Math 20E A00 Fall 2021: Homework 3

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Due Wednesday, October 20, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain.

Problem 1 Exercise 7.1.24

Compute the path integral of $f(x,y) = y^2$ over the graph of $y = e^x, x \in [0,1]$.

Hint: You can use the result from Problem 12 of Homework 2.

Problem 2 Exercise 7.1.11(a)

Evaluate the path integral $\int_{\vec{c}} f ds$ where

$$f(x,y,z) = \exp(\sqrt{z}),$$
$$\vec{c} : t \mapsto (1,2,t^2),\ t \in [0,1].$$

Problem 3 Exercise 7.1.14

(a) Show that the path integral of $f(x,y)$ along a path given in polar coordinates by $r = r(\theta)$, $\theta \in [\theta_1, \theta_2]$, is

$$\int_{\theta_1}^{\theta_2} f(r(\theta)\cos\theta, r(\theta)\sin\theta)\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Hint: Think of θ as "time".

(b) Using the result from part (a), compute the arclength of the path

$$r(\theta) = 1 + \cos\theta,\ \theta \in [0, 2\pi].$$

Hint: You should find the half-angle identity $1 + \cos\theta = 2\cos^2(\theta/2)$ useful for evaluating the integral.

Problem 4 An Elliptic Integral

For the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > 0, b > 0)$, we can parameterize this curve as $\vec{c}(\theta) = (a\cos\theta, b\sin\theta), \theta \in [0, 2\pi]$ which traverses the ellipse counterclockwise. The arclength (or "perimeter") of an ellipse can be expressed $\int_{\vec{c}} ds$. Show that the arclength is given by the integral expression

$$\int_{\vec{c}} ds = \int_0^{2\pi} b\sqrt{1 + \frac{(a^2 - b^2)}{b^2}\sin^2\theta}\, d\theta.$$

Hint: During your calculation, use the identity $\cos^2\theta = 1 - \sin^2\theta$.

Remark. Do not actually try to evaluate this integral; there is no closed form expression for this integral (unless $a = b$ in which case we have a circle). This is known as an elliptic integral.

Problem 5 Exercise 7.2.2

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (y^2, 2xy)$ where C is the entire unit circle $x^2 + y^2 = 1$ parameterized counterclockwise.

Problem 6 Exercise 7.2.4(b),(d)

Evaluate each of the following line integrals

(b) $\int_{\vec{c}} x dx + y dy$ where $\vec{c}(t) = (\cos(\pi t), \sin(\pi t)), t \in [0, 2\pi]$.

(d) $\int_{\vec{c}} x^2 dx - xy dy + dz$ where $\vec{c}$ is the parabola $z = x^2, y = 0$ from $(-1, 0, 1)$ to $(1, 0, 1)$.

Problem 7 Exercise 7.2.5

Consider the force field $\vec{F}(x, y, z) = (x, y, z)$. Compute the work done in moving a particle along the parabola $y = x^2, z = 0$ for $x \in [-1, 2]$.

Problem 8 Exercise 7.2.8

Evaluate $\int_{\vec{c}} \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = (y, 2x, y)$ and the path $\vec{c}$ is given by $\vec{c}(t) = (t, t^2, t^3),\ t \in [0, 1]$.

Problem 9 Exercise 7.2.6

Let $\vec{c}$ be a smooth path (running from time t_1 to t_2).

(a) Suppose $\vec{F}(\vec{c}(t))$ is perpendicular to $\vec{c}\,'(t)$, for each t. Show that

$$\int_{\vec{c}} \vec{F} \cdot d\vec{r} = 0.$$

Remark. In terms of the interpretation of the line integral as the work done on a particle, this is the statement that if the force is perpendicular to the direction of motion, then no work is done on the particle.

(b) If $\vec{F}(\vec{c}(t))$ is parallel to $\vec{c}\,'(t)$ for each t, show that

$$\int_{\vec{c}} \vec{F} \cdot d\vec{r} = \int_{\vec{c}} \|\vec{F}\| ds.$$

That is, when the vector field is parallel to the velocity, the line integral can be interpreted as a path integral, where the scalar function that we are integrating is the magnitude of the vector field.

Hint: By parallel to $\vec{c}\,'(t)$, we mean that $\vec{F}(\vec{c}(t))$ points in the same direction as $\vec{c}\,'(t)$, so the angle $\theta(t)$ between the vectors $\vec{F}(\vec{c}(t))$ and $\vec{c}\,'(t)$ is zero. Thus,

$$\vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t) = \|\vec{F}(\vec{c}(t))\| \|\vec{c}\,'(t)\| \cos\theta(t) = \|\vec{F}(\vec{c}(t))\| \|\vec{c}\,'(t)\|.$$

Problem 10 Practice with the FTLI

Let $\vec{F}(x,y,z) = (2xy^3, 3x^2y^2, \cos(z)e^{\sin(z)})$.

(a) Assume that we know that $\vec{F}$ is a gradient vector field. Find the potential; that is, find $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

(b) Using the fundamental theorem of line integrals, evaluate $\int_{\vec{c}} \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is as in part (a) and $\vec{c}$ is the line starting at $(0,0,0)$ and ending at $(1,1,\pi)$.

Problem 11 Exercise 7.2.11

The image of the path $t \mapsto (\cos^3 t, \sin^3 t), t \in [0, 2\pi]$ in the plane is shown in Figure 7.2.15 (see textbook). Evaluate the integral of the vector field $\vec{F}(x,y) = (x,y)$ around this curve.

Hint: You can use the fundamental theorem of line integrals; find the scalar potential whose gradient is $\vec{F}$.

Problem 12 Exercise 7.2.17

Evaluate the line integral

$$\int_C 2xyzdx + x^2zdy + x^2ydz$$

where C is *any* oriented simple curve connecting (1,1,1) to (1,2,4).

Hint: Use the fundamental theorem of line integrals.