- Asynchronous lecture replacing the lecture on Friday 10/15.

Recall: a gradient vector field is a v.f. $\vec{F} = \nabla f$,
$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$.

Theorem (Fundamental Theorem of Line Integrals [FTLI])
Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1 and let $\vec{c} : [a, b] \to \mathbb{R}^n$ be a (piecewise) C^1 path. Then,

$$\int_{\vec{c}} \nabla f \cdot d\vec{r} = f\Big|_{\vec{c}(a)}^{\vec{c}(b)} = f(\vec{c}(b)) - f(\vec{c}(a)).$$

proof:

$$\int_{\vec{c}} \nabla f \cdot d\vec{r} = \int_a^b \underbrace{\nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t)}_{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)} \, dt$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{d}{dt} f(\vec{c}(t)) \quad \text{chain rule}$$

chain rule $\frac{d}{dt} f(\vec{c}(t)) = \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t)$

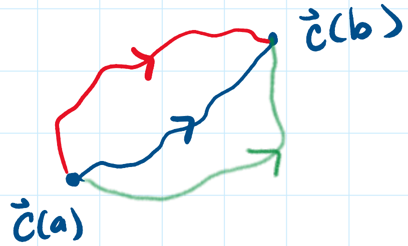
$$\Rightarrow \int_{\vec{c}} \nabla f \cdot d\vec{r} = \int_a^b \frac{d}{dt}\left[f(\vec{c}(t))\right] dt$$

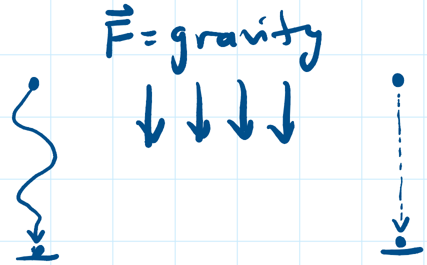
$$\underset{\text{FTC I}}{=} f(\vec{c}(t))\Big|_a^b = f(\vec{c}(b)) - f(\vec{c}(a))$$

$\square$

Line Integrals of ∇f, it only depends on the endpoints of the path

Line Integrals of ∇f, it only depends on the endpoints of the path





Conservation of Energy

Physics: Work - KE $\qquad$ Work $= \Delta KE$

change, kinetic energy

$$\vec{F} = -\nabla V$$

potential energy

$$W = \int_{\vec{c}} -\nabla V \cdot d\vec{r} = -V\Big|_{\vec{c}(a)}^{\vec{c}(b)} = -\Delta V$$

$$\Delta KE = W = -\Delta V$$

$$\Rightarrow \Delta(\underbrace{KE + V}_{\text{Energy}}) = 0$$

ex/ Let $\vec{F}(x, y, z) = (e^y, xe^y, 3z^2)$

Let $\vec{c}(t) = (t, t^2, t^3)$, $t \in [0, \pi]$.

Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{r}$

$$\vec{F} = \nabla f, \quad f = xe^y + z^3.$$

$\partial f/\partial x = e^y$
$\partial f/\partial y = xe^y$
$\partial f/\partial z = 3z^2$

$$\int_{\vec{c}} \vec{F} \cdot d\vec{r} = \int_{\vec{c}} \nabla f \cdot d\vec{r} = f\Big|_{\vec{c}(0)}^{\vec{c}(\pi)}$$

$\vec{c}(0) = (0, 0, 0)$
$\vec{c}(\pi) = (\pi, \pi^2, \pi^3)$

$$= f(\pi, \pi^2, \pi^3) - f(0, 0, 0)$$

$$= f(\pi, \pi^2, \pi^3) - f(0,0,0)$$
$$= \pi e^{\pi^2} + (\pi^3)^3 - 0 \cdot e^0 - 0^3$$
$$= \pi e^{\pi^2} + \pi^9 \qquad \square$$

ex/ Let $\vec{F}(x,y) = \left(\frac{1}{x}, \frac{1}{y}\right)$, defined on $(0,\infty) \times (0,\infty)$.

