Math 20E A00 Fall 2021: Homework 7

Instructor: Brian Tran

Due Thursday, November 18, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain (when applicable).

Problem 1 Exercise 4.4.14

Compute $\nabla \times \vec{F}$, where

$$\vec{F}(x, y, z) = (yz, xz, xy).$$

Problem 2 The product rule for the curl

Show that the product rule for the curl holds: for a differentiable scalar function $f : \mathbb{R}^3 \to \mathbb{R}$ and a differentiable vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}.$$

Problem 3 Exercise 8.2.4

Verify Stokes' theorem for the given surface S, the given boundary ∂S, and the given vector field $\vec{F}$.

$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ (oriented with upward normal), $\partial S = \{(x, y, z) : x^2 + y^2 = 1, z = 0\}$ (with the induced orientation), and

$$\vec{F}(x, y, z) = (y, z, x).$$

That is, show that the left and right hand sides of Stokes' theorem give the same answer:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

Problem 4 Exercise 8.2.8

Let C be the closed, piecewise smooth curve formed by traveling in straight lines from $(0, 0, 0)$ to $(2, 1, 5)$ to $(1, 1, 3)$ and back to the origin. Use Stokes' theorem to evaluate the integral

$$\int_C (xyz, xy, x) \cdot d\vec{r}.$$

Problem 5 Exercise 8.2.11

Verify Stokes' theorem for the upper hemisphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, oriented with upward normal, and the radial vector field $\vec{F}(x, y, z) = (x, y, z)$.

That is, show that the left and right hand sides of Stokes' theorem give the same answer:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

Problem 6 Exercise 8.2.17

Calculate the surface integral

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S},$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1, x \geq 0$ (with normal pointing in the positive x direction), and $\vec{F}(x, y, z) = (x^3, -y^3, 0)$.

Problem 7 Exercise 8.2.22

Let S be a surface and assume that $\vec{F}$ is perpendicular to the tangent vector field along the boundary ∂S (i.e., when you parametrize the curve ∂S, $\vec{F}(\vec{c}(t))$ is perpendicular to $\vec{c}\,'(t)$). Show that

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0.$$

Problem 8 Exercise 8.2.24

For a surface S and a fixed vector $\vec{v} \in \mathbb{R}^3$, prove that

$$2 \iint_S \vec{v} \cdot d\vec{S} = \int_{\partial S} (\vec{v} \times \vec{r}) \cdot d\vec{r},$$

where $\vec{r}(x, y, z) = (x, y, z)$.

Hint: Compute the curl of $(\vec{v} \times \vec{r})$.