

Practice Final:

Problem 1

Consider

$$f(x) = \frac{\ln(x) \cos(x^2)}{e^{2x}(x^2 + 1)}.$$

For what values of x is f defined? Using the derivative rules, compute the derivative of f .

• Is the denominator ever zero?

 $e^{2x} > 0$ for any $x \in \mathbb{R}$ $x^2 + 1 = 0 \iff x = \pm i$, non-real (imaginary)• Is the numerator defined for all x ? $\cos(x^2) \checkmark$ $\ln(x)$ only defined for $x > 0$

$\left. \begin{array}{l} f \text{ defined} \\ \text{on} \\ x > 0. \end{array} \right\}$

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{d}{dx} \left[\frac{\ln(x) \cos(x^2)}{e^{2x}(x^2 + 1)} \right] \\ &= \frac{g'h - h'g}{h^2} \end{aligned}$$

$$g' = \frac{d}{dx} [\ln(x) \cos(x^2)] = \frac{\cos(x^2)}{x} - \ln(x) \sin(x^2) \cdot 2x$$

$$h' = \frac{d}{dx} [e^{2x}(x^2 + 1)] = 2e^{2x}(x^2 + 1) + e^{2x} \cdot 2x$$

$$\begin{aligned} \frac{d}{dx} f &= \frac{\left(\left[\frac{\cos(x^2)}{x} - \ln(x) \sin(x^2) \cdot 2x \right] \cdot e^{2x}(x^2 + 1) \right.} \\ &\quad \left. - \left[2e^{2x}(x^2 + 1) + e^{2x} \cdot 2x \right] \cdot \ln(x) \cos(x^2) \right)}{e^{4x}(x^2 + 1)^2}. \end{aligned}$$

Problem 2

Consider $f : (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \ln(x), & 0 < x \leq e \\ \cos(\pi(x - e)), & e < x < \infty \end{cases}$$

Is f continuous on $(0, \infty)$ (why or why not)? Evaluate the limits

yes

$$\lim_{x \rightarrow e^-} e^{-\ln(x)} = \lim_{x \rightarrow e^-} e^{-f(x)}$$

$\underbrace{\ln(e) = 1}_{f(e)}$

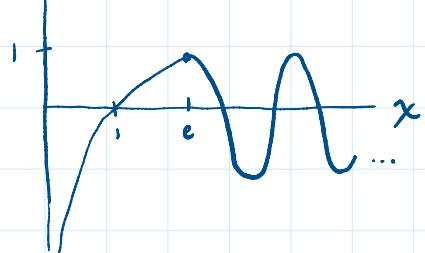
$$\lim_{x \rightarrow e^-} f(x) = f(e) = 1$$

$$\lim_{x \rightarrow 1+e} f(x) = f(1+e) = \cos(\pi(1+e-e)) = \cos(\pi) = -1.$$

↑ if f continuous

$$\left(\lim_{x \rightarrow 0^+} f(x) = -\infty \right)$$

$$\left[\begin{array}{l} \lim_{x \rightarrow 0^+} f(x) = -\infty \\ \lim_{x \rightarrow -\infty} e^x \downarrow 0. \end{array} \right]$$



$$\lim_{x \rightarrow 1+e} f(x) = f(1+e) = \cos(\pi(1+e-e)) = \cos(\pi) = -1.$$

↑ if f continuous.

$$\begin{aligned} \lim_{x \rightarrow e^+} f(x) &= \lim_{x \rightarrow e^+} \cos(\pi(x-e)) \\ &= \cos(\pi \cdot 0) = 1 = f(e) \end{aligned}$$

f continuous on $(0, e)$ because $\ln(x)$ is

f continuous on (e, ∞) because $\cos(\pi(x-e))$

f continuous at $x=e$

$$\text{because } \lim_{x \rightarrow e} f(x) = 1 = f(e).$$

Problem 3

Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x|x|$ (that is, $f(x)$ is given by the product of x and the absolute value of x).

(a) Show that f is continuous on \mathbb{R} .

(b) Show that the first derivative f' is defined on all of \mathbb{R} . Also, show that $f' : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Hint: Consider the cases $x < 0, x = 0, x > 0$ separately. For $x = 0$, use the limit definition of the derivative and split it into left and right sided limits.

(c) Show that the second derivative f'' is defined on the regions $x < 0$ and $x > 0$. Show that f'' is discontinuous at $x = 0$; what kind of discontinuity does f'' have at $x = 0$?

