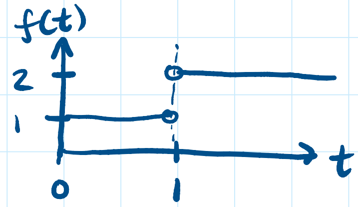
Math 20D Lecture 5: Laplace Transform

Tuesday, July 12, 2022 5:04 PM

ex/ $\dfrac{dx}{dt} = f(t)$, $\quad f(t) = \begin{cases} 1, & 0<t<1 \\ 2, & 1<t<\infty \end{cases}$

Solve the IVP $x(0)=0$ on the domain $t>0$.



Note f is continuous on $(0,1)$ and $(1,\infty)$

On $0<t<1$,

$$\frac{dx}{dt} = 1, \quad x(0)=0 \Rightarrow x(t) = t + C, \quad C = 0$$

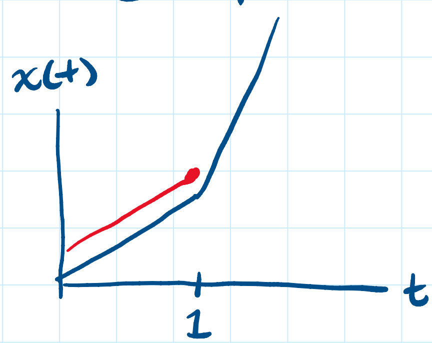
$$\Rightarrow x(t) = t, \quad 0 \le t < 1$$

$x(1)=1$, use this as an initial condition for the DE on $(1,\infty)$

$$\frac{dx}{dt} = 2, \quad x(1) = 1$$

$$x(t) = C + 2t \qquad 1 = x(1) = C + 2 \qquad C = -1$$

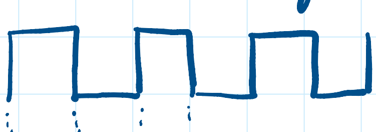
$$x(t) = 2t - 1, \quad 1 < t < \infty$$



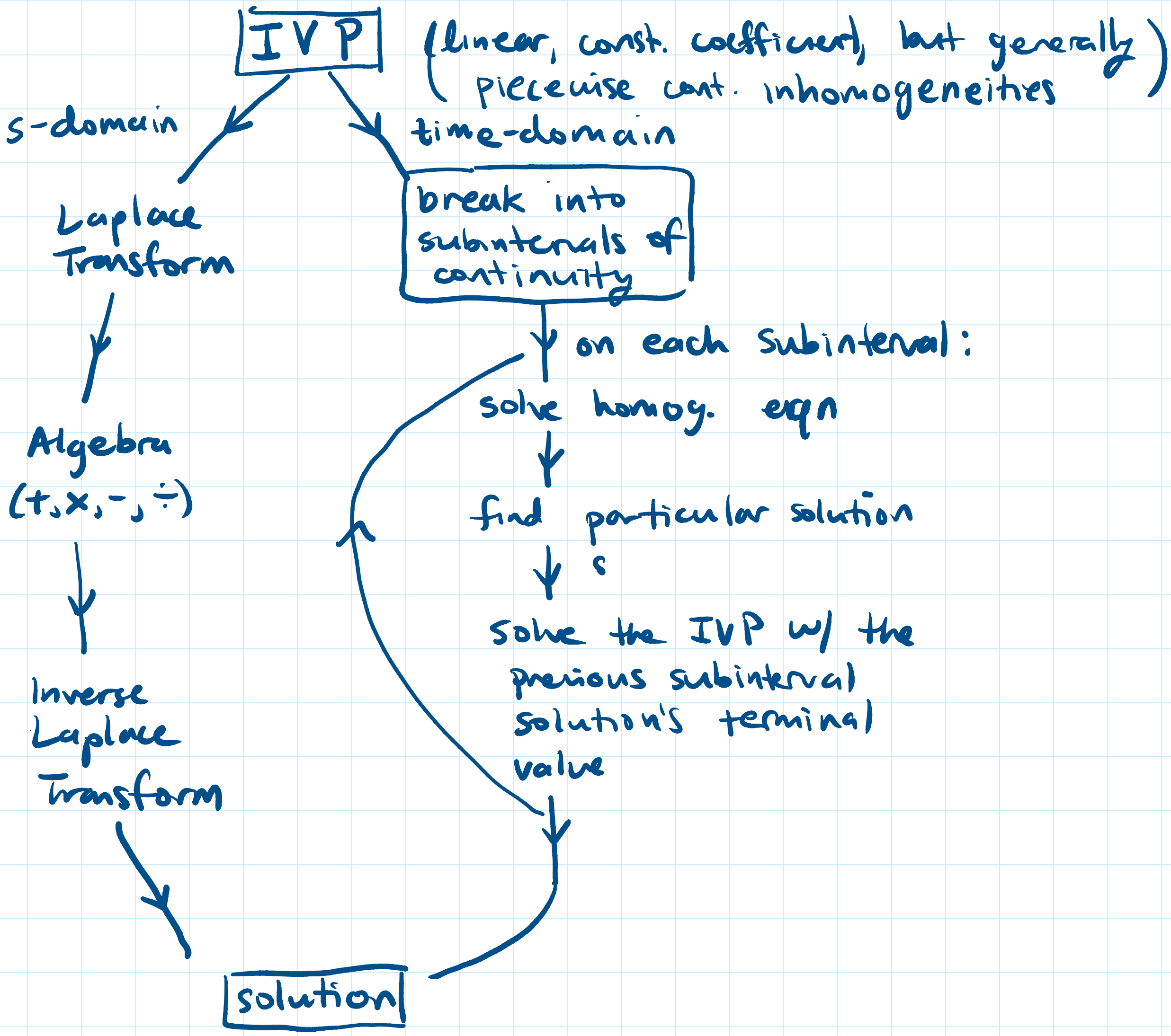
$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2t-1, & 1 \le t < \infty \end{cases}$$

