Math 20D Lecture 2: First-Order Linear ODEs

Thursday, June 30, 2022 4:30 PM

- Typo on HW1 and Lecture 1
- Discussion section starts tomorrow
- Sections 2.3, 2.4, 2.5, maybe start Ch 4

Warm-up

- Consider the following 1^{st}-order linear homogeneous ODE

$$\boxed{\frac{dx}{dt} = x.}$$

- Solve the IVP with $x(t_0) = x_0 \in \mathbb{R}$, Hint: separable.

solution

- Know $\frac{d}{dt} e^t = e^t$ $\qquad \frac{d}{dt}(Ce^t) = Ce^t$

$$\Rightarrow \frac{dx}{dt} = x$$

$$x(t) = Ce^t$$

$$x_0 = x(t_0) = Ce^{t_0} \Rightarrow C = x_0 e^{-t_0}$$

$$\Rightarrow x(t) = x_0 e^{-t_0} e^t = x_0 e^{t-t_0}$$

- Using solution method for separable equations

$$\underline{\underline{\frac{dx}{dt} = x}} \overset{(x \neq 0)}{\Rightarrow} \int \frac{1}{x} dx = \int dt$$

$$\Rightarrow \ln|x| = t + c$$

$$\Rightarrow |x| = e^c e^t.$$

$$\Rightarrow x = \underbrace{(\pm e^c)}_{C_1} \underline{\underline{e^t}} \qquad \exp : \mathbb{R} \to (0, \infty)$$

$$x = \pm C_1 e^t \qquad C_1 > 0$$

$$x = C_2 e^t \qquad C_2 \neq 0$$

adding back in the case $x = 0$,

$$x(t) = Ce^t \qquad \underline{C \in \mathbb{R}}$$

use IV: $x(t) = x_0 e^{t-t_0}$

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General form of 1st-order linear DE

$$\left[\frac{dx}{dt} + P(t)x = Q(t)\right]$$

$Q \equiv 0$ homogeneous

$Q \not\equiv 0$ inhomogeneous.

2.3 Linear DEs

consider:

$$(*) \quad \frac{dx}{dt} - x = f(t)$$

aside:

$$\frac{dy}{dt} = f(t)$$

Textbook method:

$$\left(\frac{d}{dt} g(t,x) = h(t)\right)$$

multiply by $\mu(t)$ integrating factor

$$\underline{\mu(t)\frac{dx}{dt}} - \underline{\mu(t)x} = \mu(t)f(t)$$

$$\frac{d}{dt}(\mu(t)x) = \underline{\mu' x} + \underline{\mu x'} \Rightarrow$$

want

$$\mu' x = -\mu x$$

$$\mu' = -\mu \Rightarrow \mu(t) = e^{-t}$$

$$\Rightarrow \frac{d}{dt}(e^{-t}x) = e^{-t}f(t)$$

integrate

$$e^{-t}x = \int e^{-t}f(t)\,dt + C$$

$$\Rightarrow x(t) = e^{t}\left(C + \int e^{-t}f(t)\,dt\right)$$

$$\frac{d}{dt}x - x = f(t), \qquad x(t_0) = x_0$$

$$x(t) = e^{t}\left(C + \int_{t_0}^{t} e^{-s}f(s)\,ds\right)$$

$$x_0 = x(t_0) = e^{t_0}(C + 0) \Rightarrow C = x_0 e^{-t_0}$$

$$x(t) = e^{t}\left(x_0 e^{-t_0} + \int^{t} e^{-s} f(s)\,ds\right)$$

$$x(t) = e^{t}\left(x_0 e^{-t_0} + \int_{t_0}^{t} e^{-s} f(s)\,ds\right)$$
$$= \underline{x_0 e^{t-t_0}} + \underline{\int_{t_0}^{t} e^{t-s} f(s)\,ds}$$