MTI

$$f(x) = |x|^3$$

(a) f is the product of two continuous functions

$$f = g \cdot h, \text{ where } \underbrace{g(x) = x}_{\text{continuous on } \mathbb{R}}, \underbrace{h(x) = |x|}_{\text{continuous on } \mathbb{R}}.$$

(apply product law of continuity)

$$(b) f(x) = x|x| = x \cdot \underbrace{\begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}}_{|x|} = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$\text{On } x < 0, f'(x) = \frac{d}{dx} (-x^2) = -2x$$

$$\text{On } x > 0, f'(x) = \frac{d}{dx} (x^2) = 2x.$$

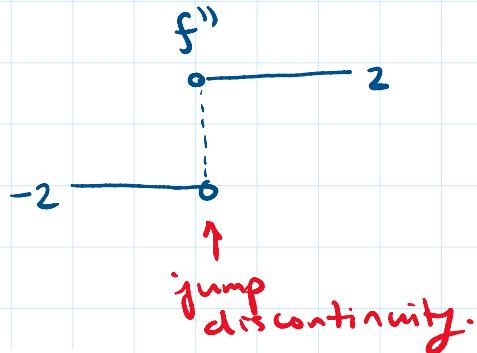
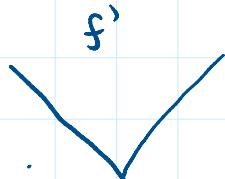
$$\text{At } x = 0,$$

$$\dots \text{r.h.s.} \text{ l.h.s.} \dots f(0) = 0 \dots \Delta h \Delta h$$

At $x = 0$,
 $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$
 $= \lim_{h \rightarrow 0} |h| = |0| = 0.$ $\Rightarrow f'(0)$ exists & equals 0.
 continuity

$$f'(x) = \begin{cases} 2x, & x > 0 \\ 0, & x = 0 \\ -2x, & x < 0 \end{cases} = 2|x| \leftarrow f' \text{ is continuous on } \mathbb{R}.$$

(c) $f'' = \frac{d}{dx}(2|x|)$

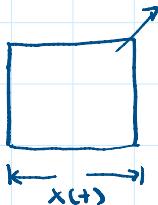


Problem 4

Imagine a square whose sides are changing in time, from $t = 0$ to $t = 3$. At time $t = 0$, the side length of the square is $x(0) = 0$. The side length of the square is given as a function of time,

$$x(t) = 3t - t^2, \quad 0 \leq t \leq 3.$$

- (a) At time $t = 1$, at what rate is the area of the square expanding?
 (b) Find the time at which the area is at a maximum (make sure to show that this is the maximum, using an appropriate theorem/method).



(a) At time $t = 1$, what is $A'(1)$?

$$\text{where } A(t) = x(t)^2.$$

$$\Rightarrow A'(t) = 2x(t) \cdot x'(t)$$

$$A'(1) = 2x(1) \cdot x'(1)$$

$$x(t) = 3t - t^2 \Rightarrow x(1) = 3(1) - 1^2 = 2$$

$$\Rightarrow x'(t) = 3 - 2t \Rightarrow x'(1) = 3 - 2 \cdot 1 = 1.$$

$$\Rightarrow A'(1) = 2(2)(1) = 4 > 0$$

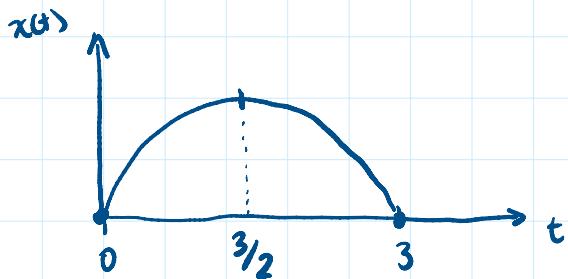
$$\Rightarrow A'(1) = 2(2)(1) = 4 > 0$$

(b) Conceptually:

[A is at a max $\Leftrightarrow x$ is at a maximum.]

$$A(t) = x(t)^2$$

$$A'(t) = 2x(t) \cdot x'(t)$$



$$x(t) = 3t - t^2, \quad x: [0, 3] \rightarrow \mathbb{R} \text{ continuous}$$

$$A(t) = (x(t))^2, \quad A: [0, 3] \rightarrow \mathbb{R} \Rightarrow \text{continuous}$$

