Math 20D Summer Session 1 2022: Homework 2

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Due Wednesday, July 13, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Problem 1 Exercise 4.2.2

Find a general solution to the given differential equation.

$$y'' + 6y' + 9y = 0.$$

Problem 2 Exercise 4.2.14

Solve the IVP

$$\begin{aligned} y'' + y' &= 0, \\ y(0) &= 2, \\ y'(0) &= 1. \end{aligned}$$

Problem 3 Exercise 4.3.10

Find a general solution to the given differential equation.

$$y'' + 4y' + 8y = 0.$$

Problem 4 Exercise 4.3.22

Solve the IVP

$$\begin{aligned} y'' + 2y' + 17y &= 0, \\ y(0) &= 1, \\ y'(0) &= -1. \end{aligned}$$

Problem 5 First-order Constant Coefficient Equations

Consider the first-order constant coefficient equation

$$ax'(t) + bx(t) = f(t),$$

where $a \neq 0$, $b \in \mathbb{R}$. We already know how to solve this type of equation using the method of integrating factors, but in this problem, we'll explore a different approach, analogous to how we solved second-order constant coefficient equations.

(a) The Solution for the Homogeneous Equation First, consider the homogeneous equation

$$ax'(t) + bx(t) = 0.$$

Analogous to what we did with second-order equations, plug in $x_1(t) = e^{rt}$. **What does r have to be in order for x_1 to be a solution of the homogeneous equation?**

With this chosen value of r, verify that $x_h(t) = Ce^{rt}$ is also a solution to the homogeneous equation, for any constant C (i.e., x_h is the general solution to the homogeneous equation). **Show that we can always find C so that x_h satisfies the initial value problem**

$$\begin{aligned} ax_h'(t) + bx_h(t) &= 0, \\ x_h(t_0) &= x_0. \end{aligned}$$

(b) The Solution for the Inhomogeneous Equation Suppose we have a particular solution of the inhomogeneous equation; i.e., we have found a function x_p such that

$$ax_p'(t) + bx_p(t) = f(t).$$

Analogous to what we did for second-order equations, we consider the solution to be the sum of the general homogeneous solution and the particular solution, $x(t) = Ce^{rt} + x_p(t)$, where r is chosen as in part (a).

Show that x satisfies the inhomogeneous equation. Furthermore, show that we can always find C so that x satisfies the initial value problem

$$\begin{aligned} ax'(t) + bx(t) &= f(t), \\ x(t_0) &= x_0. \end{aligned}$$

Problem 6 The Wronskian for Third-Order Equations

Consider the third-order, constant coefficient, linear, and homogeneous equation

$$ay''' + by'' + cy' + dy = 0,$$

where $a \neq 0, b, c, d \in \mathbb{R}$. Using again $y(t) = e^{rt}$, **what equation does r have to satisfy in order for y to be a solution?** (Hint: it is given by the roots of a cubic polynomial; the characteristic polynomial for a third-order equation is a cubic polynomial, and more generally the characteristic polynomial for an n^{th}-order equation is a polynomial of degree n). Note: Do not actually solve this equation (the roots of a cubic polynomial are given by the cubic formula, if you are interested in looking it up).

Now, suppose we have solved for the roots of the above characteristic polynomial; call them r_1, r_2, r_3. For the sake of simplicity, we will assume that all of the roots are distinct. Thus, we have three solutions for the above equation,

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}, y_3(t) = e^{r_3 t}.$$

We construct the general solution

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t).$$

We would like to see that this general solution is sufficient to solve any IVP of the form

$$\begin{aligned} ay''' + by'' + cy' + dy &= 0, \\ y(t_0) &= y_0, \\ y'(t_0) &= v_0, \\ y''(t_0) &= z_0. \end{aligned}$$

To do this, using the initial conditions, we obtain a linear system for the coefficients C_1, C_2, C_3 given by

$$\begin{pmatrix} y_0 \\ v_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) & y_3(t_0) \\ y_1'(t_0) & y_2'(t_0) & y_3'(t_0) \\ y_1''(t_0) & y_2''(t_0) & y_3''(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

Thus, we can generally solve this system for the coefficients C_1, C_2, C_3 if the 3×3 matrix appearing in the above equation is invertible. **Directly check that the above matrix is invertible for the above choices of y_1, y_2, y_3 by checking that its determinant is nonzero** (recalling that r_1, r_2, r_3 are assumed to be distinct). The determinant of the above matrix is the Wronskian for 3 functions.

