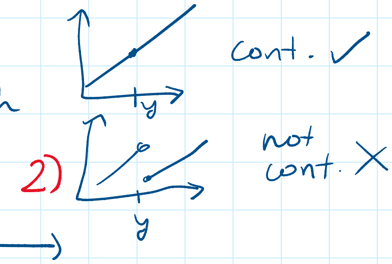
Lecture 3 - Continuity, Indeterminant Forms, & Limits at Infinity

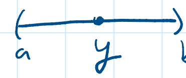
Tuesday, August 3, 2021 11:18 AM

[Read sections 2.4, 2.5, 2.7]

Continuous Function: Intuition ~ no breaks in graph



Let $f:(a,b) \to \mathbb{R}$, and let $y \in (a,b)$,



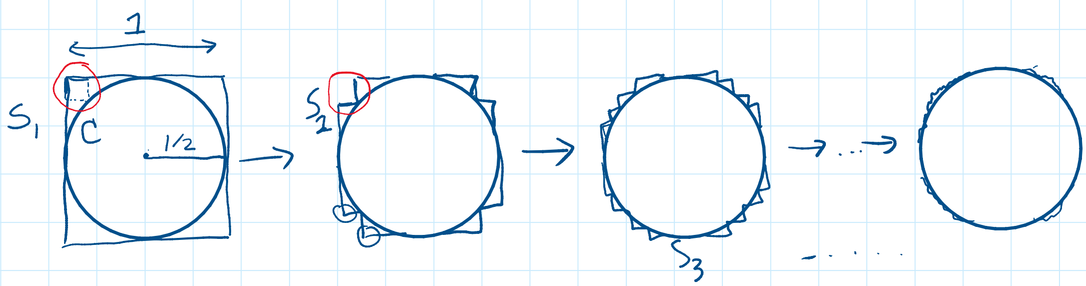
then we say f is continuous at y if: $\boxed{\lim_{x\to y} f(x) = f(y).}$

(In other terms, $\lim_{x\to y} f(x) = f(\lim_{x\to y} x)$)

if this does not hold, say f is discontinuous at y.
1) $\lim_{x\to y} f(x) \neq f(y)$
or 2) $\lim_{x\to y} f(x)$ DNE.

$A(\overset{0}{\vdash}\!\!-\!\!\overset{2}{\dashv}) = 2.$

ex/ A : curves $\longrightarrow \mathbb{R}^{\geq 0}$ A(curve) = arclength of curve



$A(C) = \pi.$ $A(S_1)=4$ $A(S_2)=4$ $A(S_n)=4$

$$\lim_{n\to\infty} A(S_n) = 4 \neq \pi = A(C) = A(\lim_{n\to\infty} S_n)$$

Arclength is not continuous!

sum from $k=0, 1, 2, \dots, n$.

ex/ Polynomial $f:\mathbb{R}\to\mathbb{R}$, $f = a_0 + a_1 x + \dots + a_n x^n = \sum_{k=0}^{n} a_k x^k$
is continuous at every $y \in \mathbb{R}$.
(use $\lim_{x\to y} x = y$ and power/sum limit laws).

Discontinuities

- Removable: discontinuity where $\lim_{x\to y} f(x)$ exists (and is finite) but does not equal $f(y)$.

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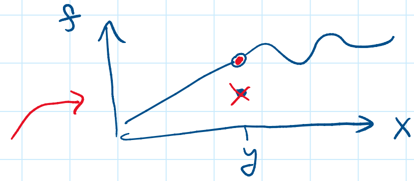
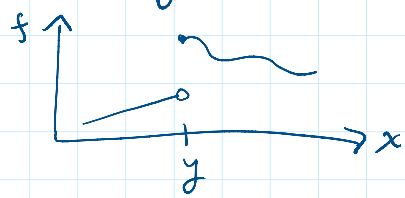
mild form of discont. because can redefine $f(y)$

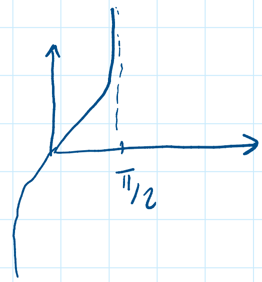


· Jump discontinuity: left & right one-sided limits exist but are unequal.



· Infinite discontinuity $\lim = \pm\infty$.