A cont. on closed domain

\Rightarrow min/max

(i) both guaranteed to exist

(ii) occur at either critical pts.
or endpoints

$$t \in (0, 3)$$

$$0 = A'(t) = 2x(t)x'(t) = 2(3t - t^2)\underbrace{(3 - 2t)}_{=0}$$

$$\Rightarrow t = 3/2$$

$$A(0) = (x(0))^2 = (3 \cdot 0 - 0^2)^2 = 0$$

$$A(3) = (x(3))^2 = (3 \cdot 3 - 3^2)^2 = 0$$

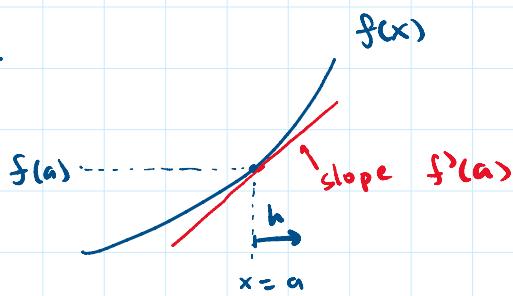
$$\begin{aligned} A\left(\frac{3}{2}\right) &= (x\left(\frac{3}{2}\right))^2 = (3 \cdot \frac{3}{2} - (\frac{3}{2})^2)^2 \\ &= \left(\frac{9}{2} - \frac{9}{4}\right)^2 > 0. \end{aligned}$$

\Rightarrow Area is a maximum at $t = 3/2$.

Problem 5

Find the linearization $L(x)$ of the function $f(x) = e^{2x-1}$ about the point $x = 1/2$. Using the linearization, estimate the value of $f(0.51)$ without a calculator.

Linearization



$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$\approx \frac{f(arh) - f(a)}{h} \quad (h \text{ small})$$

$$f(arh) \approx f(a) + h \cdot f'(a).$$

Linearization of f at $x=a$:

$$L(x) = f(a) + f'(a) \cdot (x-a)$$

$$f(x) = e^{2x-1}, \quad x=1/2.$$

$$f(1/2) = e^{2 \cdot \frac{1}{2} - 1} = e^0 = 1.$$

$$f'(x) = e^{2x-1} \cdot \frac{d}{dx}(2x-1) = 2e^{2x-1}$$

$$\Rightarrow f'(1/2) = 2$$

$$\Rightarrow L(x) = 1 + 2(x - \frac{1}{2})$$

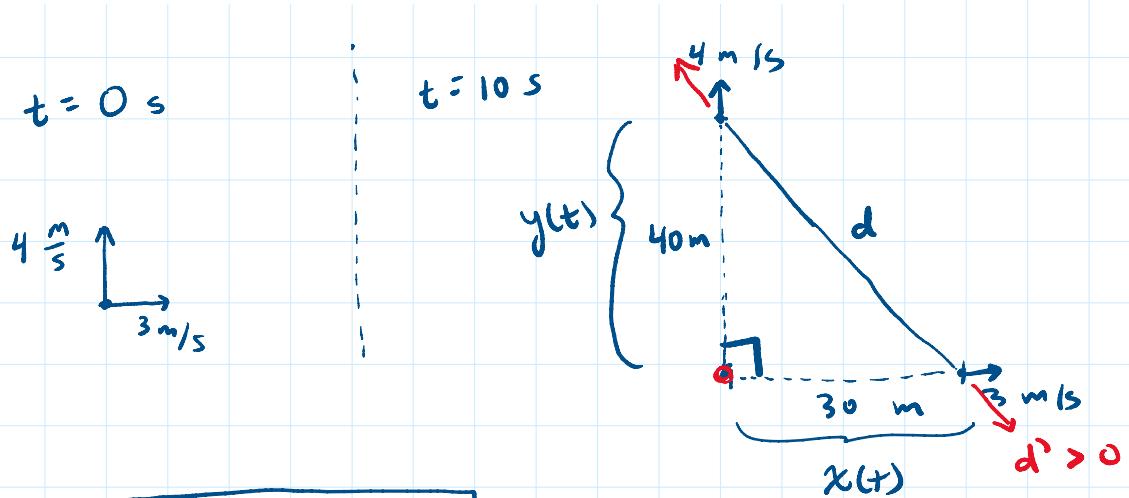
$$f(0.51) \approx L(0.51) = 1 + 2(0.51 - 0.50)$$

$$= 1 + 2(0.01) = 1 + 0.02 = 1.02.$$

1.0202...

