Math 20E A00 Fall 2021: Homework 8

Instructor: Brian Tran

Due Wednesday, November 24, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain (when applicable).

Problem 1 Exercise 4.4.23(a)

Let $\vec{F}(x, y, z) = (e^{xz}, \sin(xy), x^5y^3z^2)$.

(a) Find the divergence of $\vec{F}$.

Problem 2 The Product Rule for the Divergence

Show that the product rule for the divergence holds: for a differentiable scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ and a differentiable vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\nabla \cdot (f\vec{F}) = f\nabla \cdot \vec{F} + (\nabla f) \cdot \vec{F}.$$

Problem 3 Exercise 8.4.2

Verify the divergence theorem for the given region W, boundary ∂W with the induced normal (outward), and vector field $\vec{F}$:

$$W = [0, 1] \times [0, 1] \times [0, 1]$$
$$\vec{F}(x, y, z) = (zy, xz, xy).$$

That is, verify that the left and right hand sides of the divergence theorem give the same answer:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W (\nabla \cdot \vec{F})\, dV.$$

Problem 4 Exercise 8.4.9(b)

Let $\vec{F}(x, y, z) = (y, z, xz)$. Evaluate $\iint_{\partial W} \vec{F} \cdot d\vec{S}$ where $W = \{(x, y, z) : x^2 + y^2 \leq z \leq 1 \text{ and } x \geq 0\}$.

Problem 5 Exercise 8.4.11

Find the flux of the vector field $\vec{F}(x, y, z) = (x - y^2, y, x^3)$ out of the boundary ∂W, where $W = [0, 1] \times [1, 2] \times [1, 4]$.

Problem 6 Exercise 8.4.14

Let W be the three-dimensional solid enclosed by the surfaces $x = y^2, x = 9, z = 0, x = z$. Let $S = \partial W$. Use Gauss' theorem to find the flux of $\vec{F}(x, y, z) = (3x - 5y, 4z - 2y, 8yz)$ out of S: $\iint_S \vec{F} \cdot d\vec{S}$.

Hint: When you apply Gauss' theorem and end up with a triple integral, evaluate the integrals in the order $dz \to dx \to dy$.

Problem 7 Exercise 8.4.24

Suppose that $\vec{F}$ is tangent to the closed surface $S = \partial W$ for some region $W \subset \mathbb{R}^3$. Prove that

$$\iiint_W (\nabla \cdot \vec{F})\, dV = 0.$$

Hint: If $\vec{F}$ is tangent to S, what is its relation to the normal vector field $\hat{n}$ of S?

Problem 8 A higher-dimensional analogue of integration by parts

Let f be a (C^1) scalar function and $\vec{F}$ be a (C^1) vector field (on $\mathbb{R}^3$). Prove that, for any region $W \subset \mathbb{R}^3$,

$$\iiint_W (\nabla f) \cdot \vec{F}\, dV = \iint_{\partial W} f\vec{F} \cdot d\vec{S} - \iiint_W f\nabla \cdot \vec{F}\, dV.$$

Hint: Take the product rule from Problem 2, triple-integrate both sides over the region W, and apply the divergence theorem where appropriate.

Remark. This is a higher-dimensional analogue of integration by parts. Remember from single-variable calculus, integration by parts says that when you are integrating a function times the derivative of some other function, you can move the derivative over if you include a boundary term:

$$\int_I u\frac{dv}{dx}dx = uv\Big|_{\partial I} - \int_I \frac{du}{dx}v dx.$$

The proof of this fact just uses the single-variable product rule and the FTC I. If you look at the above equation, it is an analogous statement: you can move the derivative on f (the gradient) over to a derivative on $\vec{F}$ (the divergence) if you include an appropriate boundary term. Also, the proof in the higher-dimensional case is analogous, you use the product rule (for the divergence) and the fundamental theorem of calculus for the divergence (aka the divergence theorem).

There are other similar higher-dimensional integrations by parts formulas (for example, for the curl and using Stokes' theorem). These higher-dimensional integration by parts formulas are used extensively in higher level mathematics, especially in the area of partial differential equations (PDEs).