$f:(a,b) \to \mathbb{R}$ $\quad f(y)$ *

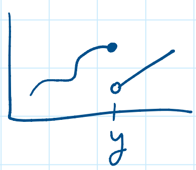


One-sided continuity:

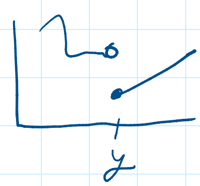
Function is left continuous at y if $\lim_{x \to y^-} f(x) = f(y)$

" " right " " $\lim_{x \to y^+} f(x) = f(y)$.

ex left



right



Continuity on intervals:

f is continuous at (a,b) if it is continuous at every $y \in (a,b)$.

f is cont. on:

$[a,b)$ if ↑ and is right continuous at a.

$(a,b]$ if ↑ and is left continuous at b.

$[a,b]$ if ↑ and is right cont. at a and left cont. at b.

<u>Laws of continuity</u>: Let f and g be continuous at y.

Sum $f+g$ is continuous

Product $f \cdot g$ is continuous (const. multiple kf is cont.)

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Quotient f/g is continuous when $g(y) \neq 0$.

ex) Polynomials ✓

Rational function $\frac{P}{Q}$ ← polynomials is continuous away from the zeroes of Q.

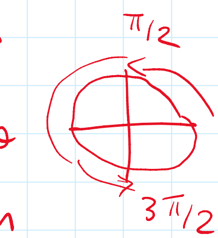
eg.) show that $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x^3+2x^2+1}{x^2+1}$ is continuous everywhere.

→ zeroes $x = \pm i$ imaginary. $\square$

Power $f(x) = x^n$ is continuous on its domain.

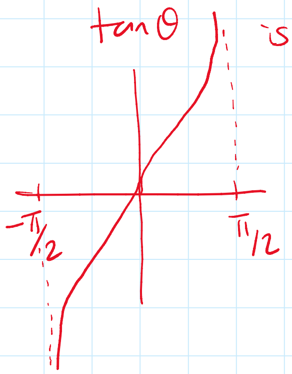
eg. $n = 1/2$ $f(x) = \sqrt{x}$ $f: [0,\infty) \to \mathbb{R}$.

Trig functions: $\sin, \cos : \mathbb{R} \to [-1,1]$ are continuous everywhere.



$\tan\theta = \frac{\sin\theta}{\cos\theta}$ ← cont. away from zeroes of $\cos\theta$

$\tan\theta$ is cont. except at $\theta = \begin{cases} \pi/2 \pm 2\pi m \\ 3\pi/2 \pm 2\pi k \end{cases}$ where m, k are any integers



Exponentials $f: \mathbb{R} \to (0,\infty)$ $f(x) = b^x$ $(b>0,\ b\neq 1)$ is cont. everywhere.

Logarithm $f^{-1}: (0,\infty) \to \mathbb{R}$ $f^{-1}(x) = \log_b x$ $(b>0,\ b\neq 1)$ is cont. everywhere.

$\mathbb{R} = (-\infty, \infty)$

<u>Thm</u>: Let $f: (a,b) \to I$ (the image of f) be invertible and continuous on (a,b). Then, $f^{-1}: I \to (a,b)$ is continuous on I.

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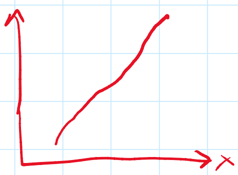
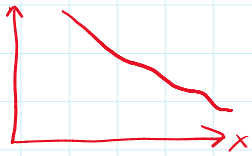
pf: f must be (strictly) monotone increasing or decreasing.

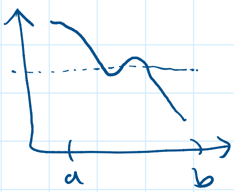




I is an interval.

$$\boxed{\lim_{y \to y_*} f^{-1}(y) = f^{-1}(y_*) = f^{-1}(f(x_*)) = x_*, \qquad y_* = f(x_*)}$$

not:



Thm If g is cont. at y and f is continuous at $g(y)$, then $F = f \circ g$ is continuous at y.