Problem 6

Two runners start at the same point. When the clock starts, one runner runs north at a constant pace of 4 m/s (meters per second) and the other runner runs east at a constant pace of 3 m/s . After 10 seconds, at what rate is the distance between the two runners increasing?



$$d(t) = \sqrt{x(t)^2 + y(t)^2} \quad \text{seconds}$$

method 1: $x(t) = (3 \text{ m/s}) \cdot t = 3t \quad (\text{meters})$

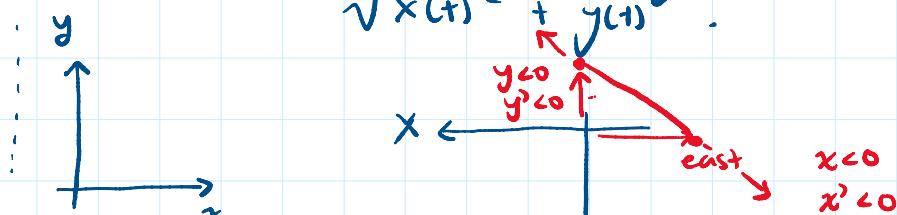
$$y(t) = (4 \text{ m/s}) \cdot t = 4t$$

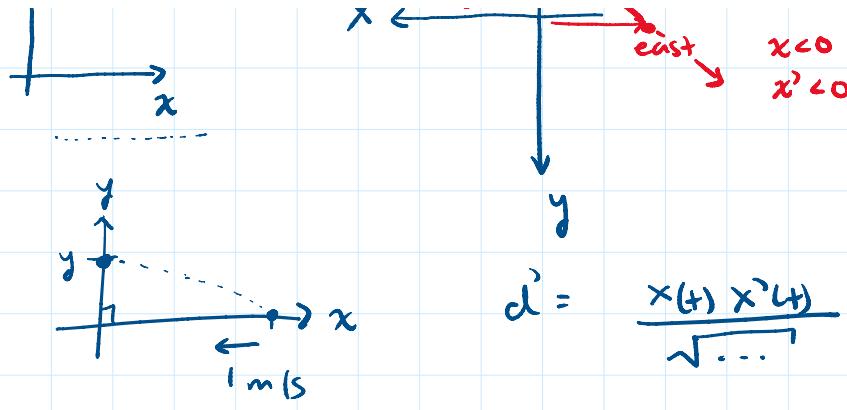
$$d(t) = \sqrt{(3t)^2 + (4t)^2} = \sqrt{9t^2 + 16t^2} = 5t$$

$$d'(t) = 5 > 0$$

method 2:

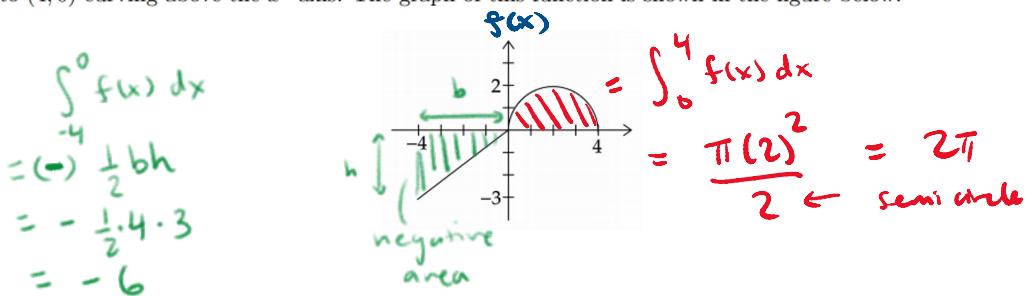
$$\begin{aligned} d'(t) &= \frac{d}{dt} \left(x(t)^2 + y(t)^2 \right)^{1/2} \\ &= \frac{1}{2} (x(t)^2 + y(t)^2)^{-1/2} \cdot \frac{d}{dt} (x(t)^2 + y(t)^2) \\ &= \frac{1}{2} \left(\frac{2x(t)x'(t) + 2y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} \right) \\ &= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} > 0 \end{aligned}$$





Problem 7

Consider a function $f : [-4, 4] \rightarrow \mathbb{R}$ given as follows. For $-4 \leq x \leq 0$, the function is a line which connects the point $(-4, -3)$ to $(0, 0)$. For $0 \leq x \leq 4$, the function is a semicircular arc connecting $(0, 0)$ to $(4, 0)$ curving above the x -axis. The graph of this function is shown in the figure below.