Variation of parameters

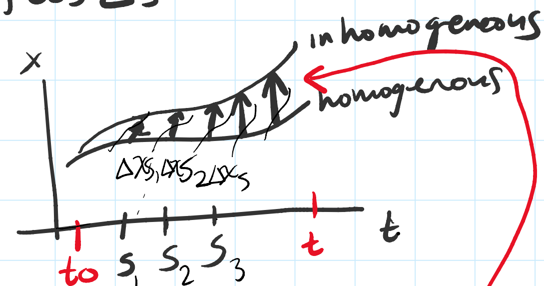
$$\frac{dx}{dt} = x + \underline{\underline{f(t)}} \qquad x(t_0) = x_0$$

in short time Δs, f adds to the position $\Delta x = f(s)\Delta s$
(starting at time s)

This is described by the homogeneous DE:

$$\frac{d}{dt}\widetilde{\Delta x_s} = \Delta x_s \qquad \Delta x_s(s) = f(s)\Delta s$$

$$\Rightarrow \Delta x_{\underline{\underline{s}}}(t) = f(s)\Delta s\, e^{t-s}$$



$$\underset{\text{inhomog.}}{X(t)} = \overset{\text{homog.}}{x_0 e^{t-t_0}} + \sum_{s} \Delta x_s(t)$$
$$= x_0 e^{t-t_0} + \sum_{s} e^{t-s} f(s)\, \Delta s \quad (\Delta s \to 0)$$
$$= x_0 \underline{\underline{e^{t-t_0}}} + \underline{\int_{t_0}^{t} \underline{\underline{e^{t-s}}}\, \underline{\underline{f(s)}}\,ds}$$

ex) $$\frac{dx}{dt} + x = t, \qquad x(0) = 2$$

multiply by e^{t}

$$e^{t}\frac{dx}{dt} + e^{t}x = e^{t}t$$

$$\underbrace{\qquad \frac{}{dt} \qquad}_{\frac{d}{dt}(e^t x)}$$

$$\Rightarrow \quad \frac{d}{dt}(e^t x) = e^t t$$

Integrate

$$\int_0^t \frac{d}{ds}(e^s x(s))\, ds = \int_0^t e^s s\, ds$$

IBP

$u = s \qquad dv = e^s ds$

$du = ds \qquad v = e^s$

$$e^t x(t) - x(0) = se^s \Big|_0^t - \int_0^t e^s ds = te^t - e^t + 1$$

$$\Rightarrow x(t) = e^{-t}(te^t - e^t + 1 + \overset{2}{x(0)})$$

$$= e^{-t}(te^t - e^t + 3)$$

$$\boxed{\frac{dy}{dt} + P(t) y = Q(t)}$$

this is equivalent to below

Multiply by $\mu(t)$

$$\underbrace{\mu(t)\frac{dy}{dt} + \mu(t) P(t) y}_{} = \mu(t) Q(t)$$

$$= \frac{d}{dt}(\mu(t) y(t)) = \mu' y + \mu y'$$

$$\Rightarrow \mu' y = \mu P y$$

$$\Rightarrow \mu' = \mu P$$

$$\Rightarrow \frac{d\mu}{dt} = P(t)\mu \qquad \text{is separable}$$

$\Rightarrow \frac{d\mu}{dt} = P(t)\mu$ is separable

$$\Rightarrow \int \frac{d\mu}{\mu} = \int P(t)\,dt$$

$$\ln|\mu| = \int P(t)\,dt + C$$

$$|\mu| = e^{C} e^{\int P(t)\,dt}$$

$$\mu = \underline{C} e^{\int P(t)\,dt}, \quad C \in \mathbb{R}$$

$$\boxed{\mu(t) = \exp\left(\int P(t)\,dt\right)}$$

$$\Rightarrow \frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)Q(t)$$ this is equivalent to above

integrate

$$\mu(t)y(t) = \int \mu(t)Q(t)\,dt + C$$

$$y(t) = \frac{1}{\underline{\mu(t)}}\left(\int \underline{\mu(t)Q(t)}\,dt + C\right)$$