$$x(t) = x(0) + \int_0^t f(s)\,ds$$

In many science & engineering applications, source/forcing terms have many discontinuous



IVP



(7.1 & 7.2)

Today: define and study how to compute the Laplace transform & its properties

<u>Def</u>: The Laplace transform (LT) of a function $f : [0, \infty) \longrightarrow \mathbb{R}$, denoted $\underline{\mathcal{L}(f)}$ or $\underset{\sim}{\tilde{f}}$,
(the textbook uses F)
is the function in the transform variable 's'
defined by

$$\tilde{f}(s) = \int_0^\infty \underline{e^{-st}} f(t)\, dt$$

OR

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$$

OR

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

The domain of $\tilde{f}$ (or $\mathcal{L}(f)$) is the set of all $s \in \mathbb{R}$ where the above integral exists. Note that this an improper integral $\int_0^\infty e^{-st} f(t)\,dt = \lim_{R\to\infty} \int_0^R e^{-st} f(t)\,dt$.

ex/ Compute $\mathcal{L}(f)$ where $f(t) = e^{at}$, $a \in \mathbb{R}$. ($f: [0,\infty) \to \mathbb{R}$)
Where is $\mathcal{L}(f)$ defined?

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt \qquad \text{exists only for } s>a$$

$$= \lim_{R\to\infty} \int_0^R e^{-(s-a)t}\,dt$$

$$= \lim_{R\to\infty} \left(\frac{-1}{s-a} e^{-(s-a)t} \Big|_0^R \right)$$

$$= \lim_{R\to\infty} \left(\frac{1}{s-a} - \frac{e^{-(s-a)R}}{s-a} \right) \qquad s>a$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$$

$$\mathcal{L} = $$

subcase $a=0$, $\mathcal{L}(1)(s) = \frac{1}{s}$, $s>0$.