(a) Using geometry, evaluate $\int_{-4}^4 f(x) dx$.

(b) Using geometry, evaluate $\int_{-4}^4 |f(x)| dx$.

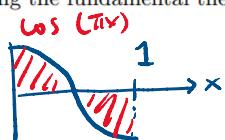
$$(a) \int_{-4}^4 f(x) dx = \underbrace{\int_{-4}^0 f(x) dx}_{=-6} + \underbrace{\int_0^4 f(x) dx}_{2\pi} = 2\pi - 6$$

$$(b) \int_{-4}^4 |f(x)| dx = \underbrace{\int_{-4}^0 |f(x)| dx}_{6} + \underbrace{\int_0^4 |f(x)| dx}_{2\pi + 2\pi} = 2\pi + 6$$

Problem 8

Find all antiderivatives for the functions $\cos(\pi x)$ and e^{2x} . Subsequently, using the fundamental theorem of calculus, evaluate the definite integral

$$\int_0^1 (2 \cos(\pi x) - 3e^{2x}) dx.$$



$$\int \cos(\pi x) dx = \frac{\sin(\pi x)}{\pi} + C$$

$$\frac{d}{dx} \left(\frac{\sin(\pi x)}{\pi} \right) = \cos(\pi x)$$

$$\int \cos(\pi x) dx = \frac{\sin(\pi x)}{\pi} + C$$

$$\frac{d}{dx} \left(\frac{\sin(\pi x)}{\pi} \right) = \cos(\pi x)$$

$$\int e^{2x} dx = \frac{e^{2x}}{2} + C$$

$$\frac{d}{dx} \left(\frac{e^{2x}}{2} \right) = e^{2x}$$

$$\begin{aligned}
& \int_0^1 [2\cos(\pi x) - 3e^{2x}] dx \\
& \stackrel{\text{linearity}}{=} 2 \int_0^1 \cos(\pi x) dx - 3 \int_0^1 e^{2x} dx \\
& = 2 \int_0^1 \frac{d}{dx} \left[\frac{\sin(\pi x)}{\pi} \right] dx - 3 \int_0^1 \frac{d}{dx} \left[\frac{e^{2x}}{2} \right] dx \\
& \stackrel{\text{FTC I}}{=} 2 \left. \frac{\sin(\pi x)}{\pi} \right|_0^1 - 3 \left. \frac{e^{2x}}{2} \right|_0^1 \\
& = \underbrace{\frac{2}{\pi} (\sin(\pi) - \sin(0))}_{= 0} - \frac{3}{2} (e^2 - e^0) \\
& = -\frac{3}{2} (e^2 - 1).
\end{aligned}$$

Problem 9

Compute the following derivative:

$$\frac{d}{dx} \int_1^{x^2} \cos(\ln(t)) dt.$$

$$\text{FTC I} \quad \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

$$\begin{aligned}
& \text{(chain rule version)} \quad \frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) \cdot g'(x) \\
& .
\end{aligned}$$

$$f(t) = \cos(\ln(t))$$

$$g(x) = x^2 \Rightarrow g'(x) = 2x$$

$$\Rightarrow \frac{d}{dx} \int_1^{x^2} \cos(\ln(t)) dt = \cos(\ln(x^2)) \cdot 2x.$$

$$\Rightarrow \frac{d}{dx} \int_1^x \cos(\ln(t)) dt = \cos(\ln(x^2)) \cdot 2x.$$

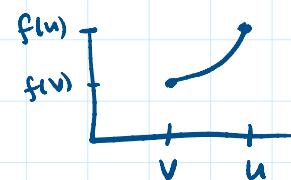
Problem 10

Using the MVT, prove the following: Let f be differentiable on the open interval (a, b) , where $b > a$, such that $f'(x) > 0$ for all $x \in (a, b)$. Then, f is a strictly monotone increasing function.

Remark. This would be something like the extra credit problem on the final, in that it is more of a proof and not computational.

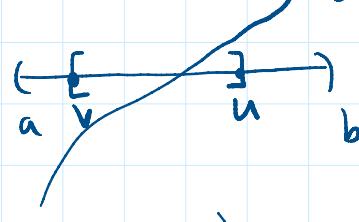
(strict)
Monotone increasing

$$u > v \quad (u, v \in (a, b)) \Rightarrow f(u) > f(v)$$



proof: Let $u, v \in (a, b)$, $u \neq v$.
Assume $u > v$.

