Math 20D Lecture 1: Intro, IVPs, and Separable Equations

Tuesday, June 28, 2022 4:34 PM

- Course Intro
- Review syllabus and course logistics, review sessions, office hours, #FinAid survey
- Canvas site
- Review Course Schedule
- Term paper overview
- Intro notation and Sections 1.1, 1.2, 2.2, maybe 2.3

Notation

· A function is a map $f: \underset{\text{domain}}{A} \longrightarrow \underset{\text{codomain}}{B}$ from a set A to a set B, which assigns to each x in A ($x \in A$) an element $f(x) \in B$.

$\mathbb{R} = \{\text{set of real numbers}\} = (-\infty, \infty)$

$\mathbb{C} = \{\text{set of complex numbers}\} = \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1}\}$

$\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R} = \{(a_1, \dots, a_n) : a_1, \dots, a_n \in \mathbb{R}\}$

In this course, we'll consider

$y : \overset{\text{time}}{I} \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is a subinterval

$y : I \to \mathbb{C}^n$

dependent variable, independent variable $x \in I$

$t \in I$

If y is differentiable, denote its derivative:

$\frac{d}{dx} y(x)$, $\frac{d}{dt} y(t)$

$y'(x)$, $\dot{y}(t)$

physics notation

If y is n times differentiable, denote n^{th} derivative

If y is n times differentiable, denote n^{th} derivative

$$\frac{d^n}{dx^n}y \quad \text{or} \quad y^{(n)}(x)$$

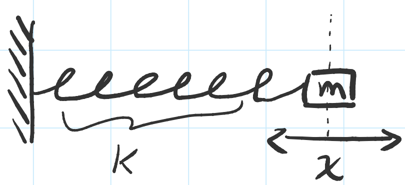
Section 1.1

- A differential equation (DE) is an equation involving an unknown functions and its derivative. These arise throughout science & engineering because they describe how a system changes through time or space.

- When the function only depends on one variable (t or x), we call the DE an ordinary differential equation (ODE) vs. partial diff. eq. (PDE) where the function depends on multiple variables

ex/ Hooke's Law

spring constant



$$m\frac{d^2x}{dt^2} = -kx$$

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = 0$$

Interested in:

Does a given ODE have a solution? existence

Is that solution unique? uniqueness

What are properties of the solution?

Is there a method to construct the solution?

Is there a method to construct the solution?

Def: The order of an ODE is the highest order derivative present in the equation

ex/ 1^{st} order $\frac{d}{dx} y = y$ $\quad a_1(x) = 1, \quad a_0(x) = -1$

ex/ 2^{nd} order $\frac{d^2}{dx^2} y = -\frac{dy}{dx} + \cos(y)$

Def: An n^{th} order ODE is linear if it can be expressed: $y \sim$ dep., $x \sim$ ind.

$$(*)\ a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x) y = f(x)$$

external force, source term

where $a_0, \dots, a_n, f$ are given.

If $f(x) \equiv 0$, the ODE is homogeneous.

If $f(x) \not\equiv 0$, the ODE is inhomogeneous.

ex/ $\frac{dy}{dx} = y^{2}$ nonlinear

$x^3 \frac{d^2y}{dx^2} + \sin(x)\frac{dy}{dx} + y = x^4$ inhomogeneous linear

$a_2(x) = x^3 \quad a_1(x) = \sin(x) \quad a_0(x) = 1 \quad f(x) = x^4$

Solutions

Solutions

- A general n^{th} order ODE can be expressed

$$\underline{F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0}$$

$$\boxed{\cos}\left(\frac{dy}{dx}\right) + y = 0$$

$$\hookrightarrow \frac{dy}{dx} = f(x,y) = \cos^{-1}(-y)$$

ex/ $F\left(x, y, \frac{dy}{dx}\right) = \frac{dy}{dx} + y$

- Any n^{th} order ODE can be transformed to the form

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right) \quad \underline{(*)}$$

(proof: implicit function theorem)

We say a function $y : I \to \mathbb{R}$ (or $\mathbb{C}$) is a solution if $(*)$ is true at every $x \in I$.

ex/ Verify $\underline{y = \ln|x|}$ solves the DE $\boxed{\frac{dy}{dx} = \frac{1}{x}}$ for $I = (0, \infty)$ and $I = (-\infty, 0)$

$$\frac{d}{dx} y(\underline{x}) = \frac{1}{\underline{x}} \text{ for all } x \in I$$

case $x > 0$

$y = \ln(x) \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

case $x < 0$

$y = \ln(-x)$ $\quad \frac{d}{dx} y = \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

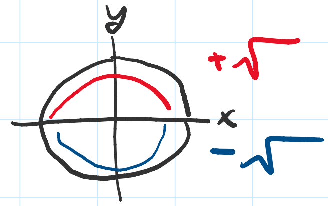
Solution not unique: $y = \ln|x| + \underline{\underline{C}}$

