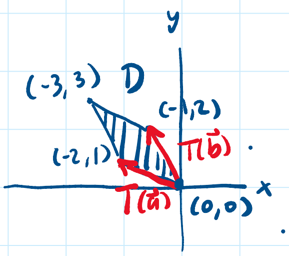
# Lecture 14 - Midterm 1 Review

- Midterm 1 is today. Available on Gradescope from 12 in the afternoon to 11:59 pm. Once viewed, you have 90 minutes to complete, scan, and upload your exam as a single PDF file. Please leave about 10 minutes to make sure you can properly upload your exam.
    - Make sure to assign your solution pages to the appropriate problem on Gradescope before submitting
- Homework 4 and 7 (the homework due the weeks of midterms 1 and 2) will be due on Thursday at 11:59 pm instead of Wednesday. Every other homework is still due on Wednesdays at 11:59 pm.

## Problem 1 (20 points)

Let $D \subset \mathbb{R}^2$ be the region contained inside the parallelogram with vertices $(0,0), (-1,2), (-2,1), (-3,3)$. Evaluate the double integral

$$\iint_D xy^2 \, dA.$$

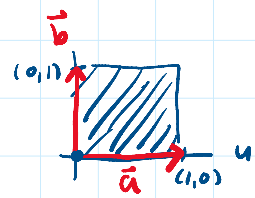



$(-1,2) + (-2,1) = (-3,3)$

Use a linear transformation

$T: D^* \longrightarrow D$

easier to integrate over




$D^* = [0,1] \times [0,1]$

$$[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = [T]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} = [T]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\Rightarrow \quad [T] = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\left|\det [T]\right| = \left|-4-(-1)\right| = 3.$$

C.O.V.

$$\iint xy^2 \, dA = \iint x(u,v)\, y(u,v)^2 \left|\det DT(u,v)\right| du\, dv$$

C.O.V.

$$\iint_D xy^2\,dA = \iint_{D^*} x(u,v)\,y(u,v)^2\,|\det DT(u,v)|\,du\,dv$$

$$[T] = \begin{pmatrix}-2 & -1\\ 1 & 2\end{pmatrix} \qquad [T]\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}-2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}-2u-v\\ u+2v\end{pmatrix}\begin{matrix}\leftarrow x\\ \leftarrow y\end{matrix}$$

$$T(u,v) = (\underbrace{-2u-v}_{x(u,v)},\ \underbrace{u+2v}_{y(u,v)}) \qquad |\det DT(u,v)| = |\det[T]| = 3$$

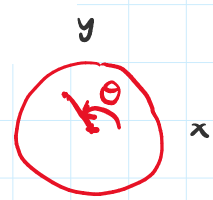
$$= \int_0^1\int_0^1 (-2u-v)(u+2v)^2\cdot 3\,du\,dv$$

$$= \ldots \text{ (algebra) } = -15$$

**Problem 2** (20 points)

Let $W = \{(x,y,z) : 1 \le z \le 2+\sin(\pi(x^2+y^2)),\ x^2+y^2 \le 9\}$. Let $f : W \to \mathbb{R}$ be given by $f(x,y,z) = 2$. Evaluate the triple integral

$$\iiint_W f\,dV.$$




$$1 \le z \le 2+\sin(\pi(x^2+y^2))$$
$$x^2+y^2 \le 9$$

Cylindrical coordinates $\quad x^2+y^2 = r^2, \quad z = z$

$$W^* = \left\{\begin{array}{l} 1 \le z \le 2+\sin(\pi r^2)\\ 0 \le r \le 3 \leftarrow \\ 0 \le \theta < 2\pi \end{array}\right\}$$

$$\iiint_W f\,dV = \iiint_{W^*} f(\text{cylindrical})\,\overbrace{r}^{\text{Jacobian}}\,dr\,d\theta\,dz$$

$$= 2\int_0^3\int_0^{2\pi}\int_1^{2+\sin(\pi r^2)} r\,dz\,d\theta\,dr$$

$$z\Big|_1^{2+\sin(\pi r^2)}\cdot r = (1+\sin(\pi r^2))\cdot r$$
$$= r + r\sin(\pi r^2)$$