Remark. The method above directly generalizes to n^{th}-order constant coefficient, linear, and homogeneous equations. In this setting, the Wronskian of n functions is

$$W(y_1(t), \dots, y_n(t)) = \det \begin{pmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{pmatrix}.$$

Problem 7 4.5.2(b)

Given that $y_1(t) = (1/4)\sin(2t)$ is a particular solution to $y'' + 2y' + 4y = \cos(2t)$ and that $y_2(t) = t/4 - 1/8$ is a particular solution to $y'' + 2y' + 4y = t$, use the superposition principle to find a solution to the following equation:

$$y'' + 2y' + 4y = 2t - 3\cos(2t).$$

Problem 8 Proving Uniqueness for Inhomogeneous Equations Using Uniqueness for Homogeneous Equations

Recall, in lecture 3, we proved that given a particular solution y_p to the inhomogeneous equation $ay_p'' + by_p' + cy_p = f(t)$, we can always construct a solution for the associated initial value problem,

$$\begin{aligned} ay'' + by' + cy &= f(t), \\ y(t_0) &= y_0, \\ y'(t_0) &= y_1, \end{aligned}$$

by taking y to be a sum of the general homogeneous solutions and the particular solution, and then appropriately choosing the coefficients. That is, we proved that a solution exists to the above initial value problem.

Now, we would like to prove that the solution to the initial value problem is unique. To do this: suppose we have two solutions $u(t)$ and $v(t)$ of the above initial value problem. Consider the difference of these two solutions $z(t) = u(t) - v(t)$. **What differential equation does z satisfy? What initial conditions does z satisfy? Subsequently, argue using uniqueness for homogeneous equations that $z(t) = 0$ for all t and hence, $u = v$; i.e., the solution is unique.**

Thus, we have proved that (given that we know that the solution to the homogeneous IVP is unique) the solution to the inhomogeneous IVP is unique.

Problem 9 Exercise 4.6.4

Find a general solution to the differential equation using the method of variation of parameters (that is, use variation of parameters to find a particular solution, and then add the general homogeneous solution to that particular solution).

$$y'' + 2y' + y = e^{-t}.$$

Problem 10 Exercise 4.6.5

Find a general solution to the differential equation using the method of variation of parameters (that is, use variation of parameters to find a particular solution, and then add the general homogeneous solution to that particular solution).

$$y''(\theta) + 16y(\theta) = \sec(4\theta).$$

Problem 11 Exercise 4.6.16

Find a general solution to the differential equation.

$$y'' + 5y' + 6y = 18t^2.$$

Problem 12 Exercise 4.7.4

Use Theorem 5 to find the largest interval I where a solution is defined for the following IVP

$$\begin{aligned} e^t y'' - \frac{y'}{t-3} + y &= \ln(t), \\ y(1) &= y_0, \\ y'(1) &= y_1. \end{aligned}$$

Problem 13 Exercise 4.7.20

Solve the given IVP for the Cauchy–Euler equation

$$\begin{aligned} t^2 y''(t) + 7ty'(t) + 5y(t) &= 0, \\ y(1) &= -1, \\ y'(1) &= 13. \end{aligned}$$

Problem 14 Exercise 4.7.32

Let y_1, y_2 be two solutions of the equation $y'' + p(t)y' + q(t)y = 0$ on an interval (a, b).

Prove that the Wronskian

$$W(y_1(t), y_2(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is given by Abel's formula

$$W(y_1(t), y_2(t)) = C \exp\Big(- \int_{t_0}^{t} p(s)ds \Big),$$

where $C = W(y_1(t_0), y_2(t_0))$ for some chosen $t_0 \in (a, b)$. To do this, see steps (a) and (b) below.

(a) Show that the Wronskian satisfies the differential equation

$$\frac{d}{dt} W + p(t)W = 0.$$

(b) Solve the separable equation in part (a).

(c) How does Abel's formula clarify that the fact that Wronskian (of two solutions) is either identically zero or never zero on (a, b)?