Def: We say a relation $g(x,y)=0$ is an implicit solution if it defines one or more solutions to $(*)$.

ex/ Show that $\overbrace{x^2+y^2-1}^{g(x,y)}=0$ is an implicit solution to the DE $y\frac{dy}{dx}+x=0$

method 1

solve for y: $y=\pm\sqrt{1-x^2}$



method 2

Implicit differentiation

$$0=x^2+y^2-1$$

$$0=\frac{d}{dx}\left(x^2+y^2-1\right)=2x+2y\frac{dy}{dx}$$

$$\Rightarrow\quad y\frac{dy}{dx}+x=0 \quad ✓$$

Initial Value Problem

Q: If I know how a car moves, i.e. its velocity, $v(t)=\frac{dx}{dt}$, can I determine its position $x(t)$? No, unless I specify the car's initial position $x(t_0)=x_0$

DE $\boxed{\frac{dx}{dt}=v(t)}$ ← x is an antiderivative of v

given

equipped w/ an initial condition $x(t_0)=x_0$.

given of v

equipped w/ an initial condition $x(t_0) = x_0$.

indef. integral

$$\Rightarrow \quad x(t) = \int v(t)\,dt + C$$

$$\int_{t_0 \leftarrow a}^{t} \frac{dx}{ds}\,ds = \underbrace{\int_{t_0 \leftarrow a}^{t} v(s)\,ds}_{\text{known } \checkmark}$$

$$\| $$

$$x(t) - x(a)$$

$$x(t) = \int_{t_0}^{t} v(s)\,ds + C$$

$$x_0 = x(t_0) = \cancel{\int_{t_0}^{t_0} v(s)\,ds}^{\,0} + C = C \Rightarrow C = x_0$$

$$\Rightarrow \quad x(t) = x_0 + \int_{t_0}^{t} v(s)\,ds$$

$$\int_1^2 f(t)\,dt = \#$$

$$x(t) = x(t_0) + \int_{t_0}^{t} \frac{dx}{ds}\,ds$$

$$\int_0^t f(s)\,ds = F(t)$$

$$\int f(t)\,dt = F(t)$$

$$\int \cos t\,dt = \sin t + C$$

$$\int_0^{\pi} \cos t\,dt = \sin t \Big|_0^{\pi} = 0$$

$$\int_0^{t} \cos s\,ds = \sin s \Big|_{s=0}^{s=t} = \sin t$$

ex) A car undergoes constant acceleration

$$\frac{d^2x}{dt^2} = \underline{a}, \quad a \in \mathbb{R}$$

$$x(t_0) = x_0 \qquad t_0 = 0$$

$$\frac{dx}{dt}(t_0) = V_0 \qquad t_0 = 0$$

integrate $\quad \dfrac{dx}{dt} = at + C_1$

again $\quad x(t) = \frac{1}{2}at^2 + C_1 t + C_2$

again $$x(t) = \frac{1}{2}at^2 + C_1 t + C_2$$

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

<u>Def</u>: An initial value problem (IVP) for an n^{th} order ODE $\frac{dx^n}{dt^n} = f\left(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$

is to solve this DE for $x(t)$ with the conditions

$$\left.\begin{aligned} x(t_0) &= x_0 \\ \frac{dx}{dt}(t_0) &= x_1 \\ &\vdots \\ \frac{d^{n-1}x(t_0)}{dt^{n-1}} &= x_{n-1} \end{aligned}\right\} \text{n initial conditions}$$

ex/ Solve the IVP by repeatedly integrating

$$\frac{d^3x}{dt^3} = t^2, \quad x(0) = 1, \quad \dot{x}(0) = 2, \quad \ddot{x}(0) = 0$$

Method 1: Indefinite integral

$$\frac{d^2x}{dt^2} = \int t^2\,dt = \frac{t^3}{3} + \overset{0}{\overset{\shortparallel}{C_1}}$$

$$\frac{dx}{dt} = \int \left(\frac{t^3}{3} + C_1\right) dt = \frac{t^4}{12} + C_1 t + \overset{2}{\overset{\shortparallel}{C_2}}$$

$$x = \int \left(\frac{t^4}{12} + C_1 t + C_2\right) = \frac{t^5}{60} + \frac{1}{2}C_1 t^2 + C_2 t + \overset{1}{\overset{\shortparallel}{C_3}}$$

$$x(t) = \frac{t^5}{60} + 2t + 1.$$

Method 2: Definite Integral

$d^3x \quad t^2$

Method 2: Definite Integral

$$\frac{d^3x}{dt^3} = t^2 \qquad\qquad t \to s$$

$$\int_0^t \frac{d^3x(s)}{ds^3}\,ds = \int_0^t s^2\,ds$$

$$\frac{d^2x(t)}{dt^2} - \cancel{\frac{d^2x(0)}{dt^2}}^{\,0 \ \text{IC}} = \frac{1}{3}t^3$$

$$\Rightarrow \frac{d^2x(t)}{dt^2} = \frac{1}{3}t^3$$

$$\frac{dx(t)}{dt} - \overset{2}{\frac{dx(0)}{dt}} = \int_0^t \frac{d^2x(s)}{ds^2}\,ds = \int_0^t \frac{1}{3}s^3\,ds = \frac{1}{12}t^4$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{12}t^4 + 2$$