ex/ $\mathcal{L}(f)$ where $f(t) = \sin(bt)$, $b \in \mathbb{R}$

$$\mathcal{L}(\sin(bt))(s)$$

$$= \int_0^\infty e^{-st} \sin(bt)\,dt$$

$$= \int_0 e \quad \sin(bt)\,dt$$

$$= \lim_{R\to\infty} \int_0^R e^{-st} \sin(bt)\,dt$$

$$I = \int_x^y e^{-st} \sin(bt)\,dt \qquad u = e^{-st} \quad dv = \sin(bt)\,dt$$

$$du = -se^{-st} \quad v = -\frac{\cos(bt)}{b}$$

$$= -e^{-st}\frac{\cos(bt)}{b}\Big|_x^y - \int_x^y \frac{s}{b} e^{-st} \cos(bt)\,dt$$

$$u = e^{-st}$$

$$du = -se^{-st}$$

$$dv = \cos(bt)\,dt$$

$$v = \frac{\sin(bt)}{b}$$

$$= -e^{-st}\frac{\cos(bt)}{b}\Big|_x^y - e^{-st}\frac{\sin(bt)}{b}\Big|_x^y$$

$$+ \frac{s}{b}\left(\frac{-s}{b}\right)\underbrace{\int_x^y e^{-st} \sin(bt)\,dt}_{I}$$

$$\frac{b^2+s^2}{b^2} \qquad -\frac{s^2}{b^2} I$$

$$\left(1+\frac{s^2}{b^2}\right) I = -e^{-st}\frac{\cos(bt)}{b}\Big|_x^y - e^{-st}\frac{\sin(bt)}{b}\Big|_x^y$$

$$I = \frac{b^2}{s^2+b^2}\left(\frac{-e^{-st}\cos(bt)}{b} - \frac{e^{-st}\sin(bt)}{b}\right)\Big|_{t=x}^{t=y}$$

$x=0, \ y=R$

$$I = \frac{b}{s^2+b^2}\left(-e^{-sR}\cos(bR) + 1 - e^{-sR}\sin(bR)\right)$$

$$\mathcal{L}(\sin(bt))(s)$$

$$= \lim_{R\to\infty} \frac{b}{s^2+b^2}\left(-e^{-sR}\cos(bR) + 1 - e^{-sR}\sin(bR)\right)$$

$$= \frac{b}{s^2+b^2}, \quad s>0$$

Table of Laplace Transform

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$f(t)$	$\mathcal{L}(f)(s)$	domain
1	$1/s$	$s>0$
e^{at}	$1/(s-a)$	$s>a$
→ $\sin(bt)$	$b/(s^2+b^2)$	$s>0$
$\cos(bt)$	$s/(s^2+b^2)$	$s>0$
$e^{at}t^n$ $(n=1,2,\dots)$	$n!/(s-a)^{n+1}$	$s>a$
→ $e^{at}\sin(bt)$	$b/((s-a)^2+b^2)$	$s>a$
$e^{at}\cos(bt)$	$s-a/((s-a)^2+b^2)$	$s>a$

(Linearity)
<u>Theorem</u>: Let f_1 and f_2 be two functions $[0,\infty)\to\mathbb{R}$ s.t. their LTs exist on $s>\alpha$ and let $c_1, c_2 \in \mathbb{R}$, then

$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2)$$

<u>Proof</u>: Follows from linearity of the integral

$$\mathcal{L}(c_1 f_1 + c_2 f_2)(s) \overset{(s>\alpha)}{}$$
$$= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$$
$$= c_1\int_0^\infty e^{-st} f_1(t)\,dt + c_2\int_0^\infty e^{-st} f_2(t)\,dt$$
$$= c_1\mathcal{L}(f_1)(s) + c_2\mathcal{L}(f_2)(s)$$

□

ex/ Compute $\mathcal{L}(3e^{2t} + 4\sin(3t))$.
Where is it defined?

$$= 3\underbrace{\mathcal{L}(e^{2t})(s)}_{} + 4\underbrace{\mathcal{L}(\sin(3t))(s)}_{}$$

$$= 3\underbrace{\mathcal{L}(e^{2t})(s)}_{s>2} + 4\underbrace{\mathcal{L}(\sin(3t))(s)}_{s>0}$$

$$= \frac{3}{s-2} + 4\,\frac{3}{s^2+9}, \quad s>2 \qquad \square$$

We want a sufficient condition for when the LT exists.