ex/ (*) $\frac{dy}{dt} - \frac{2}{t}y = t^2\sin t, \quad t > 0$

$P(t) = -2/t$

$Q(t) = t^2 \sin t$

$$\mu(t) = \exp\left(\int P(t)\,dt\right) = \exp\left(\int (-2/t)\,dt\right)$$

$$= \exp(-2\ln t) = \exp(\ln(t^{-2})) = 1/t^2$$

multiply (*) by $\mu(t) = 1/t^2$

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2}{t^3}y = \sin t$$

$$\frac{d}{dt}\left(\frac{1}{t^2}y\right) = \sin t$$

$$\frac{d}{dt}\left(\frac{1}{t^2}y\right) = \sin t$$

integrate $$\frac{1}{t^2}y = \int \sin t\,dt + C = -\cos t + C$$

$$y = Ct^2 - t^2\cos t.$$

__Theorem:__

Consider the 1[st]-order linear ODE

$$(*)\ \frac{dy}{dt} = -P(t)y + Q(t) \qquad (\text{last lecture } = \underline{f(t,y)})$$

and suppose P, Q are continuous (a,b).

Then, the IVP $(*)$ with $y(t_0) = y_0$ $\quad (t_0 \in (a,b))$

has a unique solution $y: (a,b) \to \mathbb{R}$

__proof: see above.__ (existence, uniqueness) $\square$

Formula for solving linear IVP

$$\frac{d}{dt}y + P(t)y = Q(t), \qquad y(t_0) = y_0$$

$$\mu(t) = \exp\left(\int_{t_0}^{t} P(s)\,ds\right)$$

$$y(t) = \frac{1}{\mu(t)}\left(C + \int_{t_0}^{t} \mu(r)Q(r)\,dr\right)$$

$$= Ce^{-\int_{t_0}^{t} P(s)ds} + \int_{t_0}^{t} e^{\int_{t_0}^{r} P(s)ds - \int_{t_0}^{t} P(s)ds} Q(r)\,dr$$

$t = t_0$ plug in $y(t_0) = y_0 \Rightarrow C = y_0$

$$\Rightarrow y(t) = y_0 e^{-\int_{t_0}^{t} P(s)ds} + \int^{t} e^{-\int_{r}^{t} P(s)ds} Q(r)\,dr$$

$$\Rightarrow y(t) = \underbrace{y_0 e^{-\int_{t_0}^{t} P(s)ds}}_{\substack{\text{homog. solution} \\ \text{of IVP} \\ (Q \not\equiv 0)}} + \underbrace{\int_{t_0}^{t} e^{-\int_{r}^{t} P(s)ds} Q(r)\,dr}_{\substack{\text{contribution due to} \\ \text{inhomogeneity}}}$$

Exact Equations (2.4)

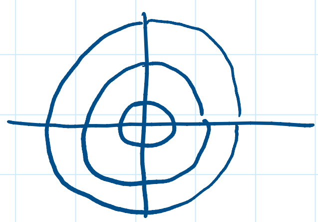
ex/ We saw $\underbrace{x^2+y^2}_{F(x,y)} = C$ satisfies the DE

$$\frac{dy}{dx} = \frac{-x}{y}$$

in other form, $y\,dy + x\,dx = 0$. $(*)$

$F(x,y) = x^2+y^2$

the level sets of F, $F(x,y)=C$, satisfies $(*)$



Total differential

$$dF(x,y) = \frac{\partial F}{\partial x}(x,y)\,dx + \frac{\partial F}{\partial y}(x,y)\,dy$$

$$= 2x\,dx + 2y\,dy$$

level set $dF = 0$

$\Rightarrow 0 = 2x\,dx + 2y\,dy$

Consider a general 1st-order ODE

$$(*)\quad P(x,y)\,dx + Q(x,y)\,dy = 0 \iff \underline{Q(x,y)\frac{dy}{dx} = -P(x,y)}$$

Def: We say the DE $(*)$ is exact if the left-hand side equals the total differential of some function $F: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$.

rectangle

$$\underline{dF} = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$\Rightarrow \quad P(x,y) = \frac{\partial F}{\partial x}(x,y) \qquad Q(x,y) = \frac{\partial F}{\partial y}(x,y)$$

Suppose we have an exact DE

$$(*)\quad \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

$$dF = 0$$

solutions of the DE $(*)$ are given implicitly by $F(x,y) = C$.

proof: implicitly diff wrt x

$$\frac{d}{dx}F(x,y) = \frac{d}{dx}C = 0$$

$$\frac{d}{dx}F(x,y) = \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$\partial x \qquad \partial y \quad dx$

Given a DE $\quad P(x,y)\,dx + Q(x,y)\,dy = 0$, is it exact?

