Math 20E Lecture 1

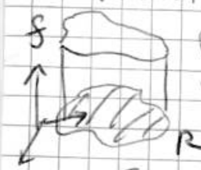
5.2 & 5.3 Double Integrals

Motivation: Vector calculus is our modern language to describe the physical world. E.g. Maxwell's equations

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c^2}\partial \vec{E}/\partial t$$

(electromagnetic waves in vacuum)

- Want to define



$$\iint_R f(x,y)\, dA \quad \text{where } f: \mathbb{R}^2 \to \mathbb{R} \text{ function of two variables}$$

(we only really need $f: R \to \mathbb{R}$ where $R \subseteq \mathbb{R}^2$)

To start, consider case where R is a rectangle $R = [a,b] \times [c,d]$.

Partition $[a,b]$ by $a = x_0 < x_1 < \dots < x_n = b$

$[c,d]$ by $c = y_0 < y_1 < \dots < y_n = d$

with $x_i - x_{i-1} = \frac{b-a}{n} = \Delta x$, $\quad y_j - y_{j-1} = \frac{d-c}{n} = \Delta y$.

Let c_{jk} be any point in $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$.

$\Rightarrow$ Riemann sum $\sum_{j,k=0}^{n-1} f(c_{jk})\,\Delta x \Delta y \equiv S_n$

The limit $\lim_{n\to\infty} S_n$, if it exists and is independent of c_{jk}, is the integral of f over R

$$\Rightarrow \iint_R f(x,y)\, dA = \lim_{n\to\infty} S_n.$$

- In this case, we say f is integrable over R.
- Ex/ continuous functions on closed rectangles are integrable. (More generally, the function can be discontinuous on curves if the function is bounded)

Properties: analogous to one-dim. integrals, double integrals are linear, monotone, and (domain) additive.

Theorem: (Fubini's)

Let $f: R \to \mathbb{R}$ be (and bounded) integrable, where $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx. \qquad \square$$

Thus, we can compute double integrals over rectangles as two one-dim. integrals

ex/ $$\int_0^1 \int_2^3 (e^x + y)\, dy\, dx = \int_0^1 \left(y e^x + \frac{y^2}{2} \right) \Big|_2^3 dx$$
$$= \int_0^1 \left(e^x + \frac{5}{2} \right) dx = \left(e^x + \frac{5}{2} x \right) \Big|_0^1 = e - 1 + \frac{5}{2}.$$

ex/ Let f be continuous on $[a,b]$ and g be continuous on $[c,d]$.

Let $R = [a,b] \times [c,d]$ and consider $fg: R \to \mathbb{R}$ defined by $(fg)(x,y) = f(x)g(y)$.

Then, fg is integrable on R and

$$\iint_R fg \, dA = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

proof: f is continuous on $[a,b]$ and g is continuous on $[c,d]$ imply fg is continuous on $R \Rightarrow fg$ is integrable over R.

By Fubini's theorem,

$$\iint_R fg\,dA = \int_c^d \left(\int_a^b f(x)g(y)\,dx\right) dy$$

pull outside, $g(y)$ independent of x

$$= \int_c^d g(y) \left(\int_a^b f(x)\,dx\right) dy$$

pull outside, independent of y

$$= \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right).$$

$\square$

ex/ Compute $\iint_R f\,dA$ where $f(x,y) = x^2\cos(y)$ and $R = [0,1] \times [0, \pi/2]$.

Compute:

$$\iint_R f\,dA = \int_0^{\pi/2}\int_0^1 x^2\cos(y)\,dx\,dy$$

$$= \left(\int_0^1 x^2\,dx\right)\left(\int_0^{\pi/2}\cos(y)\,dy\right)$$

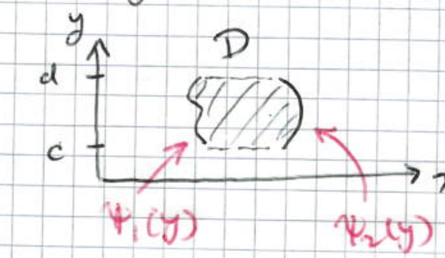
$$= \left(\frac{x^3}{3}\Big|_0^1\right)\cdot\left(\sin(y)\Big|_0^{\pi/2}\right)$$

$$= \left(\frac{1}{3}\right)\cdot(1) = 1/3.$$

$\boxed{5.3}$ Double Integrals over More General Regions

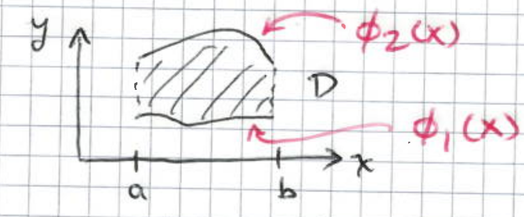
- Say a region D is x-simple if there are continuous functions ψ_1 and ψ_2 such that $\psi_1 \le \psi_2$ and

$$D = \{ (x,y) : y \in [c,d] \text{ and } \psi_1(y) \le x \le \psi_2(y) \}$$



- Similarly, a region D is y-simple if there are cont. functions ϕ_1 and ϕ_2 such that $\phi_1 \le \phi_2$ and

$$D = \{ (x,y) : x \in [a,b] \text{ and } \phi_1(x) \le y \le \phi_2(x) \}$$



Such x-simple & y-simple regions are called "elementary".