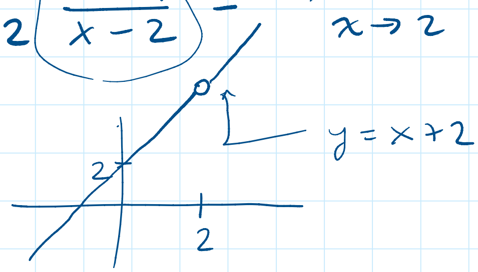
$F = f \circ g \qquad F(x) = (f \circ g)(x) \equiv f(g(x))$

$$\lim f(g(y)) \qquad \square$$

Indeterminant Forms

If $f(c)$ yields the expression $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$ we say f is an indeterminate form at c.

ex/ $\frac{0}{0}$ $\quad \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4.$

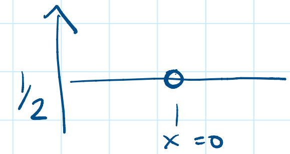


$X \to \infty$
$X^2 \to \infty$

$\frac{\infty}{\infty}$ $\quad \lim_{x \to \pi/2} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2} \frac{\sin x}{\cancel{\cos x}} \cdot \frac{1}{\left(\frac{1}{\cancel{\cos x}}\right)} = 1.$

$\frac{\infty}{\infty}$ $$\lim_{x\to\pi/2}\frac{\tan x}{\sec x} = \lim_{x\to\pi/2}\frac{\sin x}{\cancel{\cos x}}\cdot\frac{1}{\left(\frac{1}{\cancel{\cos x}}\right)} = 1.$$

$\infty\cdot 0$. $$\underbrace{\lim_{x\to 0}\cancel{x}\cdot\frac{1}{2\cancel{x}}} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}.$$



$\infty-\infty$ $$\lim_{x\to 2}\left(\frac{1}{x-2}\cdot\frac{x+2}{x+2} - \frac{4}{x^2-4}\right)$$

$$= \lim_{x\to 2}\frac{x+2-4}{x^2-4} = \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)}$$

$$= \lim_{x\to 2}\frac{1}{x+2} = 1/4.$$

Limits at Infinity

Say for

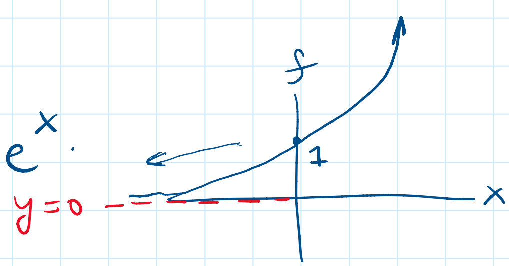
$+\infty$ $f:(a,\infty)\to\mathbb{R}$

that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets arbitrarily close to L when x is sufficiently large (and positive).

$-\infty$ $f:(-\infty,b)\to\mathbb{R}$

that $\lim_{x\to-\infty} f(x) = L$ ‖ (and negative)

ex/ $\underbrace{f:\mathbb{R}\to(0,\infty)}$ $f(x)=e^x$.



$$\lim_{x\to\infty} e^x = \infty.$$

$L<\infty$ x big enough $e^x \gg L$.

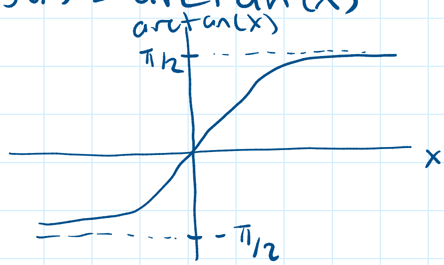
$$\lim_{x\to-\infty} e^x = 0.$$

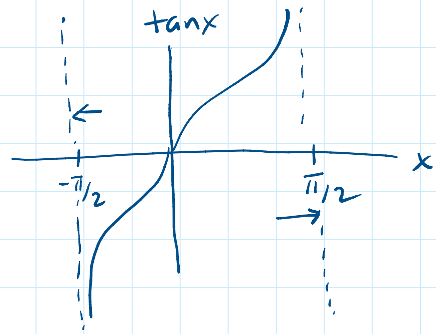
$$\lim_{x \to -\infty} e^x = 0.$$

$$\lim_{x \to -\infty} e^x = \lim_{x \to \infty} e^{-x} = \frac{1}{\lim_{x \to \infty} e^x}$$

If $\lim_{x \to +\infty \text{ (or } -)} f(x) = L$ exists, that limit is called a horizontal asymptote, $y = L$.

ex/ $f(x) = \arctan(x)$





$\tan : (-\pi/2, \pi/2) \to (-\infty, \infty) = \mathbb{R}$

$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$

$$\lim_{x \to \infty} \arctan(x) = \pi/2$$

$$\lim_{x \to -\infty} \arctan(x) = -\pi/2$$

$\Rightarrow$ arctan has horizontal asymptotes $y = \pi/2$, $y = -\pi/2$.

$$\lim_{x \to \pi/2} \tan x = \infty$$

$$\lim_{x \to -\pi/2} \tan x = -\infty$$

Polynomial & rational limits at infinity (next time).