Necessary condition:
there exists F: $\quad P = \frac{\partial F}{\partial x} \qquad Q = \frac{\partial F}{\partial y}$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial y\,\partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial x\,\partial y}$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y\,\partial x} - \frac{\partial^2 F}{\partial x\,\partial y} = 0.$$

A implies B
$A \Rightarrow B$
$B \Rightarrow A$
B implies A

<u>Theorem</u> $\left(F(x,y) = x^7 + y^5 \quad P = \frac{\partial F}{\partial x} = 7x^6, \quad Q = \frac{\partial F}{\partial y} = 5y^4 \right)$

$7x^6\,dx + 5y^4\,dy = 0$

Let P and Q be continuously differentiable functions from $D \subset \mathbb{R}^2 \to \mathbb{R}$
rectangle.

Then, the DE $\; P\,dx + Q\,dy = 0 \;$ is exact if and only if $\; \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \;$ throughout D.

<u>proof</u>: see textbook.

aside: $\left(\nabla \times \vec{G} = 0 \iff \vec{G} = \vec{\nabla} F \right)$

ex/ Solve the DE

$$(e^x \sin y + 2xy^2 + 1)\,dx + (e^x \cos y + 2x^2 y + 1)\,dy = 0$$

$$\underbrace{(e^x \sin y + 2xy^2 + 1)}_{P(x,y)} dx + \underbrace{(e^x \cos y + 2x^2y + 1)}_{Q(x,y)} dy = 0$$

Is it exact?

$$\frac{\partial P}{\partial y} = e^x \cos y + 4xy$$

$$\frac{\partial Q}{\partial x} = e^x \cos y + 4xy$$

$$\Rightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

$\Rightarrow$ eqn is exact ✓

How do we find $F(x,y)$ s.t. $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$?

$$P(x,y) = \frac{\partial F(x,y)}{\partial x}$$

integrate

1. $F(x,y) = \int P(x,y)\,dx + C_1(y)$

$$\left(\frac{\partial}{\partial x} F(x,y) = \frac{\partial}{\partial x} \int P(x,y)\,dx + \cancel{\frac{\partial}{\partial x} C(y)}^{\,0} = P(x,y)\right)$$

$$Q(x,y) = \frac{\partial F(x,y)}{\partial y}$$

2. $F(x,y) = \int Q(x,y)\,dy + C_2(x)$

$P(x,y) = e^x \sin y + 2xy^2 + 1$, $Q(x,y) = e^x \cos y + 2x^2y + 1$

(1) $F(x,y) = \int (e^x \sin y + 2xy^2 + 1)\,dx + C_1(y)$

$$= e^x \sin y + x^2y^2 + x + C_1(y)$$

(2) $F(x,y) = \int (e^x \cos y + 2x^2y + 1)\,dy + C_2(x)$

$$= e^x \sin y + x^2y^2 + y + C_2(x)$$

$$F(x,y) = \int (e^x \cos y + 2x^2 y + 1)\,dy + C_2(x)$$

$$= \underline{e^x \sin y} + \underline{x^2 y^2} + y + \underline{C_2(x)}$$

$$\Rightarrow F(x,y) = e^x \sin y + x^2 y^2 + x + y \qquad \left(\text{check: } P = \partial F/\partial x,\ Q = \partial F/\partial y\right)$$

solution to the DE is given implicitly $F(x,y) = C$.

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ex/ Solve the IVP

$$(3x^2 + y)\,dx + x\,dy = 0$$
$$y(1) = 2.$$

$$\left(\text{note linear } \frac{dy}{dx} = -\frac{1}{x} y - 3x\right)$$

Is it exact?