- Let $\vec{c}$ be a parametrization of a curve: circle of radius 1 centered at $(2,2)$ going counterclockwise; traversing 2π radians.
- Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{r}$

$$\vec{c}(t) = (2 + \cos t, \ 2 + \sin t)$$
$$t \in [0, 2\pi]$$

$\vec{F}(x,y) = \left(\frac{1}{x}, \frac{1}{y}\right)$ on $(0,\infty) \times (0,\infty)$

$\vec{F} = \nabla f$, $f = \ln(x) + \ln(y)$

$\partial f/\partial x = \frac{\partial}{\partial x} \ln(x) = \frac{1}{x}$

$\partial f/\partial y = \frac{\partial}{\partial y} \ln(y) = \frac{1}{y}$

$$\int_{\vec{c}} \vec{F} \cdot d\vec{r} = \int_{\vec{c}} \nabla f \cdot d\vec{r} = f \Big|_{\vec{c}(0)}^{\vec{c}(2\pi)}$$

$$= f(\vec{c}(2\pi)) - f(\vec{c}(0)) = 0$$

$\vec{c}(2\pi) = \vec{c}(0)$.

Theorem:
Let $\vec{F} = \nabla f$ be a (continuous) gradient vector field defined on $\mathbb{R}^n$ and let $\vec{c}: [t_0, t_1] \to \mathbb{R}^n$ be a (piecewise) C^1 path such that $\vec{c}(t_0) = \vec{c}(t_1)$ i.e. $\vec{c}$ is a closed

(piecewise) C^1 path such that $\vec{C}(t_0) = \vec{C}(t_1)$ i.e. $\vec{C}$ is a closed loop, then

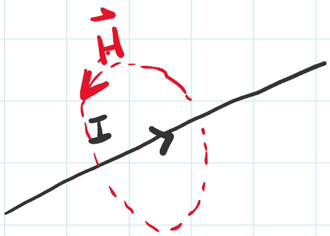
$$\oint_{\vec{C}} \nabla f \cdot d\vec{r} = 0$$

<u>proof</u>: FTLI, $\oint_{\vec{C}} \nabla f \cdot d\vec{r} = f\Big|_{\vec{C}(t_0)}^{\vec{C}(t_1)} = 0$ $\square$

············ END OF MATERIAL TESTED ON MT1 ·············

[NOT TESTED AT ALL]

Another use in physics: Ampere's law in electromagnetism



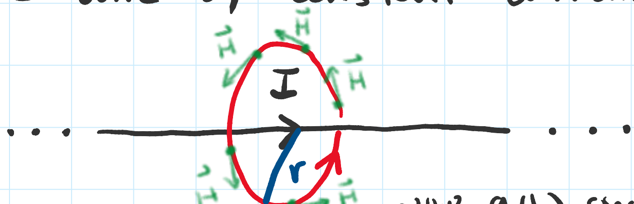
$$\oint_{\vec{C}} \vec{H} \cdot d\vec{r} = I$$

$\vec{C}$ is any closed loop surrounding the current

→ Math: given the vector field, what is the line integral.
Given $\vec{H}$ and $\vec{c}$, what is I?

→ Physics: Given I, choose $\vec{c}$ and determine $\vec{H}$?

Infinite wire w/ constant current I



$$I = \int_{\vec{C}} \vec{H} \cdot d\vec{r} \overset{\text{HW3 9(b) since } \vec{H} \parallel \vec{c}'}{=} \int_{\vec{C}} \|\vec{H}\|\, ds = \|\vec{H}(r)\| \int_{\vec{C}} ds$$

$$= \|\vec{H}(r)\|\, 2\pi r$$

$$\Rightarrow \|\vec{H}(r)\| = \frac{I}{2\pi r} \Rightarrow \vec{H}(r) = \frac{I}{2\pi r}\hat{\theta}$$ $\square$