Idea: apply MVT to $[v, u]$.



- f continuous on closed interval
- Is f continuous on $[v, u]$?

f is differentiable on (a, b)

$\Rightarrow f$ is continuous on (a, b)

$\Rightarrow f$ is continuous on $[v, u]$

is diff. on (v, u) .

~~f is cont. on $[a, b]$~~

MVT applies

\Rightarrow there exists $c \in (v, u)$ s.t.

$$f'(c) = \frac{f(u) - f(v)}{u - v}$$

$$\Rightarrow f(u) - f(v) = \underbrace{f'(c)}_{>0} \cdot \underbrace{(u - v)}_{>0} > 0$$

$$\Rightarrow f(u) > f(v).$$

We assumed $u > v$, we showed

We assumed $u > v$, we showed
 that $f(u) > f(v) \Rightarrow f$ is strictly monotone
 increasing □

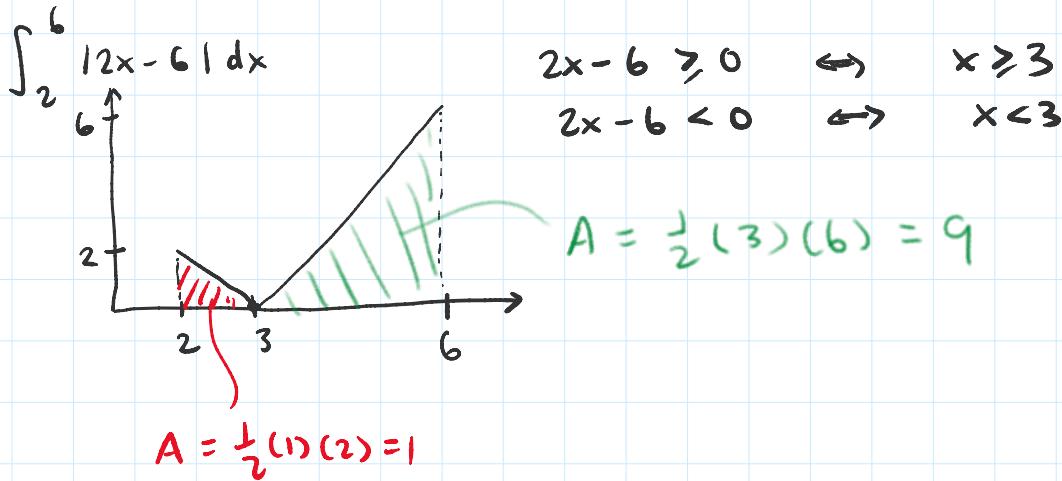
Homework 5

Problem 2 Practice Integrating Piecewise Functions

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = |2x - 6|$. Using geometry, compute

$$\int_2^6 f(x) dx.$$

Hint: Use that $|t|$ equals t for $t > 0$ and $-t$ for $t < 0$ to give a piecewise definition of $f(x)$ (when is $2x - 6$ positive and negative?). Then, split the integral from 2 to 6 into the regions where f is piecewise defined. Subsequently, use geometry to evaluate the integral in each region.



$$\int_2^6 f(x) dx = \int_2^3 f(x) dx + \int_3^6 f(x) dx = 10$$

Problem 8 Practice with the FTC I

Evaluate using FTC I:

$$\int_0^2 \left(3 \cos(\pi x) - e^{2x+4} + \frac{1}{x+1} \right) dx.$$

Hint: Use linearity to break up the integral into three separate integrals and then apply FTC I to each integral. For the exponential term, note $e^{2x+4} = e^4 e^{2x}$.

linearity

$$\rightarrow = 3 \int_0^2 \cos(\pi x) dx - e^4 \int_0^2 e^{2x} dx + \int_0^2 \frac{1}{x+1} dx$$

$$\begin{aligned}
 &= 3 \int_0^2 \frac{d}{dx} \left[\frac{\sin(\pi x)}{\pi} \right] dx - e^4 \int_0^2 \frac{d}{dx} \left[\frac{e^{2x}}{2} \right] dx + \int_0^2 \frac{d}{dx} [\ln|x+1|] dx \\
 &\stackrel{\text{FTC I}}{=} 3 \left. \frac{\sin(\pi x)}{\pi} \right|_0^2 - e^4 \cdot \left. \frac{e^{2x}}{2} \right|_0^2 + \left. \ln|x+1| \right|_0^2 \\
 &= \dots
 \end{aligned}$$

