Math 20E A00 Fall 2021: Homework 1

Instructor: Brian Tran

Due Wednesday, October 6, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 5.3.8 denotes exercise 8 of section 5.3. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Remark. Although the problem may not ask for it explicitly, it is always helpful to sketch the domain. This is especially useful for problems involving changing the order of integration, since seeing the domain will allow you to more easily interchange between x-simple and y-simple regions.

Problem 1 Exercise 5.2.2

Evaluate each of the following integrals for $R = [0,1] \times [0,1]$.

(a)

$$\iint_R (x^m y^n)\, dxdy, \ m, n > 0.$$

(b)

$$\iint_R (ax + by + c)\, dxdy, \ a, b, c \in \mathbb{R}.$$

(c)

$$\iint_R \sin(x + y)\, dxdy.$$

(d)

$$\iint_R (x^2 + 2xy + y\sqrt{x})\, dxdy.$$

Problem 2 Exercise 5.3.8

Let D be the region bounded by the positive x and y axes and the line $3x + 4y = 10$. Compute

$$\iint_D (x^2 + y^2)\, dA.$$

Problem 3 Exercise 5.4.2

Change the order of integration and evaluate

$$\int_0^1 \int_y^1 \sin(x^2)\, dxdy.$$

Problem 4 Exercise 5.4.5

Change the order of integration and evaluate

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dxdy.$$

Problem 5 Exercise 5.5.4

Evaluate the triple integral over the box $B = [0,1] \times [0,1] \times [0,1]$,

$$\iiint_B e^{-xy} y\, dxdydz.$$

Problem 6 Exercise 5.5.8

Describe the given region as an elementary region: the region cut out of the ball $x^2 + y^2 + z^2 \leq 4$ by the elliptic cylinder $2x^2 + z^2 = 1$; that is, the region inside the cylinder and the ball.

Problem 7 Exercise 5.5.11

Find the volume of the region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.

Problem 8 Exercise 5.5.18

Evaluate the triple integral

$$\iiint_W z\, dxdydz,$$

where W is the region bounded by the planes $x = 0, y = 0, z = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$ with $x \geq 0, y \geq 0$.

Problem 9 Exercise 6.1.1

Determine if the following functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one (injective) and/or onto (surjective). Do not just state an answer; explain why.

(a) $T(x,y) = (2x, y)$.

(b) $T(x,y) = (x^2, y)$.

(c) $T(x,y) = (x^{1/3}, y^{1/3})$.

(d) $T(x, y) = (\sin(x), \cos(y))$.

Problem 10 Exercise 6.1.4

Let D be (the region inside) a parallelogram with vertices $(0,0), (-1,3), (-2,0), (-1,-3)$. Let $D^* = [0,1] \times [0,1]$. Find a linear map T such that $T(D^*) = D$.

Problem 11 The Derivative Matrix of a Linear Transformation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation; letting (u, v) denote its inputs and $(x, y) = T(u, v)$ denote its outputs and letting $[T]$ denote the matrix representation of T. Recall the matrix representation of a linear transformation is just the matrix which satisfies

$$T(u, v) = [T] \begin{pmatrix} u \\ v \end{pmatrix},$$

where the right hand side of this equation is interpreted in the sense of matrix multiplication.

Letting $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = [T] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Show that the derivative matrix of T satisfies $DT(u, v) = [T]$; that is, show that the derivative matrix of a linear transformation equals the matrix which represents the linear transformation.

Subsequently, show that this implies that the Jacobian determinant of T equals the matrix determinant of $[T]$; i.e.,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det[T].$$

Problem 12 Exercise 6.2.2(a)

Suggest a substitution/transformation that will simplify the following integrand, and find its Jacobian.

$$\iint_R (5x + y)^3 (x + 9y)^4 dA.$$

Problem 13 Exercise 6.2.3

Let D be the unit disk $x^2 + y^2 \leq 1$. Evaluate

$$\iint_D \exp(x^2 + y^2) dx dy,$$

by making a change of variables to polar coordinates.

Problem 14 Changing an integral over a parallelogram to an integral over a rectangle

Consider the region D which is the area enclosed by the parallelogram with vertices $(0,0), (1,1), (1,2), (2,3)$. Find a linear transformation $T : D^* \to D$ which takes $D^* = [0,1] \times [0,1]$ to D; i.e., $T(D^*) = D$.

Subsequently, evaluate

$$\iint_D (x^2 + y)\, dxdy$$

by using change of variables to express this integral as an integral over D^*.

Problem 15 Exercise 6.2.19

Calculate

$$\iint_R (x+y)^2 e^{x-y}\, dxdy,$$

where R is the region bounded by $x+y=1, x+y=4, x-y=-1, x-y=1$

Hint: find a linear transformation such that the change of variables makes the domain easier to integrate over; e.g., a square or rectangle)

Problem 16 Exercise 6.2.15

Integrate $ze^{x^2+y^2}$ over the cylinder $x^2+y^2 \leq 4, 2 \leq z \leq 3$.

Hint: Use change of variables with cylindrical coordinates.

Problem 17 Exercise 6.2.25

Evaluate

$$\iiint_W \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}},$$

where W is the solid bounded by the two spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $0<b<a$.

Hint: Use change of variables with spherical coordinates.