$$= 4\pi\int^3 (1+\sin(\pi r^2))\,r\,dr$$

$$\int r\sin(\pi r^2)\,dr$$
$$u = \pi r^2 \qquad du = 2\pi r\,dr$$

$$= 4\pi \int_0 \left(1+\sin(\pi r^2)\right) r \, dr$$

$$= 4\pi \int_0^3 \frac{d}{dr}\left(\frac{r^2}{2} - \frac{\cos(\pi r^2)}{2\pi}\right) dr$$

$$= 4\pi\left(\frac{9}{2} - \frac{\overbrace{\cos(9\pi)}^{=-1}}{2\pi} + \frac{\overbrace{\cos(0)}^{=1}}{2\pi}\right) = 18\pi + 4\pi \cdot \frac{1}{\pi}$$

$\int r\sin(\pi r^2)\,dr$

$u = \pi r^2 \qquad du = 2\pi r\,dr$

$\int \frac{1}{2\pi}\sin(u)\,du = \frac{-\cos(u)}{2\pi} = \frac{-\cos(\pi r^2)}{2\pi}$

**Problem 3** (20 points)

Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\vec{F}(x,y) = (-y, 1)$. Let $\vec{c}$ be a parametrization of the circle of radius $r$,

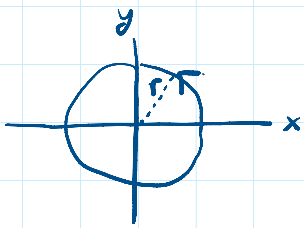
$$x^2 + y^2 = r^2,$$

$(r > 0)$ oriented counterclockwise in the $xy$ plane, which completes one full revolution (i.e., the image of $\vec{c}$ is a closed curve). Evaluate the line integral

$$\int_{\vec{c}} \vec{F} \cdot d\vec{r}.$$

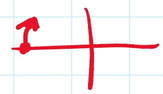
$[a,b] \quad b>a.$

$(r\cos(-t), r\sin(-t))$




$\int_{\vec{c}} \vec{F}\cdot d\vec{r} \qquad \vec{F}(x,y) = (-y, 1)$

$(-r\cos t, r\sin t)$

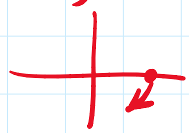


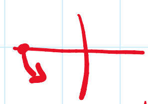
CW

$\vec{c}(t) = (r\cos t, r\sin t)$ CCW, $t \in [0, 2\pi]$

$\vec{c}(t) = (r\cos t, -r\sin t)$ CW

$-r\cos t, -r\sin t$





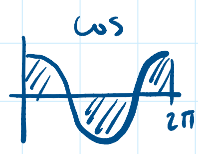
CCW

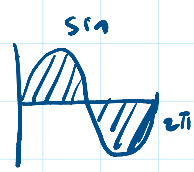
$$\int_{\vec{c}} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$\vec{F}(\vec{c}(t)) = (-y(t), 1) = (-r\sin t, 1)$

$\vec{c}\,'(t) = (-r\sin t, r\cos t)$

$\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = r^2\sin^2 t + r\cos t$




$$= \int_0^{2\pi} \left(r^2\sin^2 t + \cancel{r\cos t}\right) dt$$

$\int_0^{2\pi} \cos t\, dt = 0$

$$= r^2 \int_0^{2\pi} \left(\frac{1}{2} - \cancel{\frac{1}{2}\cos(2t)}\right) dt$$

$$= \underbrace{\pi r^2.}_{\text{Area of region}}$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\cos^2\theta = \frac{1}{2} + \frac{e^{i2\theta} + e^{-i2\theta}}{4}$

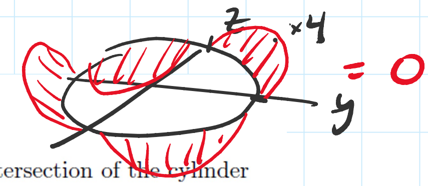
Area of region contained inside the above curve.

