Math 20E A00 Fall 2021: Homework 6

Instructor: Brian Tran

Due Wednesday, November 10, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain (when applicable).

Problem 1 The vector surface integral on the unit sphere

Let $\mathbb{S}^2$ denote the unit sphere, $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, with orientation given by the outward normal vector, and let $\vec{F} : \mathbb{S}^2 \to \mathbb{R}^3$ denote a vector field defined on $\mathbb{S}^2$.

Parametrize $\mathbb{S}^2$ using spherical coordinates, $\Phi(\theta, \phi)$, and show that the normal vector

$$\vec{T}_\phi \times \vec{T}_\theta = \sin\phi \; \Phi(\theta, \phi);$$

that is, the normal vector at (θ, ϕ) points in the same direction as the position vector $\Phi(\theta, \phi)$ (and equals the position vector $\Phi(\theta, \phi)$ multiplied by $\sin\phi$). This is a special property of the sphere (draw it to convince yourself).

Let $F_r(\theta, \phi) = \vec{F}(\Phi(\theta, \phi)) \cdot \Phi(\theta, \phi)$ denote the radial component of $\vec{F}$ at the point $\Phi(\theta, \phi)$ (since Φ is the unit vector pointing in the radial direction, the dot product of $\vec{F}$ and Φ gives the radial component of $\vec{F}$). Show that

$$\iint_{\mathbb{S}^2} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} F_r(\theta, \phi) \sin\phi \, d\phi d\theta.$$

Remark. This problem shows that when computing the surface integral of a vector field out of a sphere, only the radial component of the vector field matters, since the normal vector field to a sphere is radial. As an example, we used this in Lecture 16 to compute the electric field due to a point charge.

Problem 2 Exercise 7.6.20(a)

A uniform fluid that flows vertically downward is described by the vector field $\vec{F}(x, y, z) = (0, 0, -1)$. Find the total flux through the cone S described by $z = (x^2+y^2)^{1/2}, x^2+y^2 \le 1$ (with downward pointing normal).

Problem 3 Exercise 8.1.3

Verify Green's theorem for the indicated region D, its boundary ∂D (with CCW orientation), and functions P and Q,

$$D = [-1, 1] \times [-1, 1], \; P(x, y) = -y, \; Q(x, y) = x$$

(that is, verify the left hand side and right hand side of Section 8.1 Theorem 1 are the same).

Problem 4 Exercise 8.1.4

Verify Green's theorem for the indicated region D, its boundary ∂D (with CCW orientation), and functions P and Q,

$$D = [-1, 1] \times [-1, 1],\ P(x, y) = x,\ Q(x, y) = y$$

(that is, verify the left hand side and right hand side of Section 8.1 Theorem 1 are the same).

Problem 5 Exercise 8.1.8

A particle travels across a flat surface, starting at the origin $(0, 0)$, moving east (i.e., positive x) 3 units, then north (positive y) 4 units, and then back to the origin along a straight line. A force field $\vec{F}(x, y) = (3x+4y^2, 10xy)$ acts on the particle. Use Green's theorem to find the work done on the particle by $\vec{F}$; i.e., use Green's theorem to evaluate the line integral of $\vec{F}$ along the path of the particle as a double integral.

Problem 6 Exercise 8.1.9

Evaluate $\int_C y dx - x dy$ where C is the boundary of the square $[-1, 1] \times [-1, 1]$ oriented in the CCW direction, using Green's theorem.

Problem 7 Exercise 8.1.11(a)

Verify Green's theorem for the disc D centered at the origin with radius R and the functions

$$P(x, y) = xy^2,\ Q(x, y) = -yx^2$$

(that is, verify the left hand side and right hand side of Section 8.1 Theorem 1 are the same).

Problem 8 Exercise 8.1.13

Find the area bounded by one arc of the cycloid $x(\theta) = a(\theta - \sin\theta), y(\theta) = a(1 - \cos\theta), a > 0, \theta \in [0, 2\pi]$ and the x-axis.

Hint: Use Section 8.1 Theorem 2 to write the area of the region as a line integral over the boundary of the area that you want. Note that boundary consists of two curves: the top arc of the cycloid and the straight line given by the $x-$axis (be careful that the parametrization you use is oriented in the CCW direction). Use software to plot a single arc of the cycloid if you are having trouble visualizing the region.

Problem 9 More practice with Green's Theorem

Evaluate

$$\int_C \left(e^{x^2+\sin(x)} - y^3\right) dx + \left(x^3 + \sin(y^4)\right) dy,$$

using Green's theorem, where C is the unit circle oriented CCW (don't try to evaluate the line integral directly; it is much harder).