Problem 9 Exercise 5.4.55

Evaluate $\int_{-2}^3 f(x) dx$, where

$$f(x) = \begin{cases} 12 - x^2, & x \leq 2, \\ x^3, & x > 2. \end{cases}$$

$$\begin{aligned}
 \int_{-2}^3 f(x) dx &= \int_{-2}^2 f(x) dx + \int_2^3 f(x) dx \\
 &= \int_{-2}^2 (12 - x^2) dx + \int_2^3 x^3 dx \\
 &= \int_{-2}^2 \frac{d}{dx} \left(12x - \frac{x^3}{3} \right) dx + \int_2^3 \frac{d}{dx} \left(\frac{x^4}{4} \right) dx \\
 &\stackrel{\text{FTC I}}{=} \left. \left(12x - \frac{x^3}{3} \right) \right|_{-2}^2 + \left. \frac{x^4}{4} \right|_2^3.
 \end{aligned}$$

Problem 10 Practice with the Comparison Theorem

Recall, in lecture, we showed that $\sin(x) \leq x$ for any $x \geq 0$. Using the comparison theorem, prove the inequality (for $u \geq 0$)

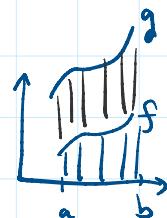
$$1 - \cos(u) \leq \frac{1}{2}u^2.$$

Hint: Recall that the comparison theorem says that if $f(x) \leq g(x)$ on an interval $[a, b]$ and if f and g are integrable on $[a, b]$, then one has

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Use the comparison theorem with $f(x) = \sin(x)$, $g(x) = x$, $a = 0$, $b = u$ (note our choices of f and g are integrable on $[0, u]$ since they are continuous) to prove the desired inequality.

$$\underbrace{\sin(x)}_0 \leq \underbrace{x}_1 \quad x \geq 0$$



$$\sin(x) \leq x \quad x \geq 0$$

On $[0, u]$, $f(x) \leq g(x)$

comparison theorem $\Rightarrow \int_0^u f(x) dx \leq \int_0^u g(x) dx$

$$\Rightarrow \int_0^u \sin x dx \leq \int_0^u x dx$$

$$\int_0^u \frac{d}{dx}[-\cos(x)] dx \leq \int_0^u \frac{d}{dx}\left(\frac{x^2}{2}\right) dx$$

FTC I $-\cos x \Big|_0^u \leq \frac{x^2}{2} \Big|_0^u$

$$-\cos u - (-\cos 0) \leq \frac{u^2}{2} - \frac{0^2}{2}$$

$$1 - \cos(u) \leq u^2/2$$

$$\rightarrow 1 - \cos(u) \geq 1 - u^2/2 \quad (u > 0)$$

□

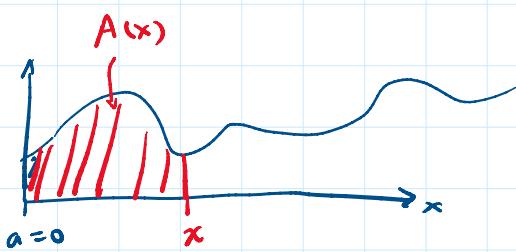
Problem 12 Exercise 5.5.6

Compute the area function $A(x)$ of $f(x)$ with lower limit a . Then, verify the FTC II relationship by checking that $A'(x)$ equals $f(x)$.

$$f(x) = 1 - x + \cos(x), [a = 0]$$

Hint: Recall that area function A of a function f with lower limit a is given by

$$A(x) = \int_a^x f(t) dt.$$



$$\begin{aligned} A(x) &= \int_0^x f(t) dt = \int_0^x (1 - t + \cos(t)) dt \\ &= \int_0^x \frac{d}{dt} \left(t - \frac{t^2}{2} + \sin(t) \right) dt \end{aligned}$$

$$\begin{aligned} \text{FTC I} &= \left[t - \frac{t^2}{2} + \sin(t) \right] \Big|_0^x \\ &= x - \frac{x^2}{2} + \sin(x). \end{aligned}$$

check $A'(x) = f(x)$.