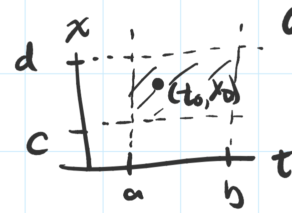
$$x(t) - \overset{1}{x(0)} = \int_0^t \frac{dx(s)}{ds}\,ds = \int_0^t \left(\frac{1}{12}s^4 + 2\right)ds = \frac{1}{60}t^5 + 2t$$

$$\Rightarrow x(t) = \frac{1}{60}t^5 + 2t + 1.$$

Theorem:

$$f: D \subset \mathbb{R}^2 \to \mathbb{R}$$

- Consider IVP $\frac{dx}{dt} = f(t,x)$, $x(t_0) = x_0$.
- Suppose f is continuously differentiable on some rectangle $D = (a,b) \times (c,d)$ containing (t_0, x_0),



then, the IVP <u>has</u> a unique solution on some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

some interval $(t_0-\varepsilon,\ t_0+\varepsilon)$ for some $\varepsilon>0$.

proof: Picard iteration
Picard-Lindelöff theorem.

Chapter 2: 1^{st}-order equations

Separable Equations (2.2)

Def.
A 1^{st}-order DE $\frac{dx}{dt}=f(t,x)$ is separable
if $f(t,x)=g(t)h(x)$. $\left(\cong \frac{g(t)}{p(x)}\right)$

$$\frac{dx}{dt}=g(t)h(x)$$

$$\int \frac{1}{h(x)}\,dx=\int g(t)\,dt$$

$$\Rightarrow \int \frac{1}{h(x)}\,dx=\int g(t)\,dt+C$$

relation
$g(t,x)=0$

proof: Let $H(x)=\int \frac{1}{h(x)}dx$, let $G(t)=\int g(t)dt$

$$\underbrace{\frac{1}{h(x)}\frac{dx}{dt}}=g(t)=\frac{d}{dt}G(t)$$

$$\frac{d}{dt}(H(x))=H'(x)\cdot\frac{dx}{dt}=\frac{1}{h(x)}\frac{dx}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(H(x)-G(t)\right)=0$$

$$\Rightarrow H(x)=G(t)+C \qquad \square$$

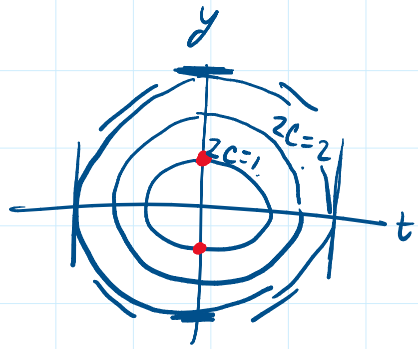
ex/ Solve $\frac{dy}{dt}=\frac{-t}{y}$

ex/ Solve $\frac{dy}{dt} = \frac{-t}{y}$

$\Rightarrow \int y\,dy = \int -t\,dt$

$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$

$\boxed{y^2 + t^2 = 2C}$



$a^2+b^2=r^2$

$2C = r^2 \Leftrightarrow r = \sqrt{2C}$

$y(0) = 1$ $\quad 1^2 + 0^2 = 2C \Rightarrow C = 1/2$

$y(0) = -1$ $\quad (-1)^2 + 0^2 = 2C \Rightarrow C = 1/2$

$y^2 + t^2 = 1$

$y(t) = \pm\sqrt{1-t^2}$

ex/ Solve IVP

$\frac{dy}{dt} = \frac{t^2+e^t}{y^2}, \quad y(0) = 3$

$y^2 dy = (t^2+e^t)\,dt$

$\int_{y(0)}^{y(t)} y^2 dy = \int_0^t (s^2+e^s)\,ds$

$\Rightarrow \frac{1}{3}y^3\Big|_{y=y(0)}^{y=y(t)} = \frac{1}{3}s^3 + e^s\Big|_0^t$

$\Rightarrow \frac{1}{3}y(t)^3 - \frac{1}{3}\underbrace{y(0)}_{=3}^3 = \frac{1}{3}t^3 + e^t - 1$

$y(t)^3 = y(0)^3 + t^3 + 3e^t - 3$

$$y(t)^3 = y(0)^3 + t^3 + 3e^t - 3$$
$$= 24 + t^3 + 3e^t$$
$$y(t) = (24 + t^3 + 3e^t)^{1/3}.$$