$P(x,y) = 3x^2 + y$

$Q(x,y) = x$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1 - 1 = 0$$

$\Rightarrow$ exact ✓

$$P = \frac{\partial F}{\partial x}, \quad Q = \frac{\partial F}{\partial y}$$

$$F(x,y) = \int P(x,y)\,dx + C_1(y) = \underline{\underline{x^3}} + \underline{xy} + \overset{0}{C_1(y)}$$

$$F(x,y) = \int Q(x,y)\,dy + C_2(x) = \underline{xy} + \underline{\underline{C_2(x)}}$$

$$\Rightarrow F(x,y) = x^3 + xy$$

Solution given $F(x,y) = C$

$$\Rightarrow x^3 + x y(x) = C \qquad \text{plugging in } y(1) = 2$$

$$1^3 + 1 \cdot y(1) = C$$

$$\Rightarrow C = 1 + 2 = 3$$

$$\Rightarrow y(x) = \frac{3}{x} - x^2$$

$$\Rightarrow \quad y(x) = \frac{3}{x} - x^2.$$

Integrating Factors and Exactness (2.5)

Consider $P(x,y)\,dx + Q(x,y)\,dy = 0$. Suppose it's not linear (section 2.3 does not apply) and suppose $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0$ (section 2.4 does not apply)

The DE is not exact, but what if

$$\overbrace{\mu(x,y)\,P(x,y)}^{\tilde{P}}\,dx + \overbrace{\mu(x,y)\,Q(x,y)}^{\tilde{Q}}\,dy = 0$$

is exact, for some μ?

We need:

$$(*)\; 0 = \frac{\partial \tilde{P}}{\partial y} - \frac{\partial \tilde{Q}}{\partial x} = \frac{\partial}{\partial y}(\mu P) - \frac{\partial}{\partial x}(\mu Q)$$

$$= \underline{\underline{\frac{\partial \mu}{\partial y}}}\,P + \mu \frac{\partial P}{\partial y} - \underline{\underline{\frac{\partial \mu}{\partial x}}}\,Q - \mu \frac{\partial Q}{\partial x} \qquad \text{PDE}$$

either assume μ only depends on x
or μ only depends on y

μ only depends on x

$(*)$: $$0 = \mu \frac{\partial P}{\partial y} - \frac{\partial \mu}{\partial x} Q - \mu \frac{\partial Q}{\partial x}$$

$$\frac{\partial \mu(x)}{\partial x} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu(x)$$

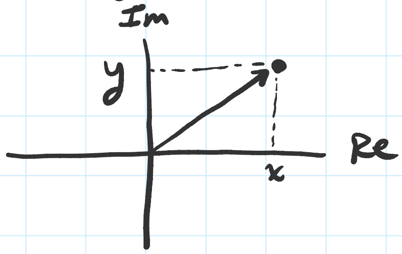
$$\frac{\partial \mu(x)}{\partial x} = \underbrace{\frac{1}{a}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}_{\text{works only if this depends only on } x.} \mu(x)$$

$$\Rightarrow \mu(x) = \exp\left(\int \frac{1}{a}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\right)$$

Review: complex numbers

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R},\ i = \sqrt{-1}\}$$

$z = x + iy$



$x = Re(z)$
$y = Im(z)$

Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

proof: $f(\theta) = e^{-i\theta}(\cos\theta + i\sin\theta)$

$$f'(\theta) = -ie^{-i\theta}(\cos\theta + i\sin\theta) + e^{-i\theta}(-\sin\theta + i\cos\theta)$$

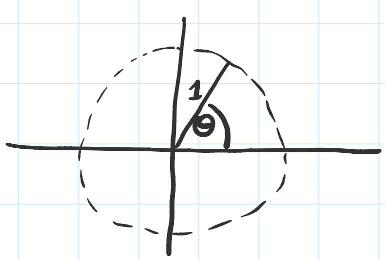
$$= e^{-i\theta}(\cancel{-i\cos\theta} + \cancel{\sin\theta} \cancel{- \sin\theta} + \cancel{i\cos\theta}) = 0$$

$f(\theta) = $ constant.

$$f(0) = e^0(\cos 0 + i\cancel{\sin 0}) = 1$$

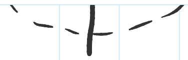
$$\Rightarrow e^{-i\theta}(\cos\theta + i\sin\theta) = 1$$

$\square$



$e^{i\theta}$

$$\cos^2\theta + \sin^2\theta = 1$$



Fundamental Theorem of Algebra

Every polynomial of degree $n \geq 1$ has n complex roots (including multiplicity)

$\square$