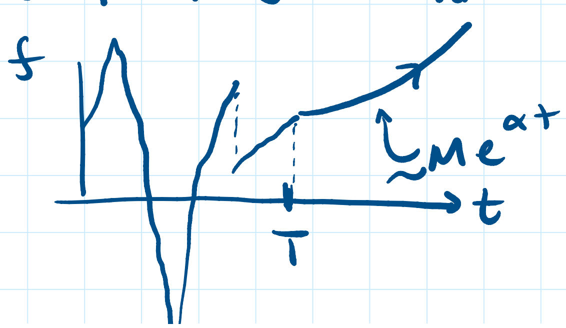
Def: A function $f: I \to \mathbb{R}$ (interval) is piecewise continuous if it is continuous at every point in I, except finitely many points where it may have a jump discontinuity

Not enough: every piecewise cont. function has a well-defined integral over a closed subinterval (of its domain I). However, $[0,\infty)$ is not closed; we need control of the integrand at ∞

We don't want f to grow too fast asymptotically as $t \to \infty$

Def: We say $f: [0,\infty) \to \mathbb{R}$ is of exponential order α if there exists constants $T \geq 0$, $M \geq 0$, $\alpha \in \mathbb{R}$ such that

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \geq T$$



e.g. logarithms, polynomials, sines, cosines, exponential, piecewise combinations of these

T

piecewise combinations of these

(e^{t^2} is not of exponential order)

Theorem: Let $f: [0,\infty) \to \mathbb{R}$ be piecewise continuous & be of exponential order α. Then, the LT of f exists for $s > \alpha$.

proof: Let T, M, α be as in the def. above

$$\int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^T e^{-st} f(t)\,dt}_{\checkmark} + \underbrace{\int_T^\infty e^{-st} f(t)\,dt}_{?}$$

Comparison

$$\left| \int_T^\infty e^{-st} f(t)\,dt \right| \le \int_T^\infty |e^{-st} f(t)|\,dt$$

$$= \int_T^\infty e^{-st} \underbrace{|f(t)|}_{\le Me^{\alpha t}}\,dt$$

$$\le M \int_T^\infty e^{-(s-\alpha)t}\,dt \quad \text{exists for } s > \alpha \qquad \square$$

Further properties (section 7.3)

Theorem: (Translation in s)

Suppose the LT of $f: [0,\infty) \to \mathbb{R}$ exists for $s > \alpha$

Then, $\mathcal{L}(e^{at} f(t))(s)$ exists for $s > a + \alpha$

and is given by $\mathcal{L}(e^{at} f(t))(s) = \mathcal{L}(f)(\underline{\underline{s-a}})$

translation of $\mathcal{L}(f)(s)$ to the right by a

proof:

$\mathcal{L}(e^{at} f(t))(s)$

$$= \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$>\alpha$

$s - a > \alpha$

$$= \int_0 e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^{\infty} e^{-\overbrace{(s-a)}^{>\alpha} t} f(t)\,dt \qquad s-a>\alpha \quad \text{i.e.,} \quad s>a+\alpha$$

$$= \mathcal{L}(f)(s-a) \qquad \square$$

Theorem (LT of a derivative)
Let $f: [0,\infty) \to \mathbb{R}$ be continuous and
$f': [0,\infty) \to \mathbb{R}$ be piecewise continuous,
both of exponential order α. Then, for $s>\alpha$,

$$\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$$

$$\mathcal{L}\left(\frac{df}{dt}(t)\right)(s) = s\,\mathcal{L}(f(t))(s) - f(0)$$

proof:

$$\mathcal{L}(f')(s) = \int_0^{\infty} e^{-st} f'(t)\,dt$$

$$= \lim_{R\to\infty} \left(\int_0^R e^{-st} f'(t)\,dt \right) \qquad \begin{matrix} u = e^{-st} & dv = f'(t)\,dt \\ du = -se^{-st} & v = f(t) \end{matrix}$$

$$= \lim_{R\to\infty} \left(f(t)\,e^{-st}\Big|_0^R + s\left(\int_0^R e^{-st} f(t)\,dt\right) \right)$$

$$= \underbrace{\lim_{R\to\infty} \underset{\sim e^{aR}}{f(R)}\, e^{-sR}}_{s>\alpha} - f(0) + s \underbrace{\int_0^{\infty} e^{-st} f(t)\,dt}_{s>\alpha\ \checkmark}$$

$$= s\,\mathcal{L}(f)(s) - f(0)$$

$= s\mathcal{L}(f)(s) - f(0)$

Corollary: (LT of Higher Derivatives)

