Math 20E A00 Fall 2021: Homework 2

Instructor: Brian Tran

Due Wednesday, October 13, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Although the problem may not ask for it explicitly, it is always helpful to sketch the domain.

Problem 1 Exercise 6.2.35(b)

The mapping $T : R^* \to R$ is defined by $T(u, v) = (u^2 - v^2, 2uv)$, where R^* is the region in the uv-plane $1 \leq u \leq 2, 1 \leq v \leq 3$ and $R = T(R^*)$ is the image of R^* under T.

Part (a) of this problem asks to show that this mapping is injective. We will just assume this (and it is also surjective since the codomain is defined to be the image of T). If you'd like, try showing that T is injective, but that won't be graded.

Find the area of R using the change of variables formula.

Hint: Note, you do not actually need to know what the image R is to solve this problem.

Problem 2 Practice computing the Jacobian

Consider the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(u, v, w) = (u^4 - v^4, uv, w)$. Compute its Jacobian determinant

$$\det DT(u, v, w).$$

Problem 3 Exercise 6.2.16

Let D be the unit disk, $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Compute

$$\iint_D (1 + x^2 + y^2)^{3/2} \, dxdy$$

using change of variables to polar coordinates.

Problem 4 Exercise 6.2.26

Use spherical coordinates to evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2 + y^2 + z^2}}{1 + (x^2 + y^2 + z^2)^2} \, dzdydx.$$

Hint: Draw the volume of integration in (x, y, z) space to determine what the bounds are in (ρ, θ, ϕ) space.

Problem 5 Ellipsoidal Spherical Coordinates

Consider the ellipsoid

$$W = \left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

where a, b, c are positive constants.

Consider the change of variables map $T : W^* \to W$ given by

$$T(\rho, \theta, \phi) = \Big(\underbrace{a\rho \sin(\phi)\cos(\theta)}_{=x(\rho,\theta,\phi)}, \underbrace{b\rho \sin(\phi)\sin(\theta)}_{=y(\rho,\theta,\phi)}, \underbrace{c\rho\cos(\phi)}_{=z(\rho,\theta,\phi)} \Big).$$

These are referred to as ellipsoidal spherical coordinates.

(a) What is the region W^* in (ρ, θ, ϕ) space that gets mapped to W?

(b) Compute $|\det DT(\rho, \theta, \phi)|$. Subsequently, use change of variables to find the volume of the ellipsoid given by

$$\mathrm{Vol}(W) = \iiint_W 1\, dxdydz.$$

Problem 6 Exercise 4.3.12

Sketch a few flow lines of the vector field

$$\vec{F}(x, y) = (x, -y).$$

(Start by sketching the vector field; then its flow lines)

Problem 7 Exercise 4.3.16

Show that the given curve $\vec{c}(t)$ is a flow line of the given velocity vector field $\vec{F}(x, y, z)$.

$$\vec{c}(t) = (t^2, 2t - 1, \sqrt{t}),\ \ t > 0;$$
$$\vec{F}(x, y, z) = \left(y + 1, 2, \frac{1}{2z} \right).$$

Problem 8 Finding the potential of a gradient vector field

Let $\vec{F}(x, y, z) = (e^x y, e^x + z, y)$. Assume that we somehow know that this vector field is actually a gradient vector field (we will see later in the course how we could know that).

Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

Problem 9 Exercise 7.1.2

Find an appropriate parametrization for the given piecewise-smooth curve in $\mathbb{R}^2$, with the implied orientation:

The curve C which goes along $y = x^2$ from the point $(0, 0)$ to the point $(2, 4)$, then in a straight line from $(2, 4)$ to $(0, 4)$, and then along the $y-$axis back to $(0, 0)$.

Problem 10 Exercise 7.1.8

Find an appropriate parametrization for the given curve in $\mathbb{R}^3$: the curve produced by the intersection of the cylinder $y^2 + z^2 = 1$ and the plane $z = x$.

Problem 11 Exercise 7.1.10

Evaluate the following path integrals $\int_{\vec{c}} f(x, y, z) ds$,

(a) $f(x, y, z) = x + y + z$ and $\vec{c} : t \mapsto (\sin t, \cos t, t), t \in [0, 2\pi]$.

(b) $f(x, y, z) = \cos(z)$ and $\vec{c}$ as in part (a).

Problem 12 Exercise 7.1.20

Show that the path integral of a function $f(x, y)$ over a path C given by the graph of $y = g(x), a \leq x \leq b$ is given by

$$\int_C f ds = \int_a^b f(x, g(x)) \sqrt{1 + [g'(x)]^2} dx.$$

Conclude that if $g : [a, b] \to \mathbb{R}$ is piecewise continuously differentiable, then the length of the graph of g on $[a, b]$ is given by

$$\int_C ds = \int_a^b \sqrt{1 + [g'(x)]^2} dx.$$

Hint: For the first part of this problem, we are free to choose the time parameter of the parameterization of the curve. For the *graph* of a function g, which is the set of points $\{(x, g(x)) : a \leq x \leq b\}$, it is natural to choose the time parameter to just be x.

For the second part of the problem concerning the length, recall that the length of a path C is just given by its path integral where the integrand is the function $f \equiv 1$.