(Green's theorem)

$\cos^2\theta = \frac{1}{2} + \frac{e \quad + e}{4}$

$= \frac{1}{2} + \frac{1}{2}\cos(2\theta)$

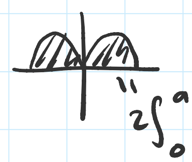
$\sin^2\theta = 1 - \cos^2\theta$




## Problem 4 (20 points)

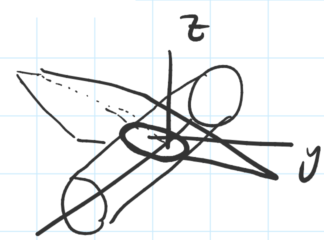
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = \boxed{yz}$. Let $C$ be the curve given by the intersection of the cylinder $\boxed{y^2 + z^2 = 1}$ with the plane $\boxed{x - z = 0}$. Evaluate the path integral

$$\int_C f\,ds.$$



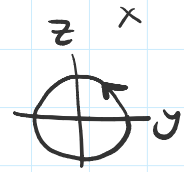

**Hint:** To parametrize this curve, use the equation for the cylinder to parametrize $y(t), z(t)$. Subsequently, the equation for the plane tells you how to parametrize $x(t)$. Also, note that the orientation does not matter, since as we showed in class, the path integral is independent of the orientation of the parametrization.

$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = yz$




$C \quad \{y^2 + z^2 = 1\} \cap \{x - z = 0\}$

$\int_C f\,ds$ $\qquad \boxed{y^2 + z^2 = 1}$




$y(t) = \cos t$
$z(t) = \sin t$ $\qquad t \in [0, 2\pi] \leftarrow$

$x = z \Rightarrow x(t) = z(t) = \sin(t)$

$\vec{c}(t) = (x(t), y(t), z(t)) = (\sin t, \cos t, \sin t) \qquad t \in [0, 2\pi].$

$$\int_C f\,ds = \int_0^{2\pi} f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt$$

$f(\vec{c}(t)) = y(t)\,z(t) = \sin t \cos t$

$\|\vec{c}\,'(t)\| = \|(\cos t, -\sin t, \cos t)\| = \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{=1} + \cos^2 t}$

$= \sqrt{\underbrace{1 + \cos^2 t}_{\geq 1 > 0}} \leftarrow$

$$\geq 1 > 0$$

$$= \int_0^{2\pi} \sin t \cos t \sqrt{1+\cos^2 t}\, dt$$

$u = 1+\cos^2 t$

$du = -2\sin t \cos t\, dt$

$-\frac{1}{2} du = \sin t \cos t\, dt$

u-sub

$$\int_2^2 \sqrt{u}\left(-\tfrac{1}{2}\right) du = 0.$$

$$= \int_0^{2\pi} \underbrace{\frac{d}{dt}\left(-\frac{1}{3}\underbrace{(1+\cos^2 t)}_{u}^{3/2}\right)}\, dt$$

$$-\frac{1}{3}u^{3/2}$$

$$= -\frac{1}{3}(1+\cos^2 t)^{3/2}\Big|_0^{2\pi} = 0.$$

**Problem 5 (Extra Credit: 10 points)**

Consider the polar coordinate map $T : (0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$ given by

$$T(r,\theta) = (r\cos\theta, r\sin\theta).$$

Show that $T$ is injective. Subsequently, compute the Jacobian determinant

$$\det DT(r,\theta).$$

**Hint:** To show that $T$ is injective, you can use the fact that $\exp(i\theta) = \exp(i\theta')$ (where $\theta, \theta'$ are real) holds if and only if $\theta$ and $\theta'$ differ by an integer multiple of $2\pi$ (this should have been discussed in Calc II when discussing complex variables).

$$T : \underbrace{(0,\infty)}_{r} \times \underbrace{[0,2\pi)}_{\theta} \to \mathbb{R}^2$$

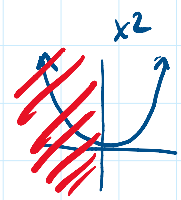
$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

