}_\phi = \frac{\partial\Phi}{\partial\phi} = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi)$$

$$\vec{T}_\phi = \frac{\partial \Phi}{\partial \phi} = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ -\sin\phi)$$

$$\vec{n}(\theta,\phi) = \vec{T}_\theta \times \vec{T}_\phi = (-\cos\theta\sin^2\phi,\ -\sin\theta\sin^2\phi,\ -\sin\phi\cos\phi)$$

$$\iint_{S,\Phi} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(\theta,\phi))\cdot\vec{n}(\theta,\phi)\,d\theta\,d\phi \qquad D = [0,2\pi]\times[0,\pi]$$

$$= \int_0^\pi\int_0^{2\pi} (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)\cdot(-\cos\theta\sin^2\phi,\ -\sin\theta\sin^2\phi,\ -\sin\phi\cos\phi)\,d\theta\,d\phi$$

$$= \int_0^\pi\int_0^{2\pi} -(\sin^3\phi + \sin\phi\cos^2\phi)\,d\theta\,d\phi$$

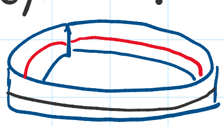
$$= \int_0^\pi\int_0^{2\pi} -\sin\phi\,d\theta\,d\phi = -2\pi\int_0^\pi \sin\phi\,d\phi = -4\pi \qquad \square$$

- We could have used $\vec{T}_\phi \times \vec{T}_\theta = -\vec{T}_\theta \times \vec{T}_\phi$ and would've gotten 4π instead.

Def:
An orientable surface is a surface s.t. at each point, there are two normal vectors $\hat{n}_1$ and $\hat{n}_2 = -\hat{n}_1$ (unit vector) and the vector $\hat{n}_1$ cannot be moved around the surface to coincide with $\hat{n}_2$.

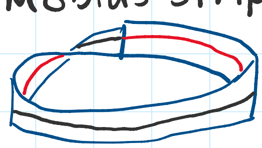
In other words, the surface has two distinct sides.

ex/ Cylinder:



Orientable ✓

Möbius strip:



"surface w/ one side"

Non-orientable surface ✗

An **oriented** surface is an orientable surface w/ a choice of "preferred" side/ normal vector field ($\hat{n}_1$ or $\hat{n}_2$)