Given an elementary region, D, extend the domain to a rectangle R such that $D \subseteq R$. Given $f : D \to \mathbb{R}$, extend f to $f^* : R \to \mathbb{R}$ by $f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D. \end{cases}$

Define the integral of f over D as

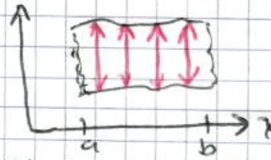
$$\iint_D f\, dA = \iint_R f^*\, dA, \quad \text{if it exists.}$$

If f^* is integrable over R, we say f is integrable over D.

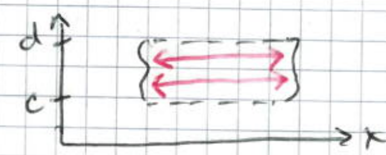
Note: $\iint_D f\, dA$ is independent of the choice of $R \supset D$.

Theorem: (General Elementary Regions by Iterated Integrals)

- Let D be y-simple, $D = \{(x,y) : x \in [a,b] \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$ and let f be integrable over D, then
$$\iint_D f\,dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$



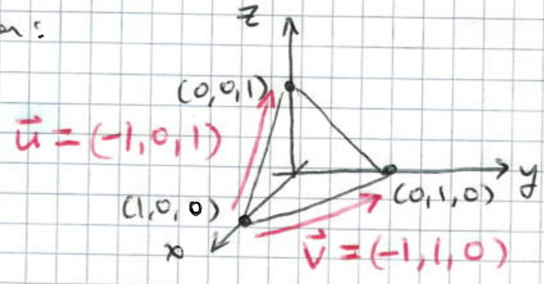
- Let D be x-simple, $D = \{(x,y) : y \in [c,d] \text{ and } \psi_1(y) \le x \le \psi_2(y)\}$ and let f be integrable over D, then
$$\iint_D f\,dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$



(show proof of one case)

ex/ Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, and the plane containing points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

Solution:



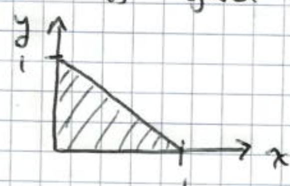
Normal vector to plane
$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1,-1,-1)$$

Equation for plane
$$\vec{n} \cdot (x-x_0, y-y_0, z-z_0) = 0$$
where $(x_0, y_0, z_0) \in$ plane.

$\Rightarrow (-1,-1,-1) \cdot (x-0, y-0, z-1) = 0$

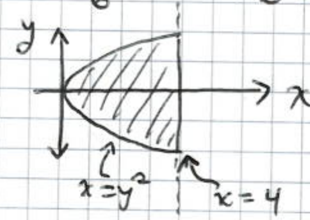
$\Rightarrow -x - y - z + 1 = 0 \Rightarrow z = 1 - x - y$. Think of z as function of x,y.

The volume is given by $\iint_D f\,dA$ where $f(x,y) = 1-x-y$ and

$D =$ 

This region is y-simple, (it is also x-simple)
$x \in [0,1]$, $0 \le y \le 1-x$

$$\Rightarrow V = \iint_D f\,dA = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$
$$= \int_0^1 \left((1-x)y - \frac{y^2}{2}\right)\Big|_0^{1-x} dx = \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2}\right) dx = \int_0^1 \frac{(x-1)^2}{2}\,dx$$
$$= \int_0^1 \frac{d}{dx}\left[\frac{(x-1)^3}{6}\right] dx = \frac{(x-1)^3}{6}\Big|_0^1 = 0 - \frac{(-1)^3}{6} = 1/6 \quad \square$$

ex/ $\iint_D f\,dA$ where $f(x,y) = x+y$, $D =$ 

x-simple region (also y-simple)

$\Rightarrow D = \{(x,y) : y \in [-2,2] \text{ and } y^2 \le x \le 4\}$

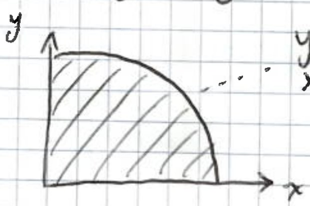
$$\Rightarrow \iint_D f\,dA = \int_{-2}^{2} \int_{y^2}^{4} (x+y)\,dx\,dy$$
$$= \int_{-2}^{2} \left(\frac{x^2}{2} + yx\right)\Big|_{y^2}^{4} dy = \int_{-2}^{2} \left(\frac{16}{2} - \frac{y^4}{2} + 4y - y^3\right) dy = \dots \quad \square$$

$\boxed{5.4}$ Changing Order of Integration

If a region is both x-simple and y-simple, one can change the order of integration.

Use this to evaluate whichever iterated integral is easier.

ex/ $$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dy\,dx \quad \text{where } a > 0.$$



$y = \sqrt{a^2-x^2}$
$x = \sqrt{a^2-y^2}$

The region, as given, is y-simple: $0 \le y \le (a^2-x^2)^{1/2}$, $0 \le x \le a$

but we see that it is also x-simple $0 \le x \le (a^2-y^2)^{1/2}$, $0 \le y \le a$.

By changing order,

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dy\,dx$$
$$= \int_0^a \int_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dx\,dy$$
$$= \int_0^a (a^2-y^2)\,dy = a^2y - \frac{y^3}{3}\Big|_0^a = a^3 - a^3/3 = \frac{2}{3}a^3$$

□

Thm
Suppose D is both x-simple and y-simple

$\psi_1(y) \le x \le \psi_2(y)$, $c \le y \le d$ $\qquad$ $\phi_1(x) \le y \le \phi_2(x)$, $a \le x \le b$

Then,
$$\iint_D f\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$
$$= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$