$$= x - \frac{x^2}{2} + \sin(x). \quad A(x) = f(x).$$

Problem 14 Practice with the (chain rule version of) FTC II

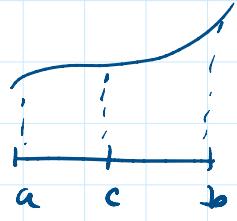
Calculate the derivative

$$\frac{d}{dx} \int_{x^2}^{e^x} \ln(t+1) dt.$$

Hint: Using the properties of the definite integral, we can write

$$\int_{x^2}^{e^x} \ln(t+1) dt = \int_{x^2}^0 \ln(t+1) dt + \int_0^{e^x} \ln(t+1) dt = - \int_0^{x^2} \ln(t+1) dt + \int_0^{e^x} \ln(t+1) dt.$$

Subsequently, differentiate the two integrals on the right hand side using the chain rule version of the FTC II.



$$\begin{aligned}
 & \underline{\frac{d}{dx} \int_{x^2}^{e^x} \ln(t+1) dt} \quad \int_a^b = \int_a^c + \int_c^b \\
 & \int_{x^2}^{e^x} \ln(t+1) dt = \int_{x^2}^0 \ln(t+1) dt + \int_0^{e^x} \ln(t+1) dt \\
 & = - \int_0^{x^2} \ln(t+1) dt + \int_0^{e^x} \ln(t+1) dt \\
 & \frac{d}{dx} \int_{x^2}^{e^x} \ln(t+1) dt \\
 & = \frac{d}{dx} \left[- \int_0^{x^2} \ln(t+1) dt + \int_0^{e^x} \ln(t+1) dt \right] \\
 & = - \frac{d}{dx} \int_0^{x^2} \ln(t+1) dt + \frac{d}{dx} \int_0^{e^x} \ln(t+1) dt \\
 & \stackrel{\text{linearity}}{=} - \ln(x^2+1) \cdot 2x + \ln(e^x+1) \cdot e^x \quad \square \\
 & \stackrel{\text{(chain rule)}}{\text{FTC I}}
 \end{aligned}$$

Problem 1 Integrating Odd and Even Functions

- (a) We say that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an odd function if $f(x) = -f(-x)$ for any x (an example of such a function is $f(x) = x$). Assume that f is integrable on the region $[-a, a]$ where $a > 0$. Graphically, explain why

$$\int_{-a}^a f(x) dx = 0.$$

- (b) We say that $f : \mathbb{R} \rightarrow \mathbb{R}$ is an even function if $f(x) = f(-x)$ for any x (an example of such a function is $f(x) = x^2$). Assume that f is integrable on the region $[-a, a]$ where $a > 0$. Graphically, explain why

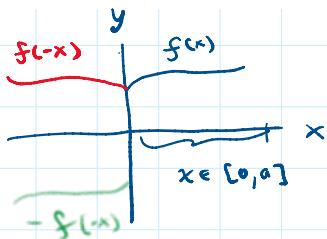
$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

y

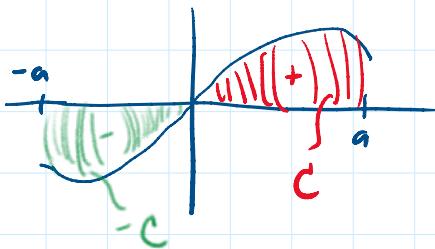
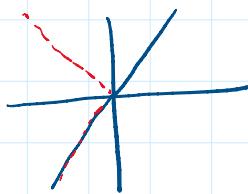
a

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx.$$

$$f(x) = -f(-x)$$



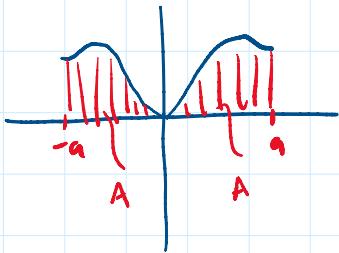
$$f(x) = x$$



$$\int_{-a}^a f(x) dx = \underbrace{\int_{-a}^0 f(x) dx}_{-C} + \underbrace{\int_0^a f(x) dx}_C = 0.$$

$$f(x) = f(-x)$$

even!



$$\begin{aligned} \int_{-a}^a f(x) dx &= \underbrace{\int_{-a}^0 f(x) dx}_A + \underbrace{\int_0^a f(x) dx}_A \\ &= 2A \\ &= 2 \int_0^a f(x) dx \end{aligned}$$