Let $f, f', \ldots, f^{(n-1)} : [0,\infty) \to \mathbb{R}$ cont. and $f^{(n)} : [0,\infty) \to \mathbb{R}$ p.w. cont., all of exponential order α. Then,

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

(proof: use induction)

$$\mathcal{L}(f'')(s) = s^2 \mathcal{L}(f)(s) - s f(0) - f'(0) \qquad n=2.$$

ex Knowing $\mathcal{L}(\sin(bt))(s)$, $s>0$, compute $\mathcal{L}(\cos(bt))(s)$, $s>0$.

$$\cos(bt) = \frac{1}{b} \frac{d}{dt} \underbrace{\sin(bt)}_{f(t)}$$

$$\mathcal{L}\left(\frac{d}{dt} f(t)\right)(s) = s\,\mathcal{L}(f(t))(s) - f(0)$$

$$\| \qquad = s \frac{b}{s^2+b^2} - \cancel{\sin(0)}$$

$\mathcal{L}(b\cos(bt))(s)$

$b\,\mathcal{L}(\cos(bt))(s)$

$$\Rightarrow \mathcal{L}(\cos(bt))(s) = \frac{s}{s^2+b^2}, \quad s>0 \qquad \square$$

Thm: Let f be p.w. continuous on $[0,\infty)$ and of exponential order α. Then, its LT is (infinitely) differentiable on $s>\alpha$, and

$$\frac{d^n}{ds^n}\mathcal{L}(f)(s) = (-1)^n \mathcal{L}(t^n f(t))(s)$$

proof:

$$\frac{d^n}{ds^n}\mathcal{L}(f)(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st} \underline{f(t)}\, dt \qquad s>\alpha$$

$$= \int_0^\infty \frac{d^n}{ds^n}(e^{-st}) f(t)\, dt$$

$$= \int_0^\infty (-1)^n e^{-st} t^n f(t)\, dt$$

$$= (-1)^n \int_0^\infty e^{-st}(t^n f(t))\, dt$$

$$= (-1)^n \mathcal{L}(t^n f(t))(s) \qquad \square$$

ex $\mathcal{L}(t^n e^{at})(s)$ $\quad n=1,2,3,\ldots$, $\quad a\in\mathbb{R}$

$$= (-1)^n \frac{d^n}{ds^n}\mathcal{L}(e^{at})(s) \qquad , s>a$$

$$= (-1)^n \frac{d^n}{ds^n}\frac{1}{(s-a)}$$

$$= \underbrace{(-1)^n(-1)^n}_{=1}\frac{n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

ex Consider the DE $(t>0)$

ex/ Consider the DE $(t>0)$

$$y'' - y = -t, \quad y(0) = 0, \quad y'(0) = 1$$

Compute the LT of both sides

$$\mathcal{L}(y''-y)(s) = \mathcal{L}(-t)(s)$$

$$\mathcal{L}(y'')(s) - \mathcal{L}(y)(s) = -\mathcal{L}(t)(s)$$

$$s^2\mathcal{L}(y)(s) - sy(0) - y'(0) - \mathcal{L}(y)(s) = -1/s^2$$

Use initial conditions

$$(s^2-1)\mathcal{L}(y)(s) - 1 = -1/s^2 \longleftarrow$$

$$(s^2-1)\mathcal{L}(y)(s) = 1 - \frac{1}{s^2} = \frac{s^2-1}{s^2}$$

$$\mathcal{L}(y)(s) = \frac{1}{s^2}$$

$$\mathcal{L}(y(t))(s) = \frac{1}{s^2}$$

$\Rightarrow$ does this necessarily mean $y(t) = t$?

Note $\mathcal{L}(t)(s) = 1/s^2$

$\square$