Injective: for $r, r' \in (0,\infty)$, $\theta, \theta' \in [0, 2\pi)$ want to show

$$T(r,\theta) \stackrel{?}{=} T(r',\theta') \Rightarrow (r,\theta) \stackrel{?}{=} (r',\theta')$$

$$(r\cos\theta, r\sin\theta) = (r'\cos\theta', r'\sin\theta')$$

$r\cos\theta = r'\cos\theta'$ $\qquad$ $r\sin\theta = r'\sin\theta'$

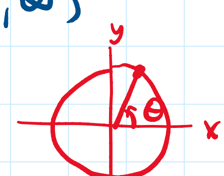



$$r^2\cos^2\theta = (r')^2\cos^2\theta'$$

$$\oplus\; r^2\sin^2\theta = (r')^2\sin^2\theta'$$

$$r^2 = (r')^2 \qquad r, r' \in (0,\infty)$$

$$\Rightarrow r = r'$$




$[0, 2\pi)$

$\mathbb{R}^2$

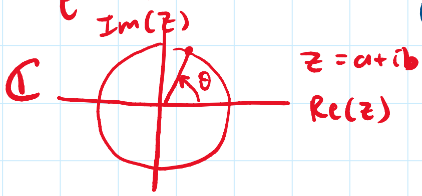
integer

$$\boxed{\cos\theta = \cos\theta' \qquad \sin\theta = \sin\theta'}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ $\qquad$ $e^{i\theta} = e^{i\theta'} \Rightarrow \theta - \theta' = 2\pi \ldots$

$\boxed{\cos\theta = \cos\theta' \qquad \sin\theta = \sin\theta'}$ integer

$e^{i\theta} = \cos\theta + i\sin\theta$




$$e^{i\theta} = e^{i\theta'} \Rightarrow \underset{[0,2\pi)}{\theta} - \underset{[0,2\pi)}{\theta'} = 2\pi \underset{0}{n}$$

$\mathbb{C} \simeq \mathbb{R}^2.$

$$\Rightarrow \theta = \theta'$$

$\Rightarrow T$ injective.

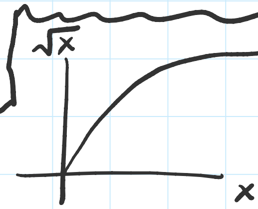
$$T(r,\theta) = (\underbrace{r\cos\theta}_{x(r,\theta)}, \underbrace{r\sin\theta}_{y(r,\theta)})$$

$$DT(r,\theta) = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det DT(r,\theta) = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

~SCRATCH



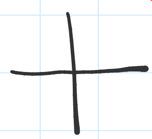

$$\sqrt{\sin^2 t \cos^2 t}$$

$t \in [0, \pi]$

$\sqrt{a^2} = |a|, \quad a \in \mathbb{R}$

$$\int f(\vec{c}(t))\|\vec{c}'(t)\|\,dt$$

$$\underbrace{\|\vec{c}'(t)\|}_{\geq 0} = \sqrt{g(t)^2} = |g(t)|$$



$t \in [0, 2\pi]$

$$\sqrt{\sin^2 t \cos^2 t} = |\sin t \cos t|$$

$[0, \pi/2]$ $\sin t \cos t \geq 0$

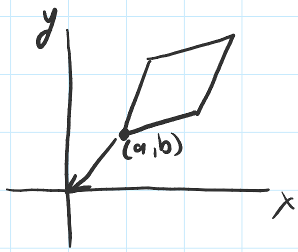
$[\pi/2, \pi]$ $-\sin t \cos t \geq 0$

$[\pi, 3\pi/2]$ $\sin t \cos t \geq 0$

$[3\pi/2, \pi]$ $-\sin t \cos t \geq 0$

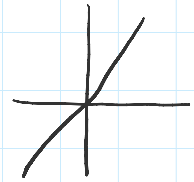
$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

$\sqrt{x^2} = |x|$




"linear transf."

$y = mx$



line

$y = mx + \underline{\underline{b}}$

"affine transf."