Math 20E A00 Fall 2021: Homework 5

Instructor: Brian Tran

Due Wednesday, November 3, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain (when applicable).

Problem 1 Exercise 7.5.7

Compute $\iint_S xy dS$ where S is the surface of the tetrahedron with sides $z = 0, y = 0, x + z = 1, x = y$.

Problem 2 Exercise 7.5.20

Evaluate the integral

$$\iint_S (1 - z) dS,$$

where S is the graph of $z = 1 - x^2 - y^2$ with $x^2 + y^2 \leq 1$.

Problem 3 The First Fundamental Form of a Surface

Let $\Phi : D \subset \mathbb{R}^2 \to \Phi(D) = S$ be a parametrization of a surface S; in coordinates, we express as usual $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$. Let

$$E(u, v) = \left\| \frac{\partial \Phi}{\partial u} \right\|^2, \quad F(u, v) = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}, \quad G(u, v) = \left\| \frac{\partial \Phi}{\partial v} \right\|^2.$$

The matrix

$$\mathrm{I}(u, v) = \begin{pmatrix} E(u, v) & F(u, v) \\ F(u, v) & G(u, v) \end{pmatrix}$$

is known as the first fundamental form of the surface S. Show that

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{\det \mathrm{I}(u, v)},$$

where as usual, $\vec{T}_u = \partial\Phi/\partial u, \vec{T}_v = \partial\Phi/\partial v$. That is, show that the "Jacobian" of the surface area element is equal to the square root of the determinant of the first fundamental form.

Hint: Use Lagrange's identity for computing the magnitude of a cross product,

$$\|\vec{a} \times \vec{b}\| = \sqrt{\|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2},$$

choosing $\vec{a}$ to be $\vec{T}_u$ and $\vec{b}$ to be $\vec{T}_v$.

Remark. This problem shows that the first fundamental form encodes the "Jacobian" of the surface area element dS and hence, encodes information about the area of the surface S. In fact, the first fundamental form encodes **all** of the intrinsic geometric properties of the surface S, such as length, area, and curvature. In the field of differential geometry, this is known as the "metric tensor" of the surface S. If you're interested in learning more about this topic, I'd recommend the courses Math 150A/B: Differential Geometry (the prereqs for Math 150A are linear algebra and vector calculus).

Problem 4 Exercise 7.6.1

Consider the closed surface S consisting of two pieces: one piece of the surface is the graph of $z = 1-x^2-y^2$ with $z \geq 0$, and the other piece of the surface is the unit disc in the xy plane ($x^2 + y^2 \leq 1$). Give this surface an outer normal. Compute

$$\iint_S \vec{F} \cdot d\vec{S},$$

where $\vec{F}(x, y, z) = (2x, 2y, z)$.

Problem 5 Exercise 7.6.2

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S},$$

where $\vec{F}(x, y, z) = (x, y, z^2)$ and S is the surface parametrized by $\Phi(u, v) = (2\sin u, 3\cos u, v)$ with $u \in [0, 2\pi]$ and $v \in [0, 1]$.

Problem 6 Exercise 7.6.4

Let $\vec{F}(x, y, z) = (2x, -2y, z^2)$. Evaluate

$$\iint_S \vec{F} \cdot d\vec{S},$$

where S is the cylinder $x^2 + y^2 = 4$ with $z \in [0, 1]$ (either orientation of normal vector is okay for this problem, since it does not specify)

Problem 7 Exercise 7.6.8

Let the velocity field of a fluid be described by $\vec{F}(x, y, z) = (\sqrt{y}, 0, 0)$ (measured in meters per second). Compute the volumetric flow rate of the fluid (measured in cubic meters per second) that crosses the surface S given by $x^2 + z^2 = 1, y \in [0, 1], x \in [0, 1]$.

That is, compute the surface integral of the vector field $\vec{F}$ across the surface, $\iint_S \vec{F} \cdot d\vec{S}$ (either orientation of normal vector is okay for this problem, since it does not specify).

Problem 8 Exercise 7.6.13

Find the flux of the vector field $\vec{V}(x, y, z) = (3xy^2, 3x^2y, z^3)$ **out of** the unit sphere $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. That is, evaluate

$$\iint_{\mathbb{S}^2} \vec{V} \cdot d\vec{S},$$

where we take the outward normal vector (since the problem asks for the flux **out of** the sphere).

Problem 9 Practice computing a surface integral over a cube

Let S be the surface of a unit cube, whose six sides are given by $x = 0, y = 0, z = 0, x = 1, y = 1, z = 1$. In other words, S is the boundary of the volume $[0, 1] \times [0, 1] \times [0, 1]$.

Let $\vec{F}(x, y, z) = (x, y, z)$. Compute

$$\iint_S \vec{F} \cdot d\vec{S},$$

where the normal vector is taken to be outward facing.

Hint: Since the surface of the cube has 6 sides, the surface integral over the whole surface can be decomposed into 6 surface integrals over each side. Each of these surface integrals can be evaluated geometrically using Theorem 5 of section 7.6 (that is, you don't need to parametrize each side of the surface, you can just